Matemática Contemporânea, Vol. 46, 156–164
http://doi.org/10.21711/231766362020/rmc4616

A Connectivity-based Decomposition for Graph Edge-colouring

Leandro M. Zatesko **João P. W. Bernardi**
Sheila M. de Almeida **Renato Carmo**
André L. P. Guedes

Abstract

We show how the problem of computing an optimal edge-colouring of a graph G can be decomposed into the problem of computing optimal edge-colourings of the biconnected components of G. That is, once optimal edge-colourings of the biconnected components are independently given, they can be adjusted in polynomial time in order to compose an optimal edge-colouring of the whole graph G with no colour conflicts. We use this decomposition strategy to show that a long-standing conjecture (proposed by Figueiredo, Meidanis, and Mello in mid-1990s) on edge-colouring chordal graphs of odd maximum degree Δ holds when $\Delta = 3$. We discuss further decomposition algorithms for graph edge-colouring.

2000 AMS Subject Classification: 05C15, 05C85, 05C40, and 68R10.

Key Words and Phrases: graph edge-colouring; graph connectivity; graph algorithms; decomposition algorithms; chordal graphs.

This work was partially supported by PPGInf/UFPR, CNPq (grant no 428941/2016-8), UFFS (grant no 23205.001243/2016-30), and PESC/CAPES (grant no 0588/2018).

1 Introduction

A *k-edge-colouring* of a graph G is a function $\varphi\colon E(G) \to \mathscr{C}$ such that $\mathscr{C}$ is a set with k colours and $\varphi(e) \neq \varphi(f)$ whenever e and f are distinct adjacent edges of G. The *chromatic index* of G, denoted $\chi'(G)$, is the least k for which G admits a *k-edge-colouring*, in which case such edge-colouring is said to be *optimal*.

A natural lower bound for $\chi'(G)$ is the maximum degree $\Delta(G)$ of G. The celebrated Vizing's Theorem, on the other hand, states that a $(\Delta(G)+1)$-edge-colouring of G can be constructed in polynomial time for any simple graph G [13]. Therefore, simple graphs have been called *Class 1* or *Class 2*, if they have chromatic index $\Delta(G)$ or $\Delta(G)+1$, respectively.

All graphs considered in this work are simple. The vertices of degree $\Delta(G)$ in a graph G are referred to as the *majors* of G. The subgraph of G induced by its majors is referred to as the *core* of G, denoted $\Lambda[G]$. The subgraph of G induced by its majors and by the neighbours of these majors is referred to as the *semi-core* of G, denoted $\Lambda\!\!\!\!\!-\,[G]$. Graphs with acyclic core are *Class 1* and admit polynomial-time optimal edge-colouring algorithms [8].

Despite the polynomiality of the algorithm implicit in Vizing's constructive proof, deciding if a graph G is *Class 1* is NP-complete [10], even when restricted to graph classes such as perfect graphs or d-regular graphs with girth at least g, for any fixed $d, g \geq 3$ [3]. Examples of graph classes in which the problem was shown to be polynomial are also known (e.g. [4, 2, 12]). For many other graph classes the complexity of the problem remains open[1]. In particular, for chordal[2] graphs the following conjecture remains open for more than 20 years.

Conjecture 1 ([5, 6, 7])**.** *All chordal graphs of odd maximum degree are Class 1.*

[1] We refer the reader to [14, Sect. 1.6] for an extensive table on the complexity of edge-colouring restricted to several graph classes.

[2] Recall that a chordal graph is a graph with no induced cycle C_k with $k \geq 4$.

While much investigation on graph classes in which edge-colouring problems can be solved efficiently has been done, there has also been much work aimed at identifying which kind of structural information of a general graph G is relevant to determine its chromatic index. In particular, the chromatic index of any graph is equal to the chromatic index of its semi-core. Furthermore, if an optimal edge-colouring of Ⱥ$[G]$ is given, then an optimal edge-colouring of the whole graph G can be constructed in polynomial time [11]. As recently observed in [9], this also leads to results on the parameterised complexity of edge-colouring problems.

This work explores results similar to the result by [11] in order to provide decomposition strategies for graph edge-colouring. In particular, we show how an optimal edge-colouring of a graph G can be efficiently constructed once optimal edge-colourings of the biconnected components[3] of G are given. We also observe some interesting corollaries which follow from our result, such as the fact that all chordal graphs with maximum degree at most 3 are *Class 1* and admit a polynomial-time optimal edge-colouring algorithm.

2 Main results and corollaries

Theorem 2 below presents our main decomposition result. In the statement, a biconnected Δ-component of a graph G is a biconnected component H of G such that $\Delta(H) = \Delta(G)$. Remark that Theorem 2 generalises the straightforward observation that the chromatic index of any disconnected graph is the maximum amongst the chromatic indices of its connected components.

Theorem 2. *The chromatic index of any graph G is the maximum amongst the degrees of the articulation points of G and the chromatic indices of its biconnected components. Moreover, being $\mathscr{B}$ the set of the biconnected Δ-components of G, if optimal edge-colourings of all $H \in \mathscr{B}$ are given,*

[3] Recall that the biconnected components of G are the maximal subgraphs of G which are biconnected.

then an optimal edge-colouring of the whole graph G can be constructed in polynomial time.

Proof. We first prove the following:

Claim 2.1. *If G_1 and G_2 are any two graphs with $V(G_1) \cap V(G_2) = \{u\}$, then $\chi'(G_1 \cup G_2) = \max\{\chi'(G_1), \chi'(G_2), d_{G_1 \cup G_2}(u)\}$.*

It should be noticed in the particular case wherein $G_2 = K_2$ that $\chi'(G_1 \cup K_2) = \max\{\chi'(G_1), d_{G_1}(u) + 1\}$.

Proof of the claim. Let $k := \chi'(G_1 \cup G_2) = \max\{\chi'(G_1), \chi'(G_2), d_{G_1 \cup G_2}(u)\}$. We assume that independent k-edge-colourings of both G_1 and G_2 are given, using the same colour set $\mathscr{C}$, and we shall demonstrate how to adjust these edge-colourings in order to obtain a valid k-edge-colouring of $G_1 \cup G_2$ in polynomial time. It is interesting to remark that we may not need optimal edge-colourings of both G_1 and G_2. If $k > \max\{\Delta(G_1), \Delta(G_2)\}$, then both k-edge-colourings of G_1 and G_2 can be obtained in polynomial time using Vizing's Theorem. If $k = \Delta(G_1) > \Delta(G_2)$, without loss of generality, then we need an optimal edge-colouring only of G_1, using Vizing's Theorem to obtain a k-edge-colouring of G_2 in polynomial time.

Since both k-edge-colourings of G_1 and G_2 use the same colour set, if we try to use these edge-colourings to compose a k-edge-colouring of $G_1 \cup G_2$, then colour conflicts may be created in the edges incident to u. However, as $k \geq d_{G_1}(u) + d_{G_2}(u)$, we can identify a set S of $d_{G_2}(u)$ colours of $\mathscr{C}$ not assigned to any edge incident to u in G_1 and simply permute $\mathscr{C}$ on G_2 so that the colours of the edges incident to u in G_2 are exactly the colours in S (see Figure 1). Observe that this operation can be carried out in polynomial time.

The proof is concluded by observing that the chromatic index of $G_1 \cup G_2$ cannot be less than k, since it cannot be less than the chromatic index of any of its subgraphs, and since $\chi'(G_1 \cup G_2) \geq \Delta(G_1 \cup G_2) \geq d_{G_1 \cup G_2}(u)$. $\Diamond$

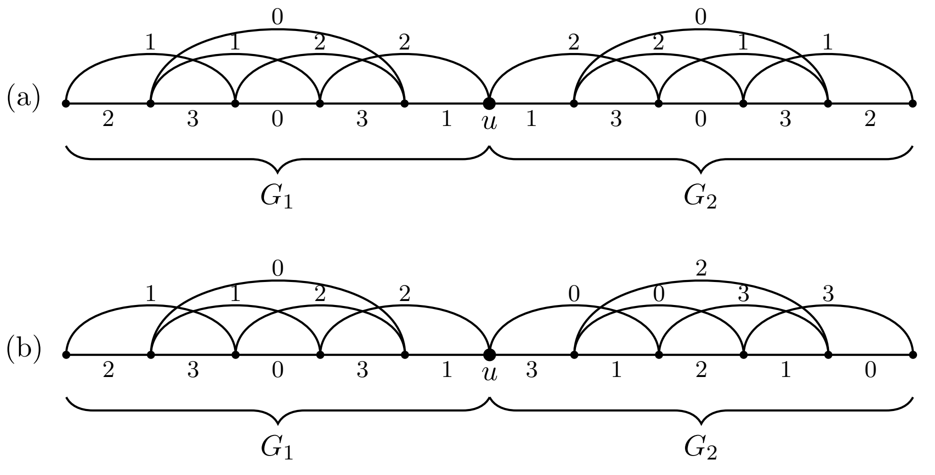

Figure 1: Permuting the colours of G_2 in order to resolve colour conflicts around vertex u (a) and obtain an optimal edge-colouring of G_2 (b)

The proof of the theorem follows by decomposing G into its (possibly many) biconnected components and its bridges (recall that K_2 is not considered a biconnected graph) and then inductively applying the constructive proof of the claim at each articulation point of G. □

From Theorem 2 follow interesting corollaries.

Corollary 3. *If no biconnected component of a graph G has the same maximum degree of G, then G is Class 1.*

Corollary 4. *Let $\mathscr{C}$ be the class of the graphs whose biconnected Δ-components have logarithmic size in the size of the graph. The problem of computing an optimal edge-colouring of a graph can be solved in polynomial time for graphs in $\mathscr{C}$.*

Proof. Follows from Theorem 2 and from the result in [1] according to which an optimal edge-colouring of any graph with m edges can be computed in $O(2^m m^{O(1)})$ time. □

Below we prove the restriction of Conjecture 1 for chordal graphs with $\Delta \leq 3$.

Theorem 5. *Except for the K_3, all chordal graphs with maximum degree $\Delta \leq 3$ are Class 1.*

Proof. Since odd cycles are the only *Class 2* graphs with $\Delta \leq 2$, and since the K_3 is the only odd cycle which is chordal, from Theorem 2 it suffices to prove that all biconnected chordal graphs with maximum degree $\Delta \leq 3$ are 3-edge-colourable. In order to do so, we shall demonstrate that if G is a biconnected chordal graph with maximum degree $\Delta \leq 3$, then G is a subgraph of the K_4, hence 3-edge-colourable.

For the sake of contradiction, assume that G has at least five vertices. Since $\Delta \leq 3$ and G is biconnected, there must be two non-adjacent vertices u and v in G such that, by Menger's Theorem, there are two internally disjoint paths between u and v, which implies that there is a cycle $C = x_0 x_1 \cdots x_t x_0$ in G for some $t \geq 3$ such that $u = x_0$ and $v = x_k$ for some $k \in \{2, \ldots, t-1\}$. Now, let i be the smallest integer in $\{1, \ldots, k-1\}$ such that $x_i v \in E(G)$, and let j be the greatest integer in $\{k+1, \ldots, t\}$ such that $x_j v \in E(G)$. Since G is chordal, the edges ux_i, ux_j, and $x_i x_j$ must all exist in G, which implies, since $\Delta \leq 3$, that C has only the four vertices u, x_i, v, x_j, which induce a diamond in G.

It is not hard to see that C is the only cycle containing u and v. If there is another cycle C', we can again demonstrate that C' has only four vertices and that these vertices induce a diamond in G. However, this would imply that $V(C) \cap V(C') = \{u, v\}$, since x_i and x_j already have degree three in C. But this would make the degrees of u and v at least four in G, a contradiction.

Since we have proved that C is the only cycle containing u and v, there must be at least one vertex x of $V(G) \backslash V(C)$ which is a neighbour of either u or v, say u, such that all paths between x and v contain u, contradicting the biconnectedness of G. □

3 Further comments

In view of Corollary 4, we encourage further investigation on phase diagrams for the size of the biconnected components in random graph models which capture the aspects of real-world networks.

As we have shown that an edge-colouring decomposition strategy can be achieved by decomposing a graph at its articulation points, we also believe that similar connectivity-based decomposition strategies can be achieved by considering separating K_2's, or separating K_ℓ's for higher values of ℓ. We remark that this can lead to proofs for restrictions of Conjecture 1 to chordal graphs with odd maximum degree at most Δ, for higher values of Δ.

Combining our results with other results from the literature, we can also obtain other decomposition strategies for edge-colouring algorithms. For example, given a graph G, which we want to optimally edge-colour, we know that we can take only the semi-core of G, leaving the other edges to be coloured later. Then, we break $\not{\mathrm{A}}[G]$ into its biconnected components. Finally, each biconnected component of $\not{\mathrm{A}}[G]$ is now a new graph for each we recursively call the decomposition algorithm. The halting criteria for this recursion can be: when we end up with a graph with smaller maximum degree, or acyclic core, or any edge-colouring result whatsoever from the literature that we may want (e.g. graph classes wherein edge-colouring is polynomial, as discussed in Sect. 1).

Acknowledgements

We would like to thank the referee for the valuable suggestions given.

References

[1] A. Björklund, T. Husfeldt, and M. Kolvisto, *Set partitioning via inclusion-exclusion*, SIAM J. Comput. **39** (2009), no 2, 546–563.

[2] H. L. Bodlaender, *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees*, J. Algorithms **11** (1990), 631–641.

[3] L. Cai and J. A. Ellis, NP-*completeness of edge-colouring some restricted graphs*, Discrete Appl. Math. **30** (1991), 15–27.

[4] A. G. Chetwynd and A. J. W. Hilton, *1-factorizing regular graphs of high degree: an improved bound*, Discrete Math. **75** (1989), 103–112.

[5] C. M. H. Figueiredo, J. Meidanis, and C. P. Mello, *A greedy method for edge-colouring odd maximum degree double chordal graphs*, Congr. Numer. **111** (1995), 170–176.

[6] C. M. H. Figueiredo, J. Meidanis, and C. P. Mello, *On edge-colouring indifference graphs*, Theor. Comput. Sci. **181** (1997), 91–106.

[7] C. M. H. Figueiredo, J. Meidanis, and C. P. Mello, *Total-chromatic number and chromatic index of dually chordal graphs*, Inf. Process. Lett. **70** (1999), 147–152.

[8] J.-C. Fournier, *Méthode et théorème générale de coloration des arêtes*, J. Math. Pures Appl. **56** (1977), 437–453.

[9] E. Galby, P. T. Lima, D. Paulusma, and B. Ries, *On the parameterized complexity of k-edge colouring*, Submitted manuscript (2019). URL: `https://arxiv.org/abs/1901.01861`

[10] I. Holyer, *The* NP-*completeness of edge-colouring*, SIAM J. Comput. **10** (1981), no 4, 718–720.

[11] R. C. S. Machado and C. M. H. Figueiredo, *Decompositions for edge-coloring join graphs and cobipartite graphs*, Discrete Appl. Math. **158** (2010), 1336–1342.

[12] C. Ortiz Z., N. Maculan, and J. L. Szwarcfiter, *Characterizing and edge-colouring split-indifference graphs*, Discrete Appl. Math. **82** (1998), 209–217.

[13] V. G. Vizing, *On an estimate of the chromatic class of a p-graph (in Russian)*, Diskret. Analiz. **3** (1964), 25–30.

[14] L. M. Zatesko, *Novel procedures for graph edge-colouring*, Doctorate thesis (2018), Federal University of Paraná. URL: `https://acervodigital.ufpr.br/handle/1884/58639`

Leandro M. Zatesko
Federal University of Fronteira Sul
Chapecó, Brazil
`leandro.zatesko@uffs.edu.br`

João P. W. Bernardi
Department of Informatics
Federal University of Paraná
Curitiba, Brazil
`winckler@ufpr.br`

Sheila M. de Almeida
Academic Department of Informatics
Federal University of Technology — Paraná
Ponta Grossa, Brazil
`sheilaalmeida@utfpr.edu.br`

Renato Carmo
Department of Informatics
Federal University of Paraná
Curitiba, Brazil
`renato@inf.ufpr.br`

André L. P. Guedes
Department of Informatics
Federal University of Paraná
Curitiba, Brazil
`andre@inf.ufpr.br`