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$\mathcal{D}^L$-integral and $\mathcal{D}^Q$-integral graphs

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Abstract

For a connected graph G, we denote by $D(G)$ the distance matrix of G and, by $T(G)$, the transmission matrix of G, the diagonal matrix of the row sums of $D(G)$. Matrices $D^L(G) = T(G) - D(G)$ and $D^Q(G) = T(G) + D(G)$ are called distance Laplacian and distance signless Laplacian of G, respectively. In this work we discuss M-integrality for some special classes of graphs, where $M = D^L(G)$ or $M = D^Q(G)$. In particular, we consider the complete split graphs, multiple complete split-like graphs, extended complete split-like graphs and multiple extended complete split-like graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph on n vertices and $A(G)$ be the adjacency matrix of G. For a connected graph G, we denote by $\mathsf{D}(G)$ the distance matrix of G, and by $T(G)$, the transmission matrix of G, the diagonal matrix of the row sums of $\mathsf{D}(G)$. Matrices $D^L(G) = T(G) - \mathsf{D}(G)$ and $\mathsf{D}^Q(G) = T(G) + \mathsf{D}(G)$ are called *distance Laplacian* and *distance*

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signless Laplacian of G, respectively. These matrices were introduced in [1] and have been studied in some articles [4, 12].

Among the aspects investigated in Spectral Graph Theory, one important problem is to characterize graphs for which all eigenvalues considering a matrix associated to a graph, are integers. Since 1974, when Harary and Schwenk [6] posed the question *Which graphs have integral spectra?*, the search for graphs G whose eigenvalues associated with the adjacency matrix are integer numbers has been considered [7]. In 2001 Balalińska et al [2] also published a survey of results on integral graphs. The same kind of problem has been addressed by considering the eigenvalues of other matrices, as the Laplacians for the adjacency matrix [5, 8] and recently this problem has been addressed to the distance matrix [10]. Also, some applications of this problem has been made in chemistry [11].

In a general way, if M is a real symmetric matrix associated to a graph G, we denote by $P_M(G, x)$ the characteristic polynomial of $M(G)$ and by $Sp_M(G)$ its spectrum. We will use exponents to exhibit the multiplicity of the eigenvalues. A graph G is called M-integral when all eigenvalues of M are integer numbers.

The aim of this work is to investigate D^L- and D^Q-integrality for some special classes of graphs. In particular, we consider the classes of complete split graphs, multiple complete split-like graphs, extended complete split-like graphs and multiple extended complete split-like The article is structured as follows: Section 2 is dedicated to discuss conditions to make some graphs D^L- or D^Q-integral. We also prove that for all connected graphs with diameter 2, the Laplacian integral graphs and the distance Laplacian integral graphs are the same and determine the D^Q-characteristic polynomial of special graphs. In Section 3 we present the unique star and wheel that are D^Q-integral. Futhermore, we construct infinite families of graphs that are D^Q-integral.

2 Results on D^L-integral and D^Q-integral graphs

In what follows, G denotes a graph with n vertices and $\overline{G}$ its complement. The diameter of a connected graph G is denoted by $diam(G)$. As usual, K_n and $K_n - e$ are, respectively, the complete graph and the complete graph minus an edge. Let G_1 and G_2 be disjoint graphs. We denote the Cartesian product of these graphs by $G_1 \times G_2$, the union by $G_1 \cup G_2$ and the join by $G_1 \vee G_2$.

We remember now some classes of graphs, that were defined in [7] and will be considered in this work:

Definition 2.1. *For a, b, $n \in \mathbb{N}$, we have the following classes of graphs: complete split graph, $CS_b^a \cong \overline{K_a} \vee K_b$; multiple complete split-like graph, $MCS_{b,n}^a \cong \overline{K_a} \vee (nK_b)$; extended complete split-like graph , $ECS_b^a \cong \overline{K_a} \vee (K_b \times K_2)$, multiple extended complete split-like graph , $MECS_{b,n}^a \cong \overline{K_a} \vee (n(K_b \times K_2))$.*

The next lemma will be useful to obtain the characteristic polynomials of some graphs:

Lemma 2.2. *[3] Let M be a square matrix of order n that can be written in blocks as*

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,k} \\ M_{2,1} & M_{2,2} & \dots & M_{2,k} \\ \vdots & \vdots & & \vdots \\ M_{k,1} & M_{k,2} & \dots & M_{k,k} \end{bmatrix},$$

where $M_{i,j}$, $1 \leq i, j \leq k$, is a $n_i \times m_j$ matrix such that its lines have constant sum equal to c_{ij}. Let $\overline{M} = [c_{ij}]_{k \times k}$.Then, the eigenvalues of $\overline{M}$ are also eigenvalues of M.

Theorem 2.3. *[1] The following results hold:*

- $Sp(\mathcal{D}^L(K_n)) = \left(n^{(n-1)}, 0\right)$ *and* $Sp(\mathcal{D}^Q(K_n)) = \left(2n-2, (n-2)^{(n-1)}\right)$.

- *If G is k-transmission regular then* $P_{\mathcal{D}^L}(G,x) = (-1)^n P_{\mathcal{D}}(G,k-x)$ *and* $P_{\mathcal{D}^Q}(G,x) = P_{\mathcal{D}}(G,x-k)$;
- $Sp(\mathcal{D}^L(K_n-e)) = \left(n+2, n^{(n-2)}, 0\right)$;
- $Sp(\mathcal{D}^Q(K_n-e)) = \left(\frac{3n-2\pm\sqrt{n^2-4n+20}}{2}, (n-2)^{(n-2)}\right)$.

As a consequence of the last theorem, we can enunciate some preliminary results:

Corollary 2.4. *The following results are valid:*

- *$G \cong K_n$ is D^L-integral and D^Q-integral;*
- *If G is transmission regular then G is D^L-integral if and only if D^Q-integral if and only if G is D-integral;*
- *$G \cong K_n-e$ is D^L-integral and it is D^Q-integral if and only if $n=5$.*

Proof. The first statements follow immediately from Theorem 2.3. To prove the last statement it suffices to see that $n^2-4n+20$ is a perfect square if and only if there is $y \in \mathbb{N}$ such that $(n-2)^2+16 = y^2$. It is easy to check that the unique possible solutions for $(n-2+y)(n-2-y) = -16$ are $n=2, y=4$ and $n=5, y=5$. Since the graph should be connected, the result holds. ■

In view of the following theorem, we can enunciate a equivalence between the set of connected graphs with diameter lower than or equal to 2 that are L-integral and D^L-integral, reobtaining that K_n and K_n-e are D^L-integral.

Theorem 2.5. *[1] Let G be a connected graph on n vertices with $diam(G) \leq 2$. Let $\mu_1 \geq \ldots \geq \mu_{n-1} > \mu_n = 0$ be the Laplacian spectrum of G. Then the distance Laplacian spectrum of G is $2n-\mu_{n-1} \geq \ldots \geq 2n-\mu_1 > \partial^L_n = 0$.*

Corollary 2.6. *Let G be a connected graph on n vertices and $diam(G) \leq 2$. The graph G is D^L-integral if and only if G is L-integral.*

Since any graph presented in Definition 2.1 has diameter 2, and they are Laplacian integral ([9], Theorem 2.1), we can enunciate:

Theorem 2.7. *$CS_b^a, MCS_{b,n}^a, ECS_b^a$ and $MECS_{b,n}^a$ are D^L-integral.*

As all graphs discussed here have diameter 2, we focus the discussion from now just on the distance signless Laplacian matrix. Moreover, henceforth, $\mathbf{1}_j$ and $\mathbf{0}_j$ are the vectors of order j with all elements equal to 1 and 0, respectively. By $\mathbf{0}_{j,k}$ and $\mathbb{J}_{j,k}$ we denote the $j\times k$ all zeros and the all ones matrices. For the $j\times j$ all ones matrix we just write $\mathbb{J}_j$.

In the following results, we determine the D^Q-characteristic polynomials for graphs $G_1\vee G_2$ and $G_1\vee(G_2\cup G_3)$, where G_i is r_i-regular, for $i=1$, 2 and 3. In each case we provide necessary and sufficient conditions for D^Q-integrality in these classes of graphs.

Theorem 2.8. *For $i=1$, 2, let G_i be a r_i-regular graph on n_i vertices. Then, the characteristic polynomial of the matrix $\mathsf{D}^Q(G_1\vee G_2)$ is equal to*

$$\frac{P_A(G_1)(-x+(2(n_1-2)+n_2-r_1))P_A(G_2)(-x+(2(n_2-2)+n_1-r_2))}{(x-(2(n_1-2)+n_2-2r_1))(x-(2(n_2-2)+n_1-2r_2))}f(x),$$

where

$$f(x)=x^2+x(-5(n_1+n_2)+2(r_1+r_2)+8)+4(n_1^2+n_2^2)+8(r_1+r_2-n_1r_2-n_2r_1)$$
$$-20(n_1+n_2)-2(n_1r_1+n_2r_2)+16n_1n_2+4r_1r_2+16. \qquad (1)$$

Proof. For $i=1$, 2, let G_i be a r_i-regular graph on n_i vertices. Then, the matrix $\mathcal{D}^Q(G)$ can be written as

$$\begin{bmatrix} (2(n_1-2)+n_2-r_1)\mathbb{I}_{n_1}-A(G_1)+2\mathbb{J}_{n_1} & \mathbb{J}_{n_1,n_2} \\ \mathbb{J}_{n_2,n_1} & (2(n_2-2)+n_1-r_2)\mathbb{I}_{n_2}-A(G_2)+2\mathbb{J}_{n_2} \end{bmatrix}.$$

It follows that, if $v_1\in\mathbb{R}^{n_1}$ is such that $A(G_1)v_1=\lambda_{v_1}v_1$ e $v_1\perp\mathbf{1}_{n_1}$, then the vector $u_1=\begin{bmatrix} v_1 & \mathbf{0}\end{bmatrix}^T\in\mathbb{R}^{n_1+n_2}$ satisfies $\mathcal{D}^Q(G)u_1=(2(n_1-2)+n_2-r_1-\lambda_{v_1})u_1$. Thus, $2(n_1-2)+n_2-r_1-\lambda_{v_1}$ is an eigenvalue of $\mathcal{D}^L(G)$.

Analogously, let $v_2\in\mathbb{R}^{n_2}$ such that $A(G_2)v_2=\lambda_{v_2}v_2$ and $v_2\perp\mathbf{1}_{n_2}$, we conclude that $u_2=\begin{bmatrix}\mathbf{0} & v_2\end{bmatrix}^T\in\mathbb{R}^{n_1+n_2}$ is an eigenvector of $\mathcal{D}^Q(G)$, corresponding to the eigenvalue $2(n_2-2)+n_1-r_2-\lambda_{v_2}$. Moreover, from

Lemma 2.2, the eigenvalues of the matrix

$$M_Q = \begin{bmatrix} 4(n_1-1)+n_2-2r_1 & n_2 \\ n_1 & 4(n_2-1)+n_1-2r_2 \end{bmatrix},$$

that are the roots of $f(x)$ presented in (1), are also eigenvalues of $\mathcal{D}^Q(G)$. ■

Corollary 2.9. *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices. Then, the graph $G_1 \vee G_2$ is $\mathcal{D}^Q$-integral if and only if G_1 and G_2 are A-integral and $(3(n_1-n_2)-2(r_1-r_2))^2+4n_1n_2$ is a perfect square.*

Theorem 2.10. *For $i = 1, 2, 3$ let G_i be a r_i-regular graph on n_i vertices. Then, the characteristic polynomial of the matrix $\mathcal{D}^Q(G_1 \vee (G_2 \cup G_3))$ is equal to*

$$\frac{P_A(G_1)(-x+(2(n_1-2)+n_2+n_3-r_1))P_A(G_2)(-x+(n_1+2(n_2-2)+2n_3-r_2))}{(x-(2(n_1-2)+n_2+n_3-2r_1))(x-(n_1+2(n_2-2)+2n_3-2r_2))}\times$$

$$\frac{P_A(G_3)(-x+(n_1+2n_2+2(n_3-2)-r_2))}{(x-(n_1+2n_2+2(n_3-2)-2r_3))}g(x),$$

where $g(x)$ is the characteristic polynomial of the matrix

$$\begin{bmatrix} 4(n_1-1)+n_2+n_3-2r_1 & n_2 & n_3 \\ n_1 & n_1+4(n_2-1)+2n_3-2r_2 & 2n_3 \\ n_1 & 2n_2 & n_1+2n_2+4(n_3-1)-2r_3 \end{bmatrix}. \tag{2}$$

Proof. Let $G = G_1 \vee (G_2 \cup G_3)$. Then, $\mathcal{D}^Q(G)$ can be written as

$$\begin{bmatrix} C_1 & \mathbb{J}_{n_1,n_2} & \mathbb{J}_{n_1,n_3} \\ \mathbb{J}_{n_2,n_1} & C_2 & 2\mathbb{J}_{n_2,n_3} \\ \mathbb{J}_{n_3,n_1} & 2\mathbb{J}_{n_3,n_2} & C_3 \end{bmatrix}, \begin{cases} C_1 = (2n_1-r_1+n_2+n_3-4)\mathbb{I}_{n_1}-A(G_1)+2\mathbb{J}_{n_1}; \\ C_2 = (n_1+2n_2+2n_3-r_2-4)\mathbb{I}_{n_2}-A(G_2)+2\mathbb{J}_{n_2}; \\ C_3 = (n_1+2n_2+2n_3-r_3-4)\mathbb{I}_{n_3}-A(G_3)+2\mathbb{J}_{n_3}. \end{cases}$$

We note that if $v_1 \in \mathbb{R}^{n_1}$ is such that $A(G_1)v_1 = \lambda_{v_1}v_1$ and $v_1 \perp \mathbf{1}_{n_1}$, then the vector $u_1 = \begin{bmatrix} v_1 & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_1+n_2+n_3}$ satisfies $\mathcal{D}^Q(G)u_1 = (2n_1+n_2+n_3-r_1-4-\lambda_{v_1})u_1$. So, $2n_1+n_2+n_3-r_1-4-\lambda_{v_1}$ is an eigenvalue of $\mathcal{D}^Q(G)$.

Analogously, choosing $v_2 \in \mathbb{R}^{n_2}$ and $v_3 \in \mathbb{R}^{n_3}$ such that $A(G_2)v_2 = \lambda_{v_2} v_2$, with $v_2 \perp \mathbf{1}_{n_2}$, and $A(G_3)v_3 = \lambda_{v_3} v_3$, with $v_3 \perp \mathbf{1}_{n_3}$, we conclude that $u_2 = \begin{bmatrix} \mathbf{0} \ v_2 \ \mathbf{0} \end{bmatrix}^T \in \mathbb{R}^{n_1+n_2+n_3}$ and $u_3 = \begin{bmatrix} \mathbf{0} \ \mathbf{0} \ v_3 \end{bmatrix}^T \in \mathbb{R}^{n_1+n_2+n_3}$ are eigenvectors of $\mathcal{D}^Q(G)$, associated to the eigenvalues $2n_2+n_1+2n_3-r_2-4-\lambda_{v_2}$ and $2n_3+n_1+2n_2-r_3-4-\lambda_{v_3}$, respectively. Moreover, from Lemma 2.2, the eigenvalues of the matrix presented in (2) are also eigenvalues of $\mathcal{D}^Q(G)$. ■

Corollary 2.11. *For $i = 1, 2, 3$ let G_i be a r_i-regular graph on n_i vertices. The graph $G_1 \vee (G_2 \cup G_3)$, is D^Q-integral if and only if the graphs G_1, G_2 and G_3 are A-integral and the eigenvalues of matrix presented in (2) are integer numbers.*

3 Infinite families of D^Q-integral graphs

In this section we build many infinity families of D^Q-integral graphs. We also determine the unique wheel and the unique star that are D^Q-integral, as immediate corollaries of precedent results.

Corollary 3.1. *Let $n \in \mathbb{N}$, $n \geq 4$. The wheel graph $W_n \cong C_{n-1} \vee K_1$ is $\mathcal{D}^Q$- integral if and only if $n = 5$, where C_{n-1} denotes the cycle on $n-1$ vertices.*

Proof. It is already well know that the graph C_{n-1} is A-integral if and only if $n = 4, 5$ or 7. So, using Corollary 2.9, we just have to check that among these values, just $n = 5$ makes $9n^2-56n+96$ a perfect square. ■

Proposition 3.2. *The complete bipartite graph $K_{a,b} \cong \overline{K}_a \vee \overline{K}_b$ is $\mathcal{D}^Q$-integral if and only if $9(a-b)^2+4ab$ is a perfect square.*

Corollary 3.3. *Let $j \in \mathbb{N}$. $K_{a,b}$ is $\mathcal{D}^Q$-integral if $a = b$ or if $a = \frac{j^2+5j+4}{2}$ and $b = \frac{j^2+3j}{2}$.*

Corollary 3.4. *The star S_n is $\mathcal{D}^Q$- integral if and only if $n = 2$.*

Proof. As $S_n \cong K_1 \vee \overline{K}_{n-1}$, it follows that the graph is D^Q-integral if and only if $9n^2-32n+32$ is a perfect square, what happens if and only if there is $x \in \mathbb{N}$ such that $9n^2-32n+32 = x^2$. This equation can be rewritten as $(6x+18n-32)(6x-18n+32) = 2^7$. As the sum of the terms in the left of the equation is equal to $12x$, it remains to analyze the possibilities $x = 3$ e $x = 2$. Thus, we conclude that the only possible solution is $n = 2$. ■

Proposition 3.5. *$CS_b^a \cong \overline{K}_a \vee K_b$ is $\mathcal{D}^Q$-integral if and only if $(3a+b+2)^2-8a(b+3)$ is a perfect square.*

Corollary 3.6. *Let $j \in \mathbb{N}$. CS_b^a is $\mathcal{D}^Q$-integral if one of the following conditions is satisfied: $a = 1$ and $b = j$; $a = j$ and $b = 2j-1$; $a = j+2$ and $b = 2j$; $a = 2j-1$ and $b = 9j-6$; $a = 2j$ and $b = 9j-6$.*

Proposition 3.7. *$MCS_{b,n}^a \cong \overline{K_a} \vee (nK_b)$ is $\mathcal{D}^Q$-integral if and only if $(3a+2b+3bn-2)^2-8bn(4a+3(b-1))$ is a perfect square.*

Corollary 3.8. *Let $j \in \mathbb{N}$. $MCS_{b,n}^a$ is $\mathcal{D}^Q$-integral if one of the following conditions is satisfied: $n = 2$, $a = 3j+2$ and $b = 2j+1$; $n = 3$, $a = 2j+3$ and $b = j+1$; $a = (n-1)j+1$ and $b = j$, for all $n \in \mathbb{N}$.*

Proposition 3.9. *$ECS_b^a \cong \overline{K}_a \vee (K_b \times K_2)$ is $\mathcal{D}^Q$-integral if and only if $(3a+4b)^2-40ab$ is a perfect square.*

Corollary 3.10. *Let $j \in \mathbb{N}$. ECS_b^a is $\mathcal{D}^Q$-integral if one of the following conditions is satisfied: $a = b$; $a = 3j$ and $b = j$ or $b = 2j$; $a = 7j$ and $b = 9j$; $a = 16j$ and $b = 7j$ or $b = 9j$; $a = 2j+1$ and $b = \frac{5j^2+7j+2}{2}$.*

Proposition 3.11. *$MECS_{b,n}^a$ is $\mathcal{D}^Q$-integral if and only if $(3a+2b+6bn)^2-16bn(4a+3b)$ is a perfect square.*

Corollary 3.12. *Let $j \in \mathbb{N}$. $MCS_{b,n}^a$ is $\mathcal{D}^Q$-integral if one of the following conditions is satisfied: $n = 2$ and $a = 8j$, $b = j$; $n = 3$ and $a = 3j$, $b = j$ or $a = 8j$, $b = j$ or $a = 5j$, $b = 2j$; $a = (2n-1)j$ and $b = j$; $a = (5n-2)j$ and $b = j$; $a = (4n-1)j$ and $b = 2j$; $a = (7n-2)j$ and $b = 5j$.*

Proposition 3.13. *$G \cong K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ is $\mathcal{D}^Q$-integral if and only if $n_1^2+16n_2n_3$ is a perfect square.*

Remark 3.14. *It follows from the last proposition that if* $G \cong K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ *is* $\mathcal{D}^Q$*-integral then it is also* $K_{cn_1} \vee (K_{cn_2} \cup K_{cn_3})$*, for all* $c \in \mathbb{N}$*. Actually, we can proceed analogously idea to get* D^Q*-integral graphs for the classes of the extended complete split-like graphs and multiple extended complete split-like graphs.*

Corollary 3.15. *Let* $j, p \in \mathbb{N}$*.* $G \cong K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ *is* $\mathcal{D}^Q$*-integral if one of the following conditions is satisfied:* $n_1 = n_2 = 2$ *and* $n_3 = \frac{j^2+3j+2}{2}$*;* $n_1 = n_2 = 2j+2$ *and* $n_3 = j+1$*;* $n_1 = j$*,* $n_2 = p$ *and* $n_3 = 4p-j$*, for* $j \leq 4p-1$*.*

We finalize by pointing out that, although for graphs of diameter 2 it could be established an equivalence between the classes of Laplacian integral graphs and distance Laplacian integral graphs (Theorem 2.7), we can not claim the same considering signless Laplacian and distance signless Laplacian matrices:

- $Sp\left(\mathcal{D}^Q(CS_2^3)\right) = \left(9, 4, 3^{(3)}\right)$ and $Sp\left(Q(CS_2^3)\right) = \left(\frac{7 \pm \sqrt{33}}{2}, 3, 2^{(2)}\right)$.
- $Sp\left(\mathcal{D}^Q(CS_4^4)\right) = \left(18, 8^{(4)}, 6^{(3)}\right)$ and $Sp\left(Q(CS_4^4)\right) = \left(12, 6^{(3)}, 4^{(3)}, 2\right)$.
- $Sp\left(\mathcal{D}^Q(CS_3^6)\right) = \left(\frac{33 \pm \sqrt{241}}{2}, 11^{(5)}, 7^{(2)}\right)$ and $Sp\left(Q(CS_3^6)\right) = \left(12, 7^{(2)}, 3^{(5)}, 1\right)$.

References

[1] M. Aouchiche and P. Hansen, *Two Laplacians for the distance matrix of a graph.* Lin. Alg. Appl., 439 (2013), 21–33.

[2] K. Balińska, D. Cvetković, Z. Radosavljević, et al. *A survey on integral graphs.* Univ. Beograd Publ. Elek. Fak. Ser. Mat., 13 (2002), 42-65.

[3] A.E. Brouwerm W.H. Haemers: *Spectra of graphs.* Springer, New York, etc., 2012.

[4] K. C. Das, *Proof of conjectures on the distance signless Laplacian eigenvalues of graphs.* Lin. Alg. Appl., 467 (2015), 100–115.

[5] M.A.A. de Freitas, N.M.M. de Abreu, R.R. Del-Vecchio, S.Jurkiewicz, *Infinite families of Q-integral graphs.* Lin. Alg. Appl., 432 (2010), 2353–2360.

[6] F. Harary, A.J. Schwenk, *Which graphs have integral spectra?* in Bari, R., Harary, F.(Eds), Graphs and Combinatorics, Springer, Berlim, (1974), 45–51.

[7] P. Hansen, H. Melot, D. Stevanović, *Integral complete split graphs.* Univ. Beograd. Publ. Elek. Fak. Ser. Mat., 13 (2002), 89–95.

[8] S. Kirkland, *Constructably Laplacian integral graphs.* Lin. Alg. Appl., 243 (2007), 3–21.

[9] R. Merris, *Laplacian graph eigenvectors.* Lin. Alg. Appl., 278 (1998), 221–236.

[10] M. Pokornýa, P. Híc, D. Stevanović, M. Milo?ević, *On distance integral graphs.* Disc. Math., 338 (2015), 1784–1792.

[11] T. Réti, I. Felde., *On Some Properties of Pseudo-Semiregular Graphs.* Acta Polytechnica Hungarica, 13 (2016), 45–65.

[12] F. Tian, D. Wong, and J. Rou, *Proof for four conjectures about the distance Laplacian and distance signless Laplacian eigenvalues of a graph.* Lin. Alg. Appl., 471 (2015), 10–20.

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