On the In-Neighbor Convexity

Alessandra A. Pereira  Carmen C. Centeno
Mitre C. Dourado  Jayme L. Szwarcfiter

Abstract

In this work, we study the in-neighbor convexity in the directed graph class. We work with the following parameters: in-neighbor convexity number, in-neighbor number and in-neighbor hull number. We present results concerning directed acyclic graphs, directed cyclic graphs and transitive graphs.

1 Introduction

The concept of graph convexity has been studied in many works and conceptions [1, 2, 3, 5]. This work propose a new convexity called in-neighbor convexity, such convexity can be applied in the process of spread of influence through a social network modeled by graphs. This process has been studied in many fields such as epidemiology [4], sociology, economics and computer science [6].

Let $D(V, A)$ be a simple directed graph, the indegree of a vertex $v$ is denoted by $d^-(v)$ and the outdegree of $v$ is denoted by $d^+(v)$. The set of direct predecessors of $v$ in $D$ is denoted by $N^-(v)$. A vertex with

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$d^-(v) = 0$ is a source. A directed graph is acyclic if it has no directed cycles. Every directed acyclic graph has at least one source. Let $f : V(D) \rightarrow \{0, 1, 2, \ldots, j\}$ be a function. A set $S \subset V(D)$ is in-neighbor convex if $\forall v \notin S$ has $|N^-(v) \cap S| < f(v)$. The in-neighbor convexity number of $D$, $c_p(D)$, is the cardinality of the largest proper in-neighbor convex set of $D$. The in-neighbor interval of a set $S \subset V(D)$, $I_p(S) = S \cup \{v \in V(D) : |N^-(v) \cap S| \geq f(v)\}$. When $I_p(S) = V(D)$, $S$ is called in-neighbor set. The in-neighbor number of $D$, $p_p(D)$, is the cardinality of the smallest in-neighbor set.

In this work we studied the In-Neighbor Convexity for directed acyclic graphs and directed cyclic graphs.

### 2 Results

**Theorem 1.** Let $D(V, A)$ be a directed acyclic graph, then:

$$c_p(D) = \begin{cases} |V(D)| - 1, & \text{if } \exists v : |N^-(v)| < f(v); \\ 0, & \text{if } \forall v : |N^-(v)| \geq f(v). \end{cases}$$

**Proof.** In the first case ($|V(D)| - 1$), in order to find the proper convex in-neighbor set of $D$ is sufficient to exclude one vertex that has $|N^-(v)| < f(v)$. The vertex $v$ will never have enough neighbors in $S$ to be contaminated. Therefore $c_p(D) = |V(D)| - 1$.

In the second case, we show that $I_p^+(\emptyset) = V(D)$, thus $c_p(D) = 0$. First, we prove that $I_p^+(\emptyset) \neq \emptyset$. Since $D$ is an acyclic directed graph, $D$ has a source, called $v$. By the property $|N^-(v)| \geq f(v)$, it follows that $f(v) = 0$. 

Assume that a vertex $v$ is contaminated if $v$ is in the subset $S$ or $I_p^+(S)$. A vertex becomes contaminated if it has at least $f(v)$ contaminated in-neighbors.
for all the sources. Observe that \( \{ x \in V(D) : f(x) = 0 \} \subseteq I^+_p(\emptyset) \). Thus, \( I^+_p(\emptyset) \) has all the sources of \( D \). Suppose that \( I^+_p(\emptyset) \neq V(D) \), therefore exists at least a vertex \( v \) of \( D \) that \( v \notin I^+_p(\emptyset) \), then \( v \) is not a source. By the property \( |N^-(v)| \geq f(v) \), we can conclude that \( |N^-(v)| + 1 - f(v) \) direct predecessors of \( v \) do not belong to \( I^+_p(\emptyset) \). The same can be concluded about these vertices and so on. Since the graph is acyclic, eventually we will find the beginning of the path, a source, that does not belong to \( I^+_p(\emptyset) \). Contradiction.

The in-neighbor number problem is NP-Complete for directed acyclic graphs. This problem can be reduced by applying the Propositional Satisfiability Problem, also known as SAT.

Consider a boolean expression written only by using the logical operators AND(\( \land \)), OR (\( \lor \)), NOT (\( \neg \)), variables (\( x_1, x_2, ..., x_i \)) and parenthesis. A literal corresponds to a single variable or its complement; a clause with the set of literals grouped by the disjunction symbol (OR). Therefore, the formulas will be a conjunction (AND) of clauses.

The following shows the reduction of it:

**Problem 1. In-Neighbor Set**

**INSTANCE:** Graph \( D = (V(D), A(D)) \), positive integer \( k \leq n \)

**QUESTION:** Is there an in-neighbor set of size \( k \) or less in \( D \), i.e., is there a subset \( V' \subseteq V(D) \), such that \( |V'| \leq k \) and every \( u \in V(D) - V' \) has at least \( f(u) \) predecessors in \( V' \)?

**Theorem 2.** The In-Neighbor Set problem is NP-complete when considering the directed acyclic graphs.

**Proof.** An in-neighbor set lies in NP, because considering a given certificate (a subset \( S \) of vertices of \( D \)), it is possible to answer in a polynomial time if \( S \) has the appropriated size and if all vertices that do not belong to \( S \) have \( f(x) \) predecessors in \( S \). In order to complete the proof, we describe a reduction from SAT. Let \( C=C_1 \land C_2 \land \ldots \land C_i \) be an arbitrary instance of SAT. We construct a graph \( D \) in the following way: for each clause

\[
C_1 \land C_2 \land \ldots \land C_i
\]
C_j of C creates a vertex c_j in V(D). For each variable C create a set of vertices \{x_l, \overline{x}_l, v_l, u_l\} and the edges \{(x_lu_l), (\overline{x}_lu_l), (v_lx_l), (v_l\overline{x}_l)\}. Assign to all vertices f(x) = 1. Finally, if a variable x_l/\overline{x}_l appears in the clause C_i, add the edge (x_lc_i)/(...c_i) in D.

First, we assume that C is satisfiable, and consider a minimum satisfying truth assignment S. Suppose S is a minimum set of true variables that satisfy C. Let P be the in-neighbor set of D and V the set of all vertices v of D. So P = S ∪ V ∪ \{u_l ∈ D | x_l, \overline{x}_l ∉ S\}. Since all clauses of C have, at least, one true variable, then we have that each c_i has at least a predecessor in P. The set of V ensures the predecessor of x and \overline{x}. The set S provides predecessors for some vertex u, those that do not have predecessor in S belongs to P. So k = 2i, which means, i vertices v, k vertices of S and i − k vertices of u.

For the converse, we assume that S is an in-neighbor set of D. Then S has all vertices v of D which ensures one predecessor for each x and \overline{x}, |S| = i. To be able to have an in-neighbor set it is required to put in S the vertices c or one of its predecessors. Since we want a minimum in-neighbor set, we will put in S one predecessor of c_i, because this predecessor is also a predecessor of u_i. There can also exist vertices u without predecessors, which for the matter of making it clear, we will add then to S. So |S| = (i) + (k) + (i − k) = 2i. Then the truth set of C is \{x/\overline{x}|x/\overline{x} ∈ S\}. ■

**Theorem 3.** Let D be a directed acyclic graph then h_p(D) = |S|, where S is the set of all vertices v of D such that |N^−(v)| < f(v).

**Proof.** Let D(V, A) be a directed acyclic graph and S the set of all vertices of D : |N^−(v)| < f(v). Suppose that S is not a in-neighbor hull of D. Then there is a vertex of D, v : v /∈ I_p^+(S). Therefore v has |N^−(v)| + 1 − f(v) direct predecessors that do not belong to I_p^+(S). These vertices clearly have |N^−(x)| ≥ f(x) and the same applies to them, and thus iteratively. Since the graph is acyclic, we will reach to the beginning of a path, a source, which or belongs to S or to I_p^+(S). Contradiction.
Clearly, $S$ is minimum.

Next, we consider directed cyclic graphs. We show algorithms that solve the problem for the in-neighbor convexity number, in-neighbor number and in-neighbor hull number.

**Algorithm 1**

**Input:** A directed cyclic transitive graph $D = (V, A)$ such that $\forall v \in V(D) : f(v) = k$, and $k$ is a constant integer greater than 1.

**Output:** $c_p(D)$.

1. Find the strongly connected components $C_i$ of the directed graph $D$.
2. Reduce each strongly connected component $C_i$ to a single vertex $v_i$ and assign a weight $P_i$ to each one, where $P_i := |V(C_i)|$.
3. For each $v_i$ create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of $C_i$ that is the direct predecessor of some vertex of $C_j$ in the original directed graph.
4. Choose a $v_i$ source with the lowest $P_i$ and do $m := P_i$.
5. Do $c_p(D) := |V(D)| - m + f(v) - 1$.

**Lemma 1.** Let $D$ be a directed, strongly connected and transitive graph, then $D$ is a complete graph.

**Proof.** Suppose that $D$ is not complete. By definition, a directed graph is strongly connected if there exists, for every pair $u, v$ of $D$, a directed path from $u$ to $v$ and vice versa. Since $D$ is not complete there is at least one edge $uv$ that does not belong to $A(D)$. However $D$ is transitive and by definition if there is a path between the vertices $u$ and $v$ then the edge $uv$ belongs to $A(D)$, therefore, there is an edge between each pair of vertices of $D$, contradiction. $D$ is complete.

When a strongly connected component $C_i$ is reduced to a source vertex $v_i$ in the Algorithm 1 let’s call it a source component.
Lemma 2. Let $D$ be a directed cyclic transitive graph, $C_i$ a strongly connected source component of $D$, $C_j$ a strongly connected non-source component of $D$, $C_i$ is predecessor of $C_j$; there is the edge $uv$ between each pair of vertices $u$ and $v$, where $u \in V(C_i)$ and $v \in V(C_j)$.

Proof. Suppose there is no edge $uv$ in $A(D)$, where $u \in V(C_i)$ and $v \in V(C_j)$. According to Lemma 1 the strongly connected components of $D$ are complete graphs, and as $C_i$ is predecessor of $C_j$ there is a path from $u$ to $v$. Since $D$ is transitive, by definition, the edge $uv$ belongs to $A(D)$ if there is a path between the vertices $u$ and $v$, contradiction. ■

As a consequence of Lemma 2 is easy to see that if $D$ is directed cyclic transitive graph, every non-source component has a source component as direct predecessor.

Theorem 4. The number $c_p(D)$ found by the Algorithm 1 is the cardinality of the largest proper in-neighbor convex set of $D$, where $D$ is a directed cyclic transitive graph such that $\forall v \in V(D) : f(v) = k$, and $k$ is a constant integer greater than 1.

Proof. Let $C_i$ be a transitive strongly connected component found by the Algorithm 1. According the Lemma 1, $C_i$ is a complete directed graph, then we can consider only $f(v) - 1$ vertices of $C_i$, otherwise, the component will become completely contaminated.

Suppose the $c_p(D)$ found by the Algorithm 1 is not the cardinality of the largest proper in-neighbor set $S$ of $D$, then there is a proper in-neighbor set $S'$ of $D$ where $|S'| > |S|$. Therefore, there is $C'_i$ such that $|C'_i| < |C_i|$, where $C_i$ is the source component with the lowest cardinality of $D$. We know that $C'_i$ is not a source, thus, according the Lemma 2, $C''_i$ will be contaminated by its predecessor, then $S'$ is not a proper in-neighbor set of $D$, contradiction. ■

Algorithm 2
**Input:** A directed cyclic transitive graph \( D = (V,A) \) where \( \forall v \in V(D) : f(v) = k \), and \( k \) is a constant integer greater than 1.

**Output:** \( p_p(D) \).

1. Find the strongly connected components \( C_i \) of the directed graph \( D \).
2. Reduce each strongly connected component \( C_i \) to a single vertex \( v_i \) and assign a weight \( P_i := |V(C_i)| \).
3. For each \( v_i \) create the oriented edge \( v_i v_j \), where \( v_i \in V(C_i) \) and \( v_j \in V(C_j) \), if there is at least one vertex of \( C_i \) that is the direct predecessor of some vertex of \( C_j \) in the original directed graph.
4. Count the amount of \( v_i \) sources that exist in the graph and do \( q \) receive that amount.
5. Do \( p_p(D) := q \ast f(v) \).

**Theorem 5.** The in-neighbor number \( p_p(D) \) found by the Algorithm 2 is the cardinality of the smallest in-neighbor set of \( D \), where \( D \) is a directed, cyclic and transitive graph such that \( \forall v \in V(D) : f(v) = k \), and \( k \) is a constant integer greater than 1.

**Proof.** According the Lemma 1, the strongly connect components of the graph are complete graphs, then it suffices to contaminate \( f(v) \) vertices of each source component to contaminate the whole component and, according the Lemma 2, these vertices will also contaminate the non-source components of the graph. \( \blacksquare \)

**Theorem 6.** Let \( D \) be a directed cyclic transitive graph where \( \forall v \in V(D) : f(v) = k \), and \( k \) is a constant integer greater than 1, then \( h_p(D) = p_p(D) \).

**Proof.** In order to get the graph contaminated we must contaminate \( f(v) \) vertices of each strongly connected source component, these vertices will contaminate the source and non-source components of the graph, as shown in Theorem 5. Therefore \( h_p(D) = p_p(D) \). \( \blacksquare \)
Algorithm 3

Input: A directed cyclic graph $D = (V, A)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $c_p(D)$.

1. Find the strongly connected components $C_i$ of the directed graph $D$.
2. Reduce each strongly connected component $C_i$ to a single vertex $v_i$ and assign a weight $P_i$ to each one, where $P_i := |V(C_i)|$.
3. For each $v_i$ create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of $C_i$ that is the direct predecessor of some vertex of $C_j$ in the original graph.
4. Choose a $v_i$ source with the lowest $P_i$ and do $m := |P_i|$.
5. Do $c_p(D) := |V(D)| - m$.

Theorem 7. The number $c_p(D)$ found by the Algorithm 3 is the cardinality of the largest proper in-neighbor convex set of $D$, where $D$ is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.

Proof. After the graph reduction performed in the Algorithm 3, it becomes an directed acyclic graph. According the Theorem 1, it suffices to remove a single vertex that has $|N^-(v)| < f(v)$, in this case, a source component. In order for $c_p(D)$ to be maximum, it is necessary to choose the lowest weight source component. Since this component is strongly connected and $\forall v \in V(D) : f(v) = 1$, clearly, none of its vertices are in $S$ the largest proper in-neighbor convex set that satisfies $c_p(D) = |S|$.

Algorithm 4

Input: A directed cyclic graph $D = (V, D)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $h_p(D)$.

1. Find the strongly connected components $C_i$ of the directed graph $D$.
2. Reduce each strongly connected component $C_i$ to a single vertex $v_i$ and assign a weight $P_i$ to each one, where $P_i := |V(C_i)|$. 
3. For each $v_i$ create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of $C_i$ that is the direct predecessor of some vertex of $C_j$ in the original directed graph.

4. Count the number of sources that exist in the directed graph and do $q$ receive that amount.

5. Do $h_p(D) := q$.

**Theorem 8.** The in-neighbor hull number $h_p(D)$ found by the Algorithm 4 is the cardinality of the smallest in-neighbor hull of $D$, where $D$ is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.

**Proof.** Suppose the $h_p(D)$ found by the Algorithm 4 is not the smallest cardinality of in-neighbor hull of $D$, then there is a $h'_p(D) < h_p(D)$, where $h'_p(D) = |S'|$ and $h_p(D) = |S|$, such that $S'$ and $S$ are in-neighbor hull sets of $D$. According the Algorithm 4, each $C_i$ source has one vertex in $S$. Since $S'$ must have one vertex less than $S$, there is a $C_i$ source that does not have vertices in $S'$. Since this $C_i$ is a source, none of its vertices has predecessors in $S'$ and will never be contaminated, therefore $S'$ is not an in-neighbor hull, contradiction. □

3 Conclusions

In this work, we show for the class of directed acyclic graphs intervals for in-neighbor convexity number and in-neighbor hull, it is also shown for this class that the in-neighbor number is NP-Complete. Moreover, for directed cyclic graphs we have created algorithms that answer the three parameters of In-Neighbor Convexity for the class of directed transitive graphs where $\forall v \in V(D) : f(v) = k$, and $k$ is a constant integer greater than 1, as well as algorithms that solve the in-neighbor convexity number and in-neighbor hull number for directed cyclic graphs that have $f(v) = 1$. 

References


Alessandra A. Pereira
Escola de Ciências Exatas e Computação - Pontifícia Universidade Católica de Goiás
Goiânia, Goiás, Brazil
alessandrapereiravps@gmail.com

Jayme L. Szwarcfiter
Instituto de Matemática, NCE, COPPE - Universidade Federal do Rio de Janeiro
Rio de Janeiro, RJ, Brazil
jayme@nce.ufrj.br

Carmen C. Centeno
Escola de Ciências Exatas e Computação - Pontifícia Universidade Católica de Goiás
Goiânia, Goiás, Brazil
cecilia@pucgoias.edu.br

Mitre C. Dourado
Instituto de Matemática, NCE, COPPE - Universidade Federal do Rio de Janeiro
Rio de Janeiro, RJ, Brazil
mitre@nce.ufrj.br