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On the In-Neighbor Convexity

Alessandra A. Pereira **Carmen C. Centeno**
Mitre C. Dourado **Jayme L. Szwarcfiter**

Abstract

In this work, we study the in-neighbor convexity in the directed graph class. We work with the following parameters: in-neighbor convexity number, in-neighbor number and in-neighbor hull number. We present results concerning directed acyclic graphs, directed cyclic graphs and transitive graphs.

1 Introduction

The concept of graph convexity has been studied in many works and conceptions [1, 2, 3, 5]. This work propose a new convexity called in-neighbor convexity, such convexity can be applied in the process of spread of influence through a social network modeled by graphs. This process has been studied in many fields such as epidemiology [4], sociology, economics and computer science [6].

Let $D(V, A)$ be a simple directed graph, the indegree of a vertex v is denoted by $d^-(v)$ and the outdegree of v is denoted by $d^+(v)$. The set of direct predecessors of v in D is denoted by $N^-(v)$. A vertex with

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$d^-(v) = 0$ is a source. A directed graph is acyclic if it has no directed cycles. Every directed acyclic graph has at least one source. Let $f : V(D) \to \{0, 1, 2, ..., j\}$ be a function. A set $S \subset V(D)$ is in-neighbor convex if $\forall v \notin S$ has $|N^-(v) \cap S| < f(v)$. The in-neighbor convexity number of D, $c_p(D)$, is the cardinality of the largest proper in-neighbor convex set of D. The in-neighbor interval of a set $S \subset V(D)$, $I_p(S) = S \cup \{v \in V(D) : |N^-(v) \cap S| \geq f(v)\}$. When $I_p(S) = V(D)$, S is called in-neighbor set. The in-neighbor number of D, $p_p(D)$, is the cardinality of the smallest in-neighbor set. Let S be a subset of $V(D)$, $I_p^+(S) = S_i$ where $S_1 = I_p(S), S_2 = I_p(S_1), S_3 = I_p(S_2), ..., S_i = I_p(S_{i-1})$ and S_i is an in-neighbor convex set. The set S is called in-neighbor convex hull. When $I_p^+(S) = V(D)$, then S is an in-neighbor hull of D. The in-neighbor hull number, $h_p(D)$, is the cardinality of the smallest in-neighbor hull of D.

Assume that a vertex v is contaminated if v is in the subset S or $I_p^+(S)$. A vertex becomes contaminated if it has at least $f(v)$ contaminated in-neighbors.

In this work we studied the In-Neighbor Convexity for directed acyclic graphs and directed cyclic graphs.

2 Results

Theorem 1. *Let $D(V, A)$ be a directed acyclic graph, then:*

$$c_p(D) = \begin{cases} |V(D)| - 1, & \text{if } \exists v : |N^-(v)| < f(v); \\ 0, \text{ if } \forall v : |N^-(v)| \geq f(v) \cdot \end{cases}$$

Proof. In the first case ($|V(D)| - 1$), in order to find the proper convex in-neighbor set of D is sufficient to exclude one vertex that has $|N^-(v)| < f(v)$. The vertex v will never have enough neighbors in S to be contaminated. Therefore $c_p(D) = |V(D)| - 1$.

In the second case, we show that $I_p^+(\emptyset) = V(D)$, thus $c_p(D) = 0$. First, we prove that $I_p^+(\emptyset) \neq \emptyset$. Since D is an acyclic directed graph, D has a source, called v. By the property $|N^-(v)| \geq f(v)$, it follows that $f(v) = 0$

for all the sources. Observe that $\{x \in V(D) : f(x) = 0\} \subseteq I_p^+(\emptyset)$. Thus, $I_p^+(\emptyset)$ has all the sources of D. Suppose that $I_p^+(\emptyset) \neq V(D)$, therefore exists at least a vertex v of D that $v \notin I_p^+(\emptyset)$, then v is not a source. By the property $|N^-(v)| \geq f(v)$, we can conclude that $|N^-(v)| + 1 - f(v)$ direct predecessors of v do not belong to $I_p^+(\emptyset)$. The same can be concluded about these vertices and so on. Since the graph is acyclic, eventually we will find the beginning of the path, a source, that does not belong to $I_p^+(\emptyset)$. Contradiction. ■

The in-neighbor number problem is NP-Complete for directed acyclic graphs. This problem can be reduced by applying the Propositional Satisfiability Problem, also known as SAT.

Consider a boolean expression written only by using the logical operators AND($\wedge$), OR ($\vee$), NOT ($\neg$), variables $(x_1, x_2, ..., x_i)$ and parenthesis. A literal corresponds to a single variable or its complement; a clause with the set of literals grouped by the disjunction symbol (OR). Therefore, the formulas will be a conjunction (AND) of clauses.

The following shows the reduction of it:

Problem 1. IN-NEIGHBOR SET
INSTANCE: Graph $D = (V(D), A(D))$, positive integer $k \leq n$
QUESTION: Is there an in-neighbor set of size k or less in D, i.e., is there a subset $V' \subseteq V(D)$, such that $|V'| \leq k$ and every $u \in V(D) - V'$ has at least $f(u)$ predecessores in V'?

Theorem 2. *The* IN-NEIGHBOR SET *problem is NP-complete when considering the directed acyclic graphs.*

Proof. An in-neighbor set lies in NP, because considering a given certificate (a subset S of vertices of D), it is possible to answer in a polynomial time if S has the appropriated size and if all vertices that do not belong to S have $f(x)$ predecessors in S. In order to complete the proof, we describe a reduction from SAT. Let $\mathcal{C}$=$C_1 \wedge C_2 \wedge \ldots \wedge C_i$ be an arbitrary instance of SAT. We construct a graph D in the following way: for each clause

C_j of $\mathcal{C}$ creates a vertex c_j in $V(D)$. For each variable $\mathcal{C}$ create a set of vertices $\{x_l, \overline{x_l}, v_l, u_l\}$ and the edges $\{(x_l u_l), (\overline{x_l} u_l), (v_l x_l), (v_l \overline{x_l})\}$. Assign to all vertices $f(x) = 1$. Finally, if a variable $x_l/\overline{x_l}$ appears in the clause C_i, add the edge $(x_l c_i)/(\overline{x_l} c_i)$ in D.

First, we assume that $\mathcal{C}$ is satisfiable, and consider a minimum satisfying truth assignment S. Suppose S is a minimum set of true variables that satisfy $\mathcal{C}$. Let P be the in-neighbor set of D and V the set of all vertices v of D. So $P = S \cup V \cup \{u_l \in D | x_l, \overline{x_l} \notin S\}$. Since all clauses of $\mathcal{C}$ have, at least, one true variable, then we have that each c_i has at least a predecessor in P. The set of V ensures the predecessor of x and $\overline{x}$. The set S provides predecessors for some vertex u, those that do not have predecessor in S belongs to P. So $k = 2i$, which means, i vertices v, k vertices of S and $i - k$ vertices of u.

For the converse, we assume that S is an in-neighbor set of D. Then S has all vertices v of D which ensures one predecessor for each x and $\overline{x}$, $|S| = i$. To be able to have an in-neighbor set it is required to put in S the vertices c or one of its predecessors. Since we want a minimum in-neighbor set, we will put in S one predecessor of c_i, because this predecessor is also a predecessor of u_i. There can also exist vertices u without predecessors, which for the matter of making it clear, we will add then to S. So $|S| = (i) + (k) + (i - k) = 2i$. Then the truth set of $\mathcal{C}$ is $\{x/\overline{x} | x/\overline{x} \in S\}$. ∎

Theorem 3. *Let D be a directed acyclic graph then $h_p(D) = |S|$, where S is the set of all vertices v of D such that $|N^-(v)| < f(v)$.*

Proof. Let $D(V, A)$ be a directed acyclic graph and S the set of all vertices of $D : |N^-(v)| < f(v)$. Suppose that S is not a in-neighbor hull of D. Then there is a vertex of D, $v : v \notin I_p^+(S)$. Therefore v has $|N^-(v)| + 1 - f(v)$ direct predecessors that do not belong to $I_p^+(S)$. These vertices clearly have $|N^-(x)| \geq f(x)$ and the same applies to them, and thus iteratively. Since the graph is acyclic, we will reach to the beginning of a path, a source, which or belongs to S or to $I_p^+(S)$. Contradiction.

Clearly, S is minimum. ■

Next, we consider directed cyclic graphs. We show algorithms that solve the problem for the in-neighbor convexity number, in-neighbor number and in-neighbor hull number.

Algorithm 1

Input: A directed cyclic transitive graph $D = (V, A)$ such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.

Output: $c_p(D)$.

1. Find the strongly connected components C_i of the directed graph D.
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.
4. Choose a v_i source with the lowest P_i and do $m := P_i$.
5. Do $c_p(D) := |V(D)| - m + f(v) - 1$.

Lemma 1. *Let D be a directed, strongly connected and transitive graph, then D is a complete graph.*

Proof. Suppose that D is not complete. By definition, a directed graph is strongly connected if there exists, for every pair u, v of D, a directed path from u to v and vice versa. Since D is not complete there is at least one edge uv that does not belong to $A(D)$. However D is transitive and by definition if there is a path between the vertices u and v then the edge uv belongs to $A(D)$, therefore, there is an edge between each pair of vertices of D, contradiction. D is complete. ■

When a strongly connected component C_i is reduced to a source vertex v_i in the Algorithm 1 let's call it a source component.

Lemma 2. *Let D be a directed cyclic transitive graph, C_i a strongly connected source component of D, C_j a strongly connected non-source component of D, C_i is predecessor of C_j; there is the edge uv between each pair of vertices u and v, where $u \in V(C_i)$ and $v \in V(C_j)$.*

Proof. Suppose there is no edge uv in $A(D)$, where $u \in V(C_i)$ and $v \in V(C_j)$. According to Lemma 1 the strongly connected components of D are complete graphs, and as C_i is predecessor of C_j there is a path from u to v. Since D is transitive, by definition, the edge uv belongs to $A(D)$ if there is a path between the vertices u and v, contradiction. ■

As a consequence of Lemma 2 is easy to see that if D is directed cyclic transitive graph, every non-source component has a source component as direct predecessor.

Theorem 4. *The number $c_p(D)$ found by the Algorithm 1 is the cardinality of the largest proper in-neighbor convex set of D, where D is a directed cyclic transitive graph such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.*

Proof. Let C_i be a transitive strongly connected component found by the Algorithm 1. According the Lemma 1, C_i is a complete directed graph, then we can consider only $f(v)-1$ vertices of C_i, otherwise, the component will become completely contaminated.

Suppose the $c_p(D)$ found by the Algorithm 1 is not the cardinality of the largest proper in-neighbor set S of D, then there is a proper in-neighbor set S' of D where $|S'| > |S|$. Therefore, there is C'_i such that $|C'_i| < |C_i|$, where C_i is the source component with the lowest cardinality of D. We know that C'_i is not a source, thus, according the Lemma 2, C'_i will be contaminated by its predecessor, then S' is not a proper in-neighbor set of D, contradiction. ■

Algorithm 2

Input: A directed cyclic transitive graph $D = (V,A)$ where $\forall v \in V(D)$: $f(v) = k$, and k is a constant integer greater than 1.
Output: $p_p(D)$.

1. Find the strongly connected components C_i of the directed graph D.
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.
4. Count the amount of v_i sources that exist in the graph and do q receive that amount.
5. Do $p_p(D) := q * f(v)$.

Theorem 5. *The in-neighbor number $p_p(D)$ found by the Algorithm 2 is the cardinality of the smallest in-neighbor set of D, where D is a directed, cyclic and transitive graph such that $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1.*

Proof. According the Lemma 1, the strongly connect components of the graph are complete graphs, then it suffices to contaminate $f(v)$ vertices of each source component to contaminate the whole component and, according the Lemma 2, these vertices will also contaminate the non-source components of the graph. ■

Theorem 6. *Let D be a directed cyclic transitive graph where $\forall v \in V(D)$: $f(v) = k$, and k is a constant integer greater than 1, then $h_p(D) = p_p(D)$.*

Proof. In order to get the graph contaminated we must contaminate $f(v)$ vertices of each strongly connected source component, these vertices will contaminate the source and non-source components of the graph, as shown in Theorem 5. Therefore $h_p(D) = p_p(D)$. ■

Algorithm 3

Input: A directed cyclic graph $D = (V, A)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $c_p(D)$.

1. Find the strongly connected components C_i of the directed graph D.
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.
3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original graph.
4. Choose a v_i source with the lowest P_i and do $m := |P_i|$.
5. Do $c_p(D) := |V(D)| - m$.

Theorem 7. *The number $c_p(D)$ found by the Algorithm 3 is the cardinality of the largest proper in-neighbor convex set of D, where D is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.*

Proof. After the graph reduction performed in the Algorithm 3, it becomes an directed acyclic graph. According the Theorem 1, it suffices to remove a single vertex that has $|N^-(v)| < f(v)$, in this case, a source component. In order for $c_p(D)$ to be maximum, it is necessary to choose the lowest weight source component. Since this component is strongly connected and $\forall v \in V(D) : f(v) = 1$, clearly, none of its vertices are in S the largest proper in-neighbor convex set that satisfies $c_p(D) = |S|$. ■

Algorithm 4

Input: A directed cyclic graph $D = (V, D)$ where $\forall v \in V(D) : f(v) = 1$.

Output: $h_p(D)$.

1. Find the strongly connected components C_i of the directed graph D.
2. Reduce each strongly connected component C_i to a single vertex v_i and assign a weight P_i to each one, where $P_i := |V(C_i)|$.

3. For each v_i create the oriented edge $v_i v_j$, where $v_i \in V(C_i)$ and $v_j \in V(C_j)$, if there is at least one vertex of C_i that is the direct predecessor of some vertex of C_j in the original directed graph.
4. Count the number of sources that exist in the directed graph and do q receive that amount.
5. Do $h_p(D) := q$.

Theorem 8. *The in-neighbor hull number $h_p(D)$ found by the Algorithm 4 is the cardinality of the smallest in-neighbor hull of D, where D is a directed cyclic graph such that $\forall v \in V(D) : f(v) = 1$.*

Proof. Suppose the $h_p(D)$ found by the Algorithm 4 is not the smallest cardinality of in-neighbor hull of D, then there is a $h'_p(D) < h_p(D)$, where $h'_p(D) = |S'|$ and $h_p(D) = |S|$, such that S' and S are in-neighbor hull sets of D. According the Algorithm 4, each C_i source has one vertex in S. Since S' must have one vertex less than S, there is a C_i source that does not have vertices in S'. Since this C_i is a source, none of its vertices has predecessors in S' and will never be contaminated, therefore S' is not a in-neighbor hull, contradiction. ■

3 Conclusions

In this work, we show for the class of directed acyclic graphs intervals for in-neighbor convexity number and in-neighbor hull, it is also shown for this class that the in-neighbor number is NP-Complete. Moreover, for directed cyclic graphs we have created algorithms that answer the three parameters of In-Neighbor Convexity for the class of directed transitive graphs where $\forall v \in V(D) : f(v) = k$, and k is a constant integer greater than 1, as well as algorithms that solve the in-neighbor convexity number and in-neighbor hull number for directed cyclic graphs that have $f(v) = 1$.

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Alessandra A. Pereira
Escola de Ciências Exatas e Computação - Pontifícia Universidade Católica de Goiás
Goiânia, Goiás , Brazil

alessandrapereiravps@gmail.com

Carmen C. Centeno
Escola de Ciências Exatas e Computação - Pontifícia Universidade Católica de Goiás
Goiânia, Goiás, Brazil

cecilia@pucgoias.edu.br

Jayme L. Szwarcfiter
Instituto de Matemática, NCE, COPPE - Universidade Federal do Rio de Janeiro
Rio de Janeiro, RJ, Brazil

jayme@nce.ufrj.br

Mitre C. Dourado
Instituto de Matemática, NCE, COPPE - Universidade Federal do Rio de Janeiro
Rio de Janeiro, RJ, Brazil

mitre@nce.ufrj.br