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The AVD-edge-coloring conjecture for some split graphs

Aloísio de Menezes Vilas-Bôas **Célia Picinin de Mello**

Abstract

Let G be a simple graph. An *adjacent vertex distinguishing edge-coloring (AVD-edge-coloring)* of G is an edge-coloring of G such that for each pair of adjacent vertices u, v of G, the set of colors assigned to the edges incident with u differs from the set of colors assigned to the edges incident with v. The *adjacent vertex distinguishing chromatic index* of G, denoted $\chi'_a(G)$, is the minimum number of colors required to produce an AVD-edge-coloring for G. The *AVD-edge-coloring conjecture* states that every simple connected graph G with at least three vertices and $G \ncong C_5$ has $\chi'_a(G) \leq \Delta(G) + 2$. The conjecture is open for arbitrary graphs, but it holds for some classes of graphs.

In this note we focus on split graphs. We prove this AVD-edge-coloring conjecture for split-complete graphs and split-indifference graphs.

1 Introduction

In this paper, G denotes a simple, undirected, finite, connected graph. The sets $V(G)$ and $E(G)$ are the vertex and edge sets of G. Let $u, v \in$

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$V(G)$. We denote an edge by uv. A *clique* is a set of vertices pairwise adjacent in G and a *stable set* is a set of vertices such that no two of which are adjacent. For each vertex v of a graph G, $N_G(v)$ denotes the set of vertices that are adjacent to v. The *degree* of a vertex v is $d(v) = |N_G(v)|$. The *maximum degree* of a graph G is then $\Delta(G) = \max_{v \in V(G)} d(v)$. We use the simplified notation $N(v)$ when there is no ambiguity. A graph G is *complete* if all its vertices are pairwise adjacent.

Let $\mathcal{C}$ be a set of colors. A *total coloring* of G is a mapping $\pi : V(G) \cup E(G) \to \mathcal{C}$ such that incident or adjacent elements (vertex or edge) receive distinct colors. An *edge-coloring* of G is a mapping $\pi : E(G) \to \mathcal{C}$ such that adjacent edges receive distinct colors. Given an edge-coloring π, the *label* of a vertex u of G is the set $L_\pi(u) = \{\pi(uv), uv \in E(G)\}$. If $L_\pi(u) \neq L_\pi(v)$ for each edge uv of G, π is an *adjacent vertex distinguishing edge-coloring (AVD-edge-coloring)* of G. If $|\mathcal{C}| = k$, we say that π is a *k-total coloring*, *k-edge-coloring* or *k-adjacent vertex distinguishing edge-coloring (k-AVD-edge-coloring)*. We denote $\mathcal{C} \setminus L_\pi(u)$ as $\overline{L}_\pi(u)$. Note that if G contains isolated edges, then G does not have any adjacent vertex distinguishing edge-coloring. The minimum number of colors required to produce a total coloring, an edge-coloring or an adjacent vertex distinghishing edge-coloring for G is the *total chromatic number*, $\chi''(G)$, the *chromatic index*, $\chi'(G)$, and the *adjacent vertex distinguishing chromatic index*, $\chi'_a(G)$, respectively.

In this paper we consider the AVD-edge-coloring problem. This problem was introduced by Zhang et al. [ZLW02] which proposed the following conjecture.

AVD-edge-coloring conjecture: *If G is a simple connected graph with at least three vertices and $G \ncong C_5$ (a 5-cycle), then $\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2$.*

The bound $\chi'_a(G) \geq \Delta(G)$ follows from $\chi'_a(G) \geq \chi(G)$. Moreover if G constains two adjacent vertices of degree $\Delta(G)$, then $\chi'_a(G) \geq \Delta(G) + 1$ [ZLW02]. In this seminal article, the authors confirmed the AVD-edge-coloring conjecture determining exact values of the adjacent vertex distinguishing chromatic index for trees, cycles, complete bipartite graphs and complete graphs. After, AVD-edge-coloring conjecture has been verified for some large families of graphs such as bipartite graphs, graphs with $\Delta(G) \leq 3$ [BGLS07], planar graphs of girth at least six [BLW11], and hypercubes [CG09]. Adjacent vertex distinguishing chromatic index of a regular graph is related with its total chromatic number. A regular graph G has $\chi'_a(G) = \Delta(G)+1$ if and only if $\chi''(G) = \Delta(G)+1$ [EHW06, ZWY$^+$09]. It is known that the problem of determining the total chromatic number of a k-regular bipartite graphs is $\mathcal{NP}$-hard, for each fixed $k \geq 3$ [MSA94]. Therefore, the problem of determining the adjacent vertex distinguishing chromatic index is also $\mathcal{NP}$-hard.

In this paper, we focus on the adjacent vertex distinguishing edge-coloring of split graphs. We prove that the AVD-edge-coloring conjecture holds for some families of split graphs.

2 Preliminaries and definitions

The *union* of simple graphs G and H is the graph $G \cup H$ with vertices set $V(G) \cup V(H)$ and edges set $E(G) \cup E(H)$. The *join* of two disjoint graphs G_1 and G_2 is the graph $G = G_1 + G_2$, called *join graph*, such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

A *split graph* $G = [Q, S]$ is a simple graph whose vertex set admits a partition into a clique Q and a stable set S. A split graph G is *split-complete* if $G = G[Q] + G[S]$.

The next theorem determines $\chi''(G)$ when $G = [Q, S]$ is a split-complete [CFK95].

Theorem 2.1. *Let $G = [Q, S]$ be a split-complete graph where $|Q| = q$ and $|S| = s$. Then,*

$$\chi''(G) = \begin{cases} \Delta(G) + 2 & \text{if } \Delta(G) \text{ is odd and } q > s^2 - s, \\ \Delta(G) + 1 & \text{otherwise.} \end{cases}$$

A *unit interval or indifference graph* is the intersection graph of a set of unit intervals of a straight line. A *split-indifference graph* is one which is simultaneously a split and an indifference graph. The following theorem due to Ortiz Z. et al. [OZMS98] characterizes the split-indifference graphs.

Theorem 2.2. *Let G be a connected graph. Graph G is split-indifference if and only if*

1. *G is a complete graph, or*
2. *G is the union of two complete graphs G_1, G_2, such that $G_1 \setminus G_2 = K_1$, or*
3. *G is the union of three complete graphs G_1, G_2, G_3, such that $G_1 \setminus G_2 = K_1$, $G_3 \setminus G_2 = K_1$, $V(G_1) \cap V(G_2) \neq \emptyset$, and $V(G_1) \cup V(G_3) = V(G)$, or*
4. *G is the union of three complete graphs G_1, G_2, G_3, such that $G_1 \setminus G_2 = K_1$, $G_3 \setminus G_2 = K_1$, and $V(G_1) \cap V(G_3) = \emptyset$.*

It remains open the edge-coloring problem and the total coloring problem for the both classes, split graphs and indifference graphs. It is known, for both classes, the chromatic index when the maximum degree is odd [CFK95, dFMdM97], and the total chromatic number when the maximum degree is even [CFK95, dFMdM99]. However, for split-indifference graphs both problems are solved [CdFMdM12, OZMS98].

In the next section we prove that AVD-edge-coloring conjecture holds for split-complete graphs and split-indifference graphs.

3 Adjacent vertex distinguishing edge-coloring of split graphs

Let $G = [Q, S]$ be a split graph. Since K_2 does not admits an AVD-edge-coloring, we have that $G \ncong K_2$. If $|Q| = 1$, then G is a star graph and $\chi'_a(G) = \Delta(G)$.

Theorem 3.1. *Let $G = [Q, S]$ be a split graph where $|Q| \geq 2$ and $d(u) = \Delta(G)$ for all $u \in Q$. If $\chi''(G) = \Delta(G) + 1$, then $\chi'_a(G) = \Delta(G) + 1$.*

Proof: Let $G = [Q, S]$ be a split graph where $|Q| \geq 2$ and $d(u) = \Delta(G)$ for all $u \in Q$. Since $|Q| \geq 2$, it follows that $\chi'_a(G) \geq \Delta(G)+1$. If G admits a $\Delta(G) + 1$-total-coloring τ, then the restriction λ of τ to the edges of G is an edge-coloring of G and $\overline{L}_\lambda(u) = \tau(u)$, for $u \in Q$. Therefore λ is a $\Delta(G) + 1$-AVD-edge-coloring. ■

From theorems 2.1 and 3.1, we have $\chi'_a(G)$ for some split-complete graphs G.

Theorem 3.2. *Let $G = [Q, S]$ be a split-complete graph where $|Q| = q \geq 2$ and $|S| = s$. If $\Delta(G)$ is even or $q \leq s^2 - s$, then $\chi'_a(G) = \Delta(G) + 1$.*

In Theorem 3.3 we proof that split-complete graphs G with $\Delta(G)$ odd admit a $\Delta(G) + 2$-AVD-edge-coloring.

Theorem 3.3. *Let $G = [Q, S]$ be a split-complete with $\Delta(G)$ odd. Then $\chi'_a(G) \leq \Delta(G) + 2$.*

Proof: Let $G = [Q, S]$ be a split-complete with $\Delta(G)$ odd. If $|Q| = 1$, G is a *star* graph and $\chi'_a(G) = \Delta(G)$. If $|S| = 1$, G is a complete graph and $\chi'_a(G) = \Delta(G) + 2$ [ZLW02]. Let $|Q| \geq 2$ and $|S| \geq 2$. Now we construct an AVD-edge-coloring for G with $\Delta(G) + 2$ colors.

Let $S = \{w, v_2, v_3, \ldots, v_{|S|}\}$ and $Q = \{v_{|S|+1}, v_{|S|+2}, \ldots, v_{|S|+|Q|}\}$. Let $\pi : E(G) \to \mathcal{C}$ be a $\Delta(G) + 2$-edge-coloring defined in the following way: $\pi(v_iv_j) = (i + j) \mod (\Delta(G) + 2)$ and $\pi(wv_j) = (2i) \mod (\Delta(G) + 2)$. Note that $\overline{L}_\pi(v_i) = \{i, (i + 1) \mod (\Delta(G) + 2)\}$, for $v_i \in Q$. Moreover, since $|S| \geq 2$, $d(v) \neq d(u)$ for $u \in Q$ and $v \in S$. Therefore, π is a $\Delta(G) + 2$-AVD-edge-coloring of G and $\chi'_a(G) \leq \Delta(G) + 2$. ■

Corollary 3.4. AVD-edge-coloring conjecture holds for split-complete graphs.

To study split-indifference graphs, for each case, we define subsets of $V(G)$. Let G be a split-indifference graph. If G satisfies Case (2), we define $A' = V(G_1) \setminus V(G_2)$, $AB = V(G_1) \cap V(G_2)$ and $B' = V(G_2) \setminus V(G_1)$. If G satisfies (3), we define $A' = V(G_1) \setminus V(G_2)$, $AB = V(G_1) \cap (V(G_2) \setminus V(G_3))$, $ABC = V(G_1) \cap V(G_2) \cap V(G_3)$, $BC = V(G_3) \cap (V(G_2) \setminus V(G_1))$ and $C' = V(G_3) \setminus V(G_2)$. If G satisfies (4), we define $A' = V(G_1) \setminus V(G_2)$, $AB = V(G_1) \cap V(G_2)$, $B' = V(G_2) \setminus (V(G_1) \cup V(G_3))$, $BC = V(G_2) \cap V(G_3)$ and $C' = V(G_3) \setminus V(G_2)$. Figure 1 shows the sets of vertices defined for each case.

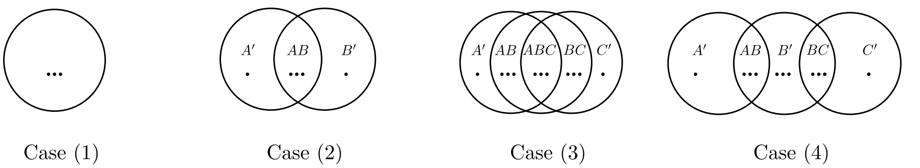

Figure 1: The split-indifference graphs.

Theorem 3.5. *Let G be a split-indifference graph. Then $\chi'_a(G) \leq \Delta(G) + 2$.*

Proof: Let G be a split-indifference graph. If G satisfies Case (1), that is, G is a complete graph, then $\chi'_a(G) \leq \Delta(G) + 2$ [ZLW02]. Assume that G satisfies Case (2), (3) or (4). If $|V(G)| = 3$, then $G \cong P_3$ and $\chi'_a(G) = \Delta(G)$. Let $|V(G)| \geq 4$ and let k be an integer where $k = 1$ if $\Delta(G)$ is even, and $k = 2$ otherwise. If G satisfies Case (2) or (3), we define $V(G) = \{w, v_k, v_{k+1}, \ldots, v_{|V(G)|+k-2}\}$. If G satisfies Case (4), $V(G) = \{w, x, v_k, v_{k+1}, \ldots, v_{|V(G)|+k-3}\}$.

The vertices v_k, v_{k+1}, w, and x are special vertices and must be distributed as follows. If G satisfies Case (2), $v_k \in A'$ and $w \in AB$. If G satisfies Case (3), $v_k \in A'$, $v_{k+1} \in C'$, and $w \in ABC$. If G satisfies Case (4), $v_k \in A'$, $x \in C'$, and $w \in AB$.

Let $\pi : E(G) \to \mathcal{C}$ be a $\Delta(G)+k$-edge-coloring of G defined as $\pi(v_i v_j) = (i+j) \mod (\Delta(G)+k)$; $\pi(wv_i) = (2i) \mod (\Delta(G)+k)$; $\pi(xv_i) = (i+1) \mod (\Delta(G)+k)$.

Note that if G satisfies Case (2), $d(v_k) \neq d(u)$ for $u \in N(v_k)$. If G satisfies Case (3), $d(v_k) \neq d(u)$ for $u \in N(v_k)$ and $d(v_{k+1}) \neq d(u)$ for $u \in N(v_{k+1})$. If G satisfies Case (4), $d(v_k) \neq d(u)$ for $u \in N(v_k)$ and $d(x) \neq d(u)$ for $u \in N(x)$. Therefore, for every case, if $v \in A'$ or $v \in C'$, then $\overline{L}_\pi(v) \neq \overline{L}_\pi(u)$, $u \in N(v)$. Note also that for all cases, we have $\overline{L}_\pi(w) = \{0\}$ when $\Delta(G)$ is even and $\overline{L}_\pi(w) = \{0, 2\}$ when $\Delta(G)$ is odd. Now we describe $\overline{L}_\pi(v_i)$ for $v_i \in V(G_2)$ according to the parity of $\Delta(G)$ in each case.

	$\boldsymbol{\Delta(G)}$ **even**			$\boldsymbol{\Delta(G)}$ **odd**		
Case (2)	$\boldsymbol{AB}$	$\boldsymbol{B'}$		$\boldsymbol{AB}$	$\boldsymbol{B'}$	
	$\{i\}$	$\{i, i+1\}$		$\{i, i+1\}$	$\{i, i+1, i+2\}$	
Case (3)	$\boldsymbol{AB}$	$\boldsymbol{ABC}$	$\boldsymbol{BC}$	$\boldsymbol{AB}$	$\boldsymbol{ABC}$	$\boldsymbol{BC}$
	$\{i, i+2\}$	$\{i\}$	$\{i, i+1\}$	$\{i, i+1, i+3\}$	$\{i, i+1\}$	$\{i, i+1, i+2\}$
Case (4)	$\boldsymbol{AB}$	$\boldsymbol{B'}$	$\boldsymbol{BC}$	$\boldsymbol{AB}$	$\boldsymbol{B'}$	$\boldsymbol{BC}$
	$\{i\}$	$\{i, i+1\}$	$\{i\}$	$\{i, i+1\}$	$\{i, i+1, i+2\}$	$\{i, i+1\}$

Table 1: $\overline{L}_\pi(v_i)$, $v_i \in V(G_2)$; $i+1$, $i+2$ and $i+3$ are modulo $\Delta(G) + k$.

Since $k \leq i \leq |V(G)| + k - 2$, then $\overline{L}_\pi(v_i) \neq \overline{L}_\pi(v_j)$, for $v_i, v_j \in V(G_2)$. Therefore, π is a $\Delta(G) + k$-AVD-edge-coloring of G. ■

Corollary 3.6. AVD-edge-coloring conjecture holds for split-indifference graphs.

4 Conclusion

In this work we proved that AVD-edge-coloring conjecture is verified for both split-complete and split-indifference graphs.

In a related work [dMVB15], we also study $\chi'_a(G)$ for some families of split graphs, including split-complete graphs and split-indifference graphs.

For a split graph $G = [Q, S]$, we have that $d(u) = |Q| - 1 + k$, for $u \in Q$ and $0 \leq k \leq |S|$. Let $\Delta(G) = |Q| - 1 + d$ and let $D = \{u : u \in V(G)$ and

$d(u) = \Delta(G)\}$. When $d(v) < d$ for every vertex $v \in S$, we proved that $\chi'_a(G) = \Delta(G)$ if $|D| = 1$ and $\chi'_a(G) = \Delta(G) + 1$ otherwise.

Studying the AVD-edge-coloring problem for split-complete graphs $G = [Q, S]$, we proved that if $\Delta(G)$ is odd and $|Q| > |S|^2$, then $\chi'_a(G) = \Delta(G) + 2$. Therefore, by Theorem 3.2, it remains to determine $\chi'_a(G)$ when $\Delta(G)$ is odd and $|S|^2 - |S| < |Q| \leq |S|^2$. In this case, we conjecture that $\chi'_a(G) = \Delta(G) + 1$.

We also determined $\chi'_a(G)$ for split-indifference graphs G, except when G satisfies Case (4), $|V(G)|$ odd, $|AB| = \frac{\Delta(G)+1}{2} + p$, $|B'| < p$, with $0 \leq p < \frac{\Delta(G)-1}{4}$, and $|AB| \geq |BC|$. In these conditions, we conjecture $\chi'_a(G) = \Delta(G) + 2$.

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Aloísio de Menezes Vilas-Bôas
IC, Universidade Estadual de Campinas
Brazil
aloisiovilasboas@gmail.com

Célia Picinin de Mello
IC, Universidade Estadual de Campinas
Brazil
celia@ic.unicamp.br