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On Contact Normal Parallel Spacelike Submanifolds in a semi-Riemannian Sasakian space form

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Abstract

In this paper we study the contact normal, spacelike parallel submanifold M^n with parallel mean curvature vector in a semi-Riemannian Sasakian space form $\widetilde{M}_q^{2m+1}(c)$ with $m > n$ and codimension $p > q$. We use a Simons type inequality to obtain a gap theorem.

1 Introduction

In the last 40 years, motivated by important physical problems, there has been an increasing interest in studying the structure of spacelike submanifolds. This goes back to 1976, when S.Y. Chen and S.T. Yau proved [2] the Calabi-Bernstein conjecture concerning complete maximal spacelike hypersurface of $\mathbb{R}_1^{n+1}$, namely, that the only ones are the spacelike hyperplanes. In [4], S. Nishikawa proved that a complete maximal spacelike hypersurface in $N_q^{n+q}(1)$ is totally geodesic. In [1], J.O. Baek, Q.M.

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Cheng and Y.J. Suh, obtained an optimal estimate of the squared norm of the second fundamental form for complete spacelike hypersurfaces with constant mean curvature in a locally symmetric Lorentz space satisfying some curvature conditions and characterized the totally umbilical hypersurfaces.

In the context of maximal submanifolds, it is well known T. Ishihara's result (see [3]) that, for an n-dimensional complete maximal spacelike submanifold M^n immersed in $N_p^{n+p}(c)$, if $c \geq 0$, then M^n is totally geodesic and if $c < 0$, then the square norm S of the second fundamental form satisfies $0 \leq S \leq -npc$.

On the other hand an interesting topic in differential geometry is the study of submanifold in space endowed with an additional structure.

Recall that if $\widetilde{M}$ is a $(2m+1)$-dimensional manifold and $\mathfrak{X}(\widetilde{M})$ the Lie algebra of vector fields on $\widetilde{M}$ then an *almost contact structure* on $\widetilde{M}$ is defined by a (1,1)-tensor φ, a vector field ξ and a 1-form η on $\widetilde{M}$ such that for any $p \in \widetilde{M}$, we have

$$\begin{gathered} \varphi_p^2 = -I + \eta_p \otimes \xi_p, \quad \eta_p(\xi_p) = 1, \\ \eta(\varphi(\widetilde{X})) = 0 \quad \widetilde{X} \in \mathfrak{X}(\widetilde{M}), \end{gathered} \tag{1}$$

where I denote the identity transformation of the tangent space $T_p\widetilde{M}$ at p. Manifolds equipped whit an almost contact structure are called *almost contact manifolds.*

If a manifold $\widetilde{M}^{2m+1}$ with a (φ, ξ, η)-structure admits a Riemannian metric $\langle\ ,\ \rangle$ such that

$$\langle \varphi\widetilde{X}, \varphi\widetilde{Y}\rangle = \langle \widetilde{X}, \widetilde{Y}\rangle - \eta(\widetilde{Y})\eta(\widetilde{Y})$$

for any $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$, then $\widetilde{M}^{2m+1}$ is said to have a $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$-structure or an *almost contact metric manifolds.*

A manifold $\widetilde{M}$ with pseudo-Riemannian metric tensor $\langle\ ,\ \rangle$ and an almost contact structure $(\varphi, \xi, \eta, \varepsilon)$ such that

$$\langle \xi, \xi \rangle = \varepsilon, \quad \varepsilon = \pm 1, \quad \eta \circ \varphi = 0,$$

$$\eta(\widetilde{X}) = \varepsilon \langle \xi, \widetilde{X} \rangle, \quad \widetilde{X} \in \mathfrak{X}(\widetilde{M}), \tag{2}$$

$$\langle \varphi \widetilde{X}, \varphi \widetilde{Y} \rangle = \langle \widetilde{X}, \widetilde{Y} \rangle - \varepsilon \eta(\widetilde{X}) \eta(\widetilde{Y}), \quad \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$$

is an *almost contact metric manifold.*

The fundamental 2-form Ψ of an almost contact metric manifold $\widetilde{M}$, with structures tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$ is defined by

$$\Psi(\widetilde{X}, \widetilde{Y}) = \langle \varphi \widetilde{X}, \widetilde{Y} \rangle,$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$. When $d\eta = \Psi$, the associated structure is a *contact metric structure* and $\widetilde{M}$ is an *almost Sasakian manifold.*

An almost Sasakian manifold $\widetilde{M}$ with structure tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$ is called a *Sasakian manifold* if

$$[\varphi \widetilde{X}, \varphi \widetilde{Y}] + \varphi^2 [\widetilde{X}, \widetilde{Y}] - \varphi[\widetilde{X}, \varphi \widetilde{Y}] - \varphi[\varphi \widetilde{X}, \widetilde{Y}] = -2 d\eta(\widetilde{X}, \widetilde{Y}) \xi \tag{3}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$. A necessary and sufficient condition for an almost contact metric manifold to be a Sasakian manifold is (see [8])

$$\left(\widetilde{\nabla}_{\widetilde{X}} \varphi \right) \widetilde{Y} = \varepsilon \eta(\widetilde{Y}) \widetilde{X} - \langle \widetilde{X}, \widetilde{Y} \rangle \xi, \tag{4}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$, where $\widetilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian metric $\langle\ ,\ \rangle$.

Example 1.1 ([8])**.** Let $\mathbb{R}^{2m+2}_{2s}$ be the pseudo-Euclidian space with the indefinite standard Kaehler structure. The pseudo-sphere

$$S^{2m+1}_{2s}(1) = \{ p \in \mathbb{R}^{2m+2}_{2s} \ ; \ \langle p, p \rangle = 1 \}$$

and the pseudo-hyperbolic space

$$H^{2m+1}_{2s-1} = \{ p \in \mathbb{R}^{2m+2}_{2s} \ ; \ \langle p, p \rangle = -1 \}$$

are hyperquadrics of $\mathbb{R}^{2m+2}_{2s}$, both of dimension $2m+1$ of index $2s$ an $2s-1$ and of constant sectional curvature 1 and -1 respectively. They have a canonical structure of Sasakian indefinite manifolds, with characteristic vector field ξ spacelike and timelike respectively.

Suppose that $\widetilde{M}^{2m+1}_q$ is a Sasakian manifold with structures tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$. Let

$$\Omega_p = \{\widetilde{X} \in T_p\widetilde{M}\ ;\ \eta(\widetilde{X}) = 0\}.$$

For a non-null vector $\widetilde{X}$ in Ω_p, $\widetilde{X}$ and $\varphi\widetilde{X}$ span a non-degenerate 2-plane and hence, we can consider a sectional curvature $K(\widetilde{X}) = K(\widetilde{X}, \varphi\widetilde{X})$. If $K(\widetilde{X})$ is constant for all non-null vectors $\widetilde{X} \in \Omega_p$, we call M^{2m+1}_q to be of constant φ-sectional curvature at p. If $K(\widetilde{X})$ is constant φ-sectional curvature at every point, $K(\widetilde{X})$ is a function of $p \in \widetilde{M}^{2m+1}_q$, say $c(p)$. In this case, if $c(p) = c$ is constant on M^{2m+1}_q, we call $\widetilde{M}^{2m+1}_q$ to be a *Sasakian space form* and is denoted by $\widetilde{M}^{2m+1}_q(c)$.

The curvature tensor of a Sasakian space form $\widetilde{M}^{2m+1}_q(c)$ is given by [8]

$$\begin{aligned} \widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \ & \tfrac{1}{4}(c+3\varepsilon)\{\langle\widetilde{Y}, \widetilde{Z}\rangle\widetilde{X} - \langle\widetilde{X}, \widetilde{Z}\rangle\widetilde{Y}\} \\ & + \tfrac{1}{4}(\varepsilon c - 1)\{\eta(\widetilde{X})\eta(\widetilde{Z})\widetilde{Y} - \eta(\widetilde{Y})\eta(\widetilde{Z})\widetilde{X}\} \\ & + \tfrac{1}{4}(c-\varepsilon)\{\langle\widetilde{X}, \widetilde{Z}\rangle\eta(\widetilde{Y})\xi - \langle\widetilde{Y}, \widetilde{Z}\rangle\eta(\widetilde{X})\xi \\ & + \langle\varphi\widetilde{Y}, \widetilde{Z}\rangle\varphi\widetilde{X} + \langle\varphi\widetilde{Z}, \widetilde{X}\rangle\varphi\widetilde{Y} - 2\langle\varphi\widetilde{X}, \widetilde{Y}\rangle\varphi\widetilde{Z}\}, \end{aligned} \tag{5}$$

for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}\left(\widetilde{M}^{2m+1}_q(c)\right)$.

A *Lorentzian Sasakian manifold* is a Sasakian manifolds, with a Lorentz metric such that the characteristic vector field ξ is timelike.

In this paper we study parallel spacelike submanifolds M^n of codimension p, isometrically immersed in a Sasakian space form $\widetilde{M}^{2m+1}_q(c)$ with pseudo-Riemannian metric of index q $(q < p)$, such that the structure vector field ξ is timelike and normal to M^n everywhere.

Our result can be stated as:

Theorem 1.1. Let M^n be an n-dimensional ($n \geq 3$) complete not maximal contact normal spacelike submanifold of codimension p, with parallel mean curvature vector in a Sasakian space form $\widetilde{M}_q^{2m+1}(c)$ with structure tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$. Let $Q(x) = (n-2)^2 x + 4(n-1)$ and S denotes the squared norm of the second fundamental form of M^n. Suppose that M^n is parallel.

(a) If $H^2 < \frac{(n-1)}{Q(q)}(c-3)$ then $c > 3$, $S = nH^2$ and M^n is totally umbilical.

(b) If $H^2 = \frac{(n-1)}{Q(q)}(c-3)$ then $c > 3$ and either $S = nH^2$ and M^n is totally umbilical or $\sup S = n\dfrac{(c-3)}{4}\dfrac{Q(q^2)}{Q(q)}$.

(c) If $H^2 > \frac{(n-1)}{Q(p)}(c-3)$ and $c < 3$ then either $S = nH^2$ and M^n is totally umbilical or $nH^2 < \sup S \leq (\vartheta_H^+)^2 + nH^2$, where

$$\vartheta_H^+ = \frac{q}{2}\sqrt{\frac{n}{(n-1)}}\left\{(n-2)H + \sqrt{\frac{Q(q)H^2 - (n-1)(c-3)}{q}}\right\}.$$

(d) If $H^2 > \frac{(n-1)}{Q(q)}(c-3)$ and $c > 3$ then either $S = nH^2$ and M^n is totally umbilical or $(\vartheta_H^-)^2 + nH^2 < \sup S \leq (\vartheta_H^+)^2 + nH^2$, where

$$\vartheta_H^- = \frac{q}{2}\sqrt{\frac{n}{(n-1)}}\left\{(n-2)H - \sqrt{\frac{Q(q)H^2 - (n-1)(c-3)}{q}}\right\}.$$

This article is organized as follows: in section 2 we will introduce some basic facts notations that will appear in the paper. In section 3 we obtain a Simons type inequality for a complete not maximal contact normal spacelike submanifold with parallel mean curvature vector and second fundamental form parallel in a Sasakian space form $\widetilde{M}_q^{2m+1}(c)$. Finally, in section 3, we prove the theorem 1.1.

In this paper, manifolds and tensor fields are supposed to be class C^∞.

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2 Preliminaries

Throughout this section we will introduce some basic facts and notations that will appear in the paper.

We have the following result:

Lemma 2.1. *If $\widetilde{M}_q^{2m+1}$ is a Sasakian manifold with structure tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$ then*

$$\widetilde{\nabla}_{\widetilde{X}}\xi = \varepsilon\varphi(\widetilde{X})$$

for any $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$, where $\widetilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian metric $\langle\ ,\ \rangle$.

Proof. From $\eta(\varphi(\widetilde{Y})) = \langle \xi, \varphi\widetilde{Y}\rangle = 0$ we get

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle + \langle \xi, (\widetilde{\nabla}_{\widetilde{X}}\varphi)\widetilde{Y}\rangle = 0, \tag{6}$$

and using (4) in (6) we obtain

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle = -\varepsilon\{\varepsilon\eta(\widetilde{X})\eta(\widetilde{Y}) - \langle \widetilde{X}, \widetilde{Y}\rangle\}. \tag{7}$$

Hence, from (2) we have

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle = \varepsilon\langle \varphi\widetilde{X}, \varphi\widetilde{Y}\rangle.$$

□

Let $\widetilde{M}_q^{2m+1}(c)$ be a Sasakian space form with structure tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$ whose index is q and M^n an n-dimensional complete spacelike submanifold isometrically with parallel mean curvature in $\widetilde{M}_q^{2m+1}(c)$.

As usual, $\widetilde{\nabla}$ (resp. ∇) be Levi-Civita connection with respect to $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle|_M$) and $\nabla^{\perp}$ the connection in the normal bundle on M^n. The Gauss and the Weingarten formulas are given respectively by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\ \widetilde{\nabla}_X N &= -A_N X + \nabla^{\perp}_X N, \end{aligned} \tag{8}$$

for any X, Y vectors tangent to M^n and any N vector normal to M^n, where A_N is the shape operator in direction N and σ is the second fundamental form of M. The shape operator and the second fundamental form are related by

$$\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle. \tag{9}$$

Let R and $\widetilde{R}$ the curvature tensors of ∇ and $\widetilde{\nabla}$, respectively. Then, the Gauss equation is given by

$$\begin{aligned} \langle R(X,Y)Z,W\rangle = \ & \langle \widetilde{R}(X,Y)Z,W\rangle + \langle \sigma(X,Z), \sigma(Y,W)\rangle \\ & -\langle \sigma(X,W), \sigma(Y,Z)\rangle, \end{aligned} \tag{10}$$

If M^n is a spacelike submanifold normal to the structure vector field ξ of $\widetilde{M}_q^{2m+1}$ then M^n is called a *contact normal spacelike submanifold.* Note that, if M^n is a contact normal spacelike submanifold, from (2), for all $X \in \mathfrak{X}(M)$, $\varphi(X)$ is a spacelike vector field.

Lemma 2.2. *If M^n is a contact normal spacelike submanifold in a Sasakian manifold $\widetilde{M}_q^{2m+1}$ with structure tensors $(\varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon)$, then M^n is an anti-invariant submanifold of $\widetilde{M}_q^{2m+1}$, that is $\varphi(T_pM) \subset T_pM^\perp$ for all $p \in M^n$, $n \leq m$ and $A_\xi X = 0$ for all $X \in \mathfrak{X}(M)$.*

Proof. For any vector field $X, Y \in \mathfrak{X}(M)$, from (2) and Lemma 2.1 we have

$$\begin{aligned} \langle \varphi(X), Y\rangle = \ & \langle \varepsilon \widetilde{\nabla}_X \xi, Y\rangle = -\varepsilon\langle A_\xi X, Y\rangle = -\varepsilon\langle A_\xi Y, X\rangle \\ = \ & \langle \varepsilon \widetilde{\nabla}_Y \xi, X\rangle = \langle \varphi(Y), X\rangle = -\langle \varphi(X), Y\rangle. \end{aligned} \tag{11}$$

Hence, we have $A_\xi X = 0$ and $\varphi(X)$ is normal to M^n that is, M^n is anti-invariant and $n \leq m$. □

Observe that, if $X \in \mathfrak{X}(M)$, from lemma 2.2, $A_\xi X = 0$. Hence, from the Gauss and the Weingarten formulas, we get

$$\langle \sigma(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle = 0.$$

Thus, we have

Lemma 2.3. *Let M^n be a contact normal, spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then, for any $X, Y \in \mathfrak{X}(M)$, $\sigma(X,Y) \in <\xi>^\perp$, where $<\xi>$ is the 1-dimensional distribution spanned by ξ on $\widetilde{M}_q^{2m+1}$.*

Now, we assume that M^n is a contact normal spacelike submanifold of a semi-Rieannian Sasakian space form $\widetilde{M}_q^{2m+1}(c)$. We choose a local field of semi-Riemannian orthonormal vector frames

$$e_1, e_2, ..., e_n, e_{1^*} = \varphi e_1, ..., e_{n^*} = \varphi e_n,$$
$$e_{2n+1}, ..., e_{2n+(m-n)}, e_{(2n+1)^*} = \varphi e_{2n+1}, ..., e_{(2n+(m-n))^*} = \varphi e_{2n+(m-n)}, e_{0^*} = \xi$$

in $\widetilde{M}_q^{2m+1}(c)$ such that at each point of M^n, $e_1, ..., e_n$ span the tangent space of M^n.

We use the following convention of indices:

$$A, B, C, D = 1, ...n, 1^*, ..., n^*, 2n+1, ..., (2n+(m-n)), (2n+1)^*,$$
$$..., (2n+(m-n))^*, 0^*.$$
$$i, j, k, l, r, s = 1, ..., n$$
$$i^*, j^*, k^*, l^*, r^*, s^* = 1^*, ..., n^*.$$
$$a, b, c, d = 1^*, ..., n^*, 2n+1, ..., (2n+(m-n)), (2n+1)^*, ..., (2n+(m-n))^*, 0^*$$
$$\alpha, \beta, \gamma = 2n+1, ..., (2n+(m-n)), (2n+1)^*, ..., (2n+(m-n))^*, 0^*.$$

Let $\omega_1, ..., \omega_{(2n+p)^*}, \omega_{0^*} = \eta$ be its dual frame field so that the semi-Riemannian metric of $\widetilde{M}_q^{2m+1}$ is given by $ds = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = \epsilon_{i^*} = 1$, $\epsilon_\alpha = -1$ and $\epsilon_{0^*} = \varepsilon$. Then the structure equations of $\widetilde{M}_q^{2m+1}$ are given by

$$\begin{aligned} d\omega_A = & -\textstyle\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0 \\ d\omega_{AB} = & -\textstyle\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \omega_C \wedge \omega_D, \end{aligned} \tag{12}$$

where K_{ABCD} denotes the components of the curvature tensor of $\widetilde{M}_q^{2m+1}$.

Restricting those forms to M^n, we get ω_α and from Cartan's lemma, we write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{13}$$

Hence, we obtain the structure equations of M^n,

$$\begin{aligned} d\omega_i = & -\textstyle\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} = & -\textstyle\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \end{aligned} \tag{14}$$

where R_{ijkl} denotes the components of the curvature tensor of M^n.

From (5) and (10), we have

$$R_{ijkl} = \frac{(c+3\varepsilon)}{4}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_r (h_{il}^{r^*} h_{jk}^{r^*} - h_{ik}^{r^*} h_{jl}^{r^*}) - \sum_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha) \tag{15}$$

The Ricci curvature tensor $\{R_{ij}\}$ and the scalar curvature τ are expressed repectiveley, as follows:

$$\begin{aligned} R_{jk} = & \tfrac{(n-1)}{4}(c+3\varepsilon)\delta_{jk} + \textstyle\sum_r \left(\left(\sum_i h_{ii}^{r^*}\right) h_{jk}^{r^*} - \sum_i h_{ik}^{r^*} h_{ji}^{r^*} \right) \\ & - \textstyle\sum_\alpha \left(\left(\sum_i h_{ii}^\alpha\right) h_{jk}^\alpha - \sum_i h_{ik}^\alpha h_{ji}^\alpha \right) \end{aligned} \tag{16}$$

and

$$\begin{aligned} \tau = & \tfrac{n(n-1)}{4}(c+3\varepsilon) + \textstyle\sum_r \left(\sum_i h_{ii}^{r^*}\right)^2 - \sum_r \sum_i \left(h_{ij}^{r^*}\right)^2 \\ & - \textstyle\sum_\alpha \left(\sum_i h_{ii}^\alpha\right)^2 + \sum_\alpha \sum_i \left(h_{ij}^\alpha\right)^2 . \end{aligned} \tag{17}$$

We also have

$$R_{abij} = \frac{(c-\varepsilon)}{4}(\delta_{j^*a}\delta_{i^*b} - \delta_{i^*a}\delta_{j^*b}) - \sum_l (h_{il}^a h_{lj}^b - h_{lj}^a h_{il}^b) \tag{18}$$

Now, if $Z \in \mathfrak{X}(M)$, from (4),

$$\varphi\widetilde{\nabla}_Z Y = \langle Z, Y\rangle\xi + \widetilde{\nabla}_Z(\varphi Y). \tag{19}$$

Hence, if $X, Y \in \mathfrak{X}(M)$, from the Gauss and the Weingarten formulas, by using (19), for any $Z \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned} \langle A_{\varphi X} Y, Z\rangle = & \langle \sigma(Y, Z), X\rangle = \langle \widetilde{\nabla}_Z Y, \varphi X\rangle \\ = & -\langle \varphi\widetilde{\nabla}_Z Y, X\rangle = \langle A_{\varphi Y} Z, X\rangle \\ = & \langle A_{\varphi Y} X, Z\rangle. \end{aligned}$$

This proves the following result:

Lemma 2.4. *Let M^n be a contact normal, spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then, $A_{\varphi X}Y = A_{\varphi Y}X$ for all $X, Y \in \mathfrak{X}(M)$.*

Using $h_{ij}^{\alpha} = \langle \sigma(e_i, e_j), e_\alpha \rangle$, we get

$$h_{ij}^{k^*} = h_{jk}^{i^*} = h_{ki}^{j^*} \quad \text{and} \quad h_{ij}^{0^*} = 0 \quad \text{for all} \quad 1 \leq i, j, k \leq n. \tag{20}$$

The first covariant derivative of σ is defined by

$$\begin{aligned} (\overline{\nabla}\sigma)(X, Y, Z) = & \ (\widetilde{\nabla}_Z \sigma)(X, Y) \\ = & \ \nabla_Z^{\perp}[\sigma(X, Y)] - \sigma(\nabla_Z Y, X) - \sigma(Y, \nabla_Z X), \end{aligned} \tag{21}$$

and the second covariant derivative is defined by

$$\begin{aligned} (\widetilde{\nabla}^2\sigma)(X, Y, Z, W) = & \ (\widetilde{\nabla}_W \widetilde{\nabla}_Z \sigma)(X, Y) \\ = & \ \nabla_W^{\perp}[(\widetilde{\nabla}_Z\sigma)(X, Y)] - (\widetilde{\nabla}_Z\sigma)(\nabla_W X, Y) \\ & -(\widetilde{\nabla}_Z\sigma)(X, \nabla_W Y) - (\widetilde{\nabla}_{\widetilde{\nabla}_W Z}\sigma)(X, Y). \end{aligned} \tag{22}$$

The components of the first covariant derivative of σ are given by

$$h_{ijk}^a = \left\langle \left(\widetilde{\nabla}_{e_k}\sigma\right)(e_i, e_j), e_a \right\rangle = \widetilde{\nabla}_{e_k} h_{ij}^a, \tag{23}$$

and the components of the second covariant derivative of σ are given by

$$h_{ijkl}^a = \left\langle \left(\widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k}\sigma\right)(e_i, e_j), e_a \right\rangle = \widetilde{\nabla}_{e_l} h_{ijk}^a = \widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k} h_{ij}^a. \tag{24}$$

We assume that $\widetilde{M}_q^{2m+1}(c)$ is a Sasakian space form. Hence,

$$\begin{gathered} \langle \xi, \xi \rangle = \varepsilon, \quad \varepsilon = \pm 1, \quad \eta \circ \varphi = 0, \\ \eta(\widetilde{X}) = \varepsilon\langle \xi, \widetilde{X} \rangle, \quad \widetilde{X} \in \mathfrak{X}(\widetilde{M}), \\ \langle \varphi\widetilde{X}, \varphi\widetilde{Y} \rangle = \langle \widetilde{X}, \widetilde{Y} \rangle - \varepsilon\eta(\widetilde{X})\eta(\widetilde{Y}), \quad \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M}), \\ \left(\widetilde{\nabla}_{\widetilde{X}}\varphi\right)\widetilde{Y} = \varepsilon\eta(\widetilde{Y})\widetilde{X} - \langle \widetilde{X}, \widetilde{Y} \rangle \xi \quad \text{and} \quad \widetilde{\nabla}_{\widetilde{X}}\xi = \varepsilon\varphi(\widetilde{X}) \end{gathered} \tag{25}$$

Lemma 2.5. *Let M^n be a contact normal, spacelike submanifold of in $\widetilde{M}_q^{2m+1}(c)$.*

1. *If $m = n$ and the second fundamental form of M^n in $\widetilde{M}_q^{2m+1}(c)$ is parallel, then M^n is totally geodesic.*

2. *If $m = n$ and the mean curvature vector of M^n is parallel, then M^n is maximal.*

Proof. From (24), using Lemma 2.3, (8) and (25), we have

$$\begin{aligned} h_{ijk}^{0^*} &= \left\langle \left(\widetilde{\nabla}_{e_k}\sigma\right)(e_i,e_j),\xi\right\rangle = \left\langle \nabla_{e_k}^{\perp}\sigma(e_i,e_j),\xi\right\rangle \\ &= -\left\langle \sigma(e_i,e_j),\nabla_{e_k}^{\perp}\xi\right\rangle = -\varepsilon\langle\sigma(e_i,e_j),\varphi e_k\rangle \qquad (26) \\ &= -\varepsilon h_{ij}^{k^*} \end{aligned}$$

If the second fundamental form of M^n is parallel, $h_{ijk}^{0^*} = 0$. Then, from (26), $h_{ij}^{k^*} = 0$ for all $1 \le i,j,k \le n$. Hence, if $m = n$, M^n is totally geodesic.

Now, from (20) and (26) we obtain $\sum_k h_{kki}^{0^*} = \sum_k h_{kk}^{i^*}$. This implies that if mean curvature vector field is parallel, then $\sum_k h_{kk}^{i^*} = 0$. Hence, if $m = n$ then M^n is maximal. □

Suppose that M^n is not maximal and that the second fundamental form of M^n in $\widetilde{M}_q^{2m+1}(c)$ is parallel. In this case $h_{ij}^{k^*} = 0$ for all $1 \le i,j,k \le n$. Therefore, from (15), (16), (17) and (18) we get,

$$R_{ijkl} = \frac{(c+3\varepsilon)}{4}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum_{\alpha=2n+1}^{2m+1}(h_{il}^{\alpha}h_{jk}^{\alpha} - h_{ik}^{\alpha}h_{jl}^{\alpha}). \qquad (27)$$

$$R_{jk} = (n-1)\frac{(c+3\varepsilon)}{4}\delta_{jk} - \sum_{\alpha}\left(\left(\sum_i h_{ii}^{\alpha}\right)h_{jk}^{\alpha} - \sum_i h_{ik}^{\alpha}h_{ji}^{\alpha}\right), \qquad (28)$$

$$\tau = n(n-1)\frac{(c+3\varepsilon)}{4} - \sum_{\alpha}\sum_i (h_{ii}^{\alpha}) + \sum_{\alpha}\sum_{i,j}\left(h_{ij}^{\alpha}\right)^2, \qquad (29)$$

$$R_{k^*m^*ij} = \frac{(c-\varepsilon)}{4}(\delta_{j^*k^*}\delta_{i^*m^*} - \delta_{i^*m^*}\delta_{j^*m^*}) \quad \text{and} \quad R_{\alpha\beta ij} = -\sum_l (h^\alpha_{il}h^\beta_{lj} - h^\alpha_{lj}h^\beta_{il}). \tag{30}$$

Let S be the squared norm of second fundamental form of M^n and H the mean curvature of M^n, that is,

$$S = \sum_\alpha \sum_{i,j} \left(h^\alpha_{ij}\right)^2 \quad \text{and} \quad H = \frac{1}{n}\sqrt{\sum_\alpha \left(\sum_i h^\alpha_{ii}\right)^2}. \tag{31}$$

Hence,

$$\tau = n(n-1)\frac{(c+3\varepsilon)}{4} + \left(S - n^2H^2\right). \tag{32}$$

The components h^α_{ij} satisfies

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{33}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_r h^\alpha_{rjk}\omega_{ri} - \sum_r h^\alpha_{irk}\omega_{rj} - \sum_r h^\alpha_{ijr}\omega_{rk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{34}$$

By exterior differentiation of (13), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik} \tag{35}$$

and to get the following Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_r h^\alpha_{ir}R_{rjkl} - \sum_r h^\alpha_{jr}R_{rikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}. \tag{36}$$

The Laplacian Δh^α_{ij} of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (36) we have

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{r,k} h^\alpha_{kr}R_{rijk} - \sum_{r,k} h^\alpha_{ri}R_{rkjk} - \sum_{\beta,k} h^\beta_{ki}R_{\alpha\beta jk}. \tag{37}$$

3 A Simons type formula

Now, we assume that the mean curvature vector h of M^n is parallel (i.e., $\nabla^{\perp}h = 0$), and M is a complete contact normal spacelike submanifold with parallel second fundamental form in $\widetilde{M}_q^{2m+1}(c)$. Suppose that ξ is timelike, that is $\varepsilon = -1$.

If we assuming $H \neq 0$, we can choose $e_{2n+1} = \dfrac{h}{H}$. Hence

$$\sum_k h^{\alpha}_{kki} = 0, \quad \omega_{\alpha,n+1} = 0, \quad H^{\alpha}H^{2n+1} = H^{n+1}H^{\alpha}, \tag{38}$$

$$trH^{2n+1} = nH, \quad trH^{\alpha} = 0, \ \ \alpha \neq 2n+1 \quad \text{and} \quad R_{(2n+1)\alpha ij} = 0. \tag{39}$$

where H^{α} denotes the matrix (h^{α}_{ij}). Let us define

$$\Phi^{2n+1}_{ij} = h^{2n+1}_{ij} - H\delta_{ij}, \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \ \ \alpha \neq 2n+1. \tag{40}$$

Therefore

$$\Phi^{2n+1} = H^{2n+1} - HI, \ \Phi^{\alpha} = H^{\alpha}, \ \ \alpha \neq 2n+1, \tag{41}$$

where Φ^{α} denotes the matrix (Φ^{α}_{ij}). Then

$$|\Phi^{2n+1}|^2 = \operatorname{tr}\left(H^{2n+1}\right)^2 - nH^2, \tag{42}$$

$$\sum_{\alpha \neq 2n+1} |\Phi^{\alpha}|^2 = \sum_{\beta \neq 2n+1} \left(h^{\beta}_{ij}\right)^2, \tag{43}$$

and

$$tr\left(\Phi^{\alpha}\right) = 0, \ \ \forall\alpha. \tag{44}$$

Thus,

$$S = \sum_{\alpha} |\Phi^{\alpha}|^2. \tag{45}$$

By inserting (18), (27), (38) and (39) into (37) we get

$$\begin{aligned}\Delta h_{ij}^{2n+1} = \; & n\tfrac{(c-3)}{4}h_{ij}^{2n+1} - nH\tfrac{(c-3)}{4}\delta_{ij} + \textstyle\sum_{\beta,k,m} h_{km}^{2n+1}h_{mk}^{\beta}h_{ij}^{\beta} \\ & -2\textstyle\sum_{\beta,k,m} h_{km}^{2n+1}h_{mj}^{\beta}h_{ik}^{\beta} + \textstyle\sum_{\beta,k,m} h_{mi}^{2n+1}h_{mk}^{\beta}h_{kj}^{\beta} \\ & -nH\textstyle\sum_{m} h_{mi}^{2n+1}h_{mj}^{2n+1} + \textstyle\sum_{\beta,k,m} h_{jm}^{2n+1}h_{mk}^{\beta}h_{ki}^{\beta}\end{aligned} \tag{46}$$

and

$$\begin{aligned}\Delta h_{ij}^{\alpha} = \; & n\tfrac{(c-3)}{4}h_{ij}^{2n+1} + \textstyle\sum_{\beta,k,m} h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta} - 2\textstyle\sum_{\beta,k,m} h_{km}^{\alpha}h_{mj}^{\beta}h_{ik}^{\beta} \\ & + \textstyle\sum_{\beta,k,m} h_{mi}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta} - nH\textstyle\sum_{m} h_{mi}^{\alpha}h_{mj}^{2n+1} + \textstyle\sum_{\beta,k,m} h_{jm}^{\alpha}h_{mk}^{\beta}h_{ki}^{\beta}.\end{aligned} \tag{47}$$

Since

$$\frac{1}{2}\Delta S = \frac{1}{2}\sum_{\alpha,i,j}\Delta(h_{ij}^{\alpha})^2 = \sum_{\alpha,i,j} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} + \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2. \tag{48}$$

Since the second fundamental form is parallel, from (46), (47) and (38)

$$\begin{aligned}\tfrac{1}{2}\Delta S = \; & n\tfrac{(c-3)}{4}S - n^2H^2\tfrac{(c-3)}{4} - nH\textstyle\sum_{\alpha} tr\left(H^{2n+1}(H^{\alpha})^2\right) \\ & + \textstyle\sum_{\alpha,\beta}[tr(H_{\alpha}H_{\beta})]^2 + \textstyle\sum_{\alpha,\beta\neq 2n+1} N\left(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha}\right),\end{aligned} \tag{49}$$

where $N(A) = tr(AA^t)$, for all matrix $A = (a_{ij})$.

The symmetric tensor Φ is defined by $\Phi = \sum_{i,j,\alpha\geq 2n+1}\Phi_{ij}^{\alpha}\omega_i\omega_j e_{\alpha}$, where Φ_{ij}^{α} was given by (40).

Hence from (45), we have that

$$S = |\Phi|^2 + nH^2. \tag{50}$$

Now we recall a fundamental property for the generalized maximum principle due to H. Omori [6] and S. T. Yau [9].

Lemma 3.1. *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below on M^n. Let F be a C^2-function bounded from below on M^n. Then, for any $\varepsilon > 0$, there exist a point p in M^n such that*

$$|\nabla F(p)| < \varepsilon, \quad \Delta F(p) > -\varepsilon, \quad \inf F + \varepsilon > F(p),$$

where $|\nabla F|$ denotes the norm of the gradient of F and ΔF the Laplacian of F.

Recall also an algebraic lemma due to M. Okumura [5].

Lemma 3.2. *Let μ_i, $i = 1, \ldots, n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = constant \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{51}$$

and the equality holds in (51) if and only if at least $(n-1)$ of the μ_i are equal to $\beta\sqrt{\frac{n}{n-1}}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta\sqrt{\frac{n-1}{n}}$.

We also need the following algebraic lemma whose proof can be found in [7].

Lemma 3.3. *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $trA = trB = 0$. Then*

$$|tr(A^2B)| \leq \frac{(n-2)}{\sqrt{n(n-1)}} trA^2\sqrt{trB^2}. \tag{52}$$

The equality holds if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i| = \left(\frac{trA^2}{n(n-1)}\right)^{\frac{1}{2}} \quad and \quad |y_i| = \left(\frac{trB^2}{n(n-1)}\right)^{\frac{1}{2}}, \ x_i x_j \geq 0.$$

Lemma 3.4 (A Simons type inequality)**.** *Let M^n be a complete not maximal contact normal spacelike submanifold with parallel mean curvature* *vector and second fundamental form parallel in a Sasakian space form $\widetilde{M}_q^{2m+1}(c)$. Then the following inequality holds:*

$$\frac{1}{2}\Delta|\Phi|^2 \geq |\Phi|^2\left(\frac{|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n\left(\frac{(c-3)}{4} - H^2\right)\right). \quad (53)$$

Proof. We have that $h_{ij}^{k^*} = 0$ for all $1 \leq i,j,k \leq n$. Hence the second fundamental form is locally timelike. Because h is parallel we can choose a local field of orthonormal frames $\{e_A\}$ such that $e_{2n+1} = \frac{h}{H}$. With this choose, we have that

$$|\Phi|^2 = \sum_{\alpha=2n+1}^{2m+1} tr(\Phi^\alpha)^2 \quad (54)$$

Note that,

$$\begin{aligned} nH\textstyle\sum_{\alpha\geq 2n+1} tr(H^{2n+1}(H^\alpha)^2) = \; & nH\textstyle\sum_{\alpha\geq 2n+1} tr(\Phi^{2n+1}(\Phi^\alpha)^2) \\ & +nH^2|\Phi|^2 + 2nH^2 tr(\Phi^{2n+1})^2 \quad (55) \\ & +n^2H^4, \end{aligned}$$

and

$$\sum_{\alpha,\beta}[tr(H^\alpha H^\beta)]^2 = \sum_{\alpha,\beta}[tr(\Phi^\alpha\Phi^\beta)]^2 + 2nH^2 tr(\Phi^{2n+1})^2 + n^2H^4. \quad (56)$$

Hence, from (49) we obtain

$$\begin{aligned} \frac{1}{2}\triangle|\Phi|^2 \geq \; & n\left(\tfrac{(c+3)}{4} - H^2\right)|\Phi|^2 - nH\textstyle\sum_\alpha tr\left(\Phi^{2n+1}(\Phi^\alpha)^2\right) \\ & + \textstyle\sum_{\alpha,\beta}[tr(\Phi^\alpha\Phi^\beta)]^2 + \sum_{\alpha,\beta\neq 2n+1} N(\Phi^\alpha\Phi^\beta - \Phi^\beta\Phi^\alpha) \end{aligned} \quad (57)$$

By applying a Lemma 3.3 to Φ^α and Φ^{2n+1} we get

$$|\, tr(\Phi^{2n+1}(\Phi^\alpha)^2)\, | \leq \frac{n-2}{\sqrt{n(n-1)}} \,|\, \Phi^{2n+1} \,||\, \Phi^\alpha \,|^2$$

and so

$$\sum_{\alpha} tr(\Phi^{2n+1}(\Phi^{\alpha})^2) \leq \frac{n-2}{\sqrt{n(n-1)}} \mid \Phi^{2n+1} \mid\mid \Phi \mid^2 \leq \frac{n-2}{\sqrt{n(n-1)}} \mid \Phi \mid^3 . \tag{58}$$

Using Cauchy-Schwarz inequality, it is easy to prove that

$$\mid \Phi \mid^4 \leq q \sum_{\alpha} [tr(\Phi^{\alpha}\Phi^{\beta})]^2. \tag{59}$$

This completes the proof of Simon's type inequality. □

Now, consider the following polynomial:

$$P_{H,q}(x) = \frac{x^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx + n\left(\frac{(c-3)}{4} - H^2\right). \tag{60}$$

We state without proof some elementary properties of $P_{H,q}$.

Lemma 3.5. *Let $Q(x) = (n-2)^2 x + 4(n-1)$ and $P_{H,q}$ the polynomial defined in (60). Then:*

1. *If $H^2 < \frac{(n-1)}{Q(q)}(c-3)$, then $c > 3$ and $P_{H,q}(x) > 0$ for any $x \in \mathbb{R}$.*
2. *If $H^2 = \frac{(n-1)}{Q(q)}(c-3)$, then $c > 3$ and the (double) root of $P_{H,q}$ is*

$$\vartheta_H^{\pm} = \frac{n(n-2)q}{2\sqrt{n}} \sqrt{\frac{c-3}{Q(q)}}$$

so that $P_{H,q}(x) = \left(x - \frac{n(n-2)q}{2\sqrt{n}} \sqrt{\frac{c-3}{Q(q)}}\right)^2 \geq 0$

3. *If $H^2 > \frac{(n-1)}{Q(q)}(c-3)$, then $P_{H,q}$ has two real roots ϑ_H^- and ϑ_H^+ given by*

$$\vartheta_H^{\pm} = \frac{q}{2}\sqrt{\frac{n}{(n-1)}} \left\{ (n-2)H \pm \sqrt{\frac{Q(q)H^2 - (n-1)(c-3)}{q}} \right\}.$$

ϑ_H^+ is always positive; on the other hand, $\vartheta_H^- < 0$ if only if, $H^2 > \frac{(c-3)}{4}$, $\vartheta_H^- = 0$, if only if, $H^2 = \frac{(c-3)}{4}$, and $\vartheta_H^- > 0$ if only if, $\frac{(n-1)}{Q(q)}(c-3) \leq H^2 < \frac{(c-3)}{4}$.

Note that $Q(q) > 0$ and that $q \geq 3$.

4 Proof of theorem 1.1

Consider the positive smooth function f on M^n defined by

$$f = \frac{1}{\sqrt{1+|\Phi|^2}}.$$

We have

$$|\nabla f|^2 = -\frac{1}{4}\frac{|\triangle|\Phi|^2|^2}{(1+|\Phi|^2)^3} \tag{61}$$

and

$$f\triangle f = -\frac{1}{2}\frac{\triangle|\Phi|^2}{(1+|\Phi|^2)^2} + 3|\nabla f|^2. \tag{62}$$

From (53) and (62), we have

$$f\triangle f \leq \frac{-|\Phi|^2}{(1+|\Phi|^2)^2}\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n\frac{(c-3)}{4} - nH^2\right) + 3|\nabla f|^2. \tag{63}$$

Since $trH^\alpha = 0$ and $h_{ij}^{i^*} = 0$ (which implies that the second fundamental form is locally timelike), using formula (28), we get

$$Ric(e_i) = (n-1)\frac{(c-3)}{4} - (trH^{2n+1})h_{ik}^{2n+1} + \sum_{\alpha,j}(h_{ij}^\alpha)^2. \tag{64}$$

Assume that the second fundamental form of M^n with respect to e_{2n+1} has been diagonalized so that the eigenvalues are λ_i^{2n+1}. Then we have

$$\begin{aligned} Ric(e_i) &\geq (n-1)\tfrac{(c-3)}{4} - nHh_{ii}^{2n+1} + \textstyle\sum_k \left(h_{ik}^{2n+1}\right)^2 \\ &= (n-1)\tfrac{(c-3)}{4} - nH\lambda_i^{2n+1} + (\lambda_i^{2n+1})^2 \geq (n-1)\tfrac{(c-3)}{4} - \tfrac{n^2H^2}{4} \end{aligned} \tag{65}$$

So the Ricci curvature of M^n is bounded from below. Since M^n is spacelike and $f > 0$, we can apply Lemma 3.1 to the function f. Therefore there is a sequence of points p_k in M^n such that

$$\lim_{k\to\infty} f(p_k) = \inf f, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0, \quad \lim_{k\to\infty} \inf \triangle f(p_k) \geq 0. \tag{66}$$

From (65), we have $\inf(f) \neq 0$, so $\lim_{k\to\infty} |\Phi|^2(p_k) = \sup|\Phi|^2 < \infty$.

By inserting (66) into (63) we get

$$\frac{\sup|\Phi|^2}{(1+\sup|\Phi|^2)^2}\left(\frac{\sup|\Phi|^2}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sup|\Phi| + n\left(\frac{(c-3)}{4} - H^2\right)\right) \leq 0. \tag{67}$$

By hypothesis M^n is not maximal. Then from lemmas 2.2 and 2.5, $m > n$.

(a) If $H^2 < \dfrac{(n-1)}{Q(q)}(c-3)$, then $c > 3$ and $P_{H,q}(x) > 0$ for any $x \in \mathbb{R}$. Hence

$$P_{H,q}(\sup|\Phi|) = \frac{\sup|\Phi|}{q} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sup|\Phi| + n\left(\frac{(c-3)}{4} - H^2\right) > 0. \tag{68}$$

Therefore, from (67), we have either $\sup|\Phi|^2 = 0$. Then $|\Phi|^2 = 0$ and from (50), $S = nH^2$, which asserts that M^n is totally umbilical.

(b) If $H^2 = \dfrac{(n-1)}{Q(q)}(c-3)$ we have $c > 3$ and from lemma 3.5 we get

$$P_{H,q}(\sup|\Phi|) = \left(\sup|\Phi| - \frac{n(n-2)q}{2\sqrt{n}}\sqrt{\frac{c-3}{Q(q)}}\right)^2 \geq 0.$$

Then either $\left(\sup|\Phi| - \dfrac{n(n-2)q}{2\sqrt{n}}\sqrt{\dfrac{c-3}{Q(q)}}\right)^2 > 0$, and from (67) we have either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical or $\sup|\Phi| = \dfrac{n(n-2)q}{2\sqrt{n}}\sqrt{\dfrac{c-3}{Q(q)}} = \vartheta_H^{\pm}$. If $\sup|\Phi| = \dfrac{n(n-2)q}{2\sqrt{n}}\sqrt{\dfrac{c-3}{Q(q)}}$, from (50) $\sup S = n\frac{(c-3)}{4}\dfrac{Q(q^2)}{Q(q)}$.

(c) If $H^2 > \dfrac{(n-1)}{Q(q)}(c-3)$ we have

$$P_{H,p}(\sup|\Phi|) = (\sup|\Phi| - \vartheta_H^-)(\sup|\Phi| - \vartheta_H^+).$$

If $c < 3$ we infer that $\vartheta_H^- < 0$. Hence, from (67) either $\sup|\Phi|^2 = 0$ and in this case M^n is totally umbilical or $0 < \sqrt{\sup|\Phi|^2} \leq \vartheta_H^+$. Therefore, from (50) we obtain $nH^2 < \sup S \leq \vartheta_H^+ + nH^2$.

(d) Suppose that $H^2 > \dfrac{(n-1)}{Q(q)}(c-3)$ and $c > 3$. If $H^2 > \frac{c-3}{4}$ we infer that $\vartheta_H^- < 0$ and from (67), we have that $\sup|\Phi|^2 = 0$ and, in this case M^n is totally umbilical or $0 < \sqrt{\sup|\Phi|^2} \leq \vartheta_H^+$.

When $\frac{(n-1)}{Q(p)}(c-3) < H^2 < \frac{(c-3)}{4}$ we infer that $\vartheta_H^- > 0$. In this case (67) implies that either $\sup|\Phi|^2 = 0$ and M^n is totally umbilical or $\vartheta_H^- \leq \sup|\Phi| \leq \vartheta_H^+$. Therefore, from (50) we have that $(\vartheta_H^-)^2 + nH^2 \leq \sup S \leq (\vartheta_H^+)^2 + nH^2$.

References

[1] Baek, J.O., Cheng, Q.M. and Suh, Y.J., *Complete spacelike hypersurfaces in locally symmetric Lorentz space*, J. Geom. Phys. 49 (2004), 231–247.

[2] Cheng, S.Y. and Yau, S.T., *Maximal spacelike hypersurfaces in the Lorentz-Minkowski space*, Ann. of Math. (2) 104 (1976), 407–419.

[3] Ishihara, T., *Maximal space-like submanifolds of a pseudo-Riemannian space of constant curvature*, Mich. Math. J. 35 (1988), 345–352.

[4] Nishikawa, S., *On spacelike hypersurfaces in a Lorentz manifold*, Nagoya Math. J. 95 (1984), 117–124.

[5] Okumura, M., *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math. **96** (1974), 207-213.

[6] Omori, H., *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205-214.

[7] Santos, W., *Submanifolds with parallel mean curvature vector in spheres*, Tôhoku Math. J. **46** (1994), 403-415.

[8] Takahashi, T., *Sasakian manifold with pseudo-Riemannian metrics*, Tohoku Math. J. 21 (1969), 271–290.

[9] Yau, S.T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201-228.

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