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Stability Properties of Rotational Catenoids in the Heisenberg Groups

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Abstract

In this paper, we determine the maximally stable, rotationally invariant domains on the catenoids $\mathcal{C}_a$ (minimal surfaces invariant by rotations) in the Heisenberg group with a left-invariant metric. We show that these catenoids have Morse index at least 3 and we bound the index from above in terms of the parameter a. We also show that the index of $\mathcal{C}_a$ tends to infinity with a. Finally, we study the rotationally symmetric stable domains on the higher dimensional catenoids.

1 Introduction

Minimal surfaces in the Heisenberg group equiped with a left-invariant metric have been studied by several authors, see [7, 8, 4, 5] and the references therein. *Catenoids* in the Heisenberg group Nil(3) are complete minimal surfaces which are invariant under a one-parameter subgroup of rotations with axis the center of the group. They come in a one-parameter family $\{\mathcal{C}_a, a > 0\}$ of complete minimal surfaces and were first described in [7] and [8] where the authors provide the classification of constant mean

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curvature surfaces in the Heisenberg group, invariant under certain subgroups of isometries (the parameter a is the neck size of the catenoid, see (9)).

In this paper, we study the stability properties of the catenoids $\{\mathcal{C}_a, a > 0\}$. More precisely, we determine the rotationally invariant stable domains of the catenoids in Nil$(2n+1)$, $n \geq 1$, with a different behaviour (Lindeloef's property) when $n = 1$ and when $n \geq 2$. We also study the Morse index of the catenoids in Nil(3). As in [3], the proofs rely in part on a detailed analysis of the Jacobi fields induced from the Killing fields of the ambient Heisenberg space and from the variation of the parameter a.

The paper is organized as follows. In Section 2, we give some preliminary results. We first recall the basic geometry of the Heisenberg group Nil(3) equiped with a left-invariant metric $\hat{g}$ (see [8] for more details). In order to keep our paper self-contained, we derive the differential equation satisfied by the generating curves of the catenoids, using a flux formula. In Section 3, we describe the stable rotationally invariant domains on $\{\mathcal{C}_a\}$ (Theorem 3.1). The proof uses Jacobi fields. We also give some information on the Gauss map of the catenoids $\{\mathcal{C}_a\}$. In Section 4, Theorem 4.1, we prove that the catenoids $\mathcal{C}_a, a > 0$ have Morse index at least 3. We bound the index from above in terms of a, and we also show that its goes to infinity with a. The proof uses Jacobi fields, Fourier analysis and an adapted perturbation of the original parametrization of the catenoids. Finally, in Section 5, we study the maximally stable, rotationally invariant domains on the higher dimensional catenoids (Theorem 5.1).

In the sequel our functions will often depend on the parameter a. We will occasionally omit a to keep the notations simpler. In this paper, we only consider left-invariant Riemannian metrics on the Heisenberg groups.

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2 Preliminaries

2.1 The 3-dimensional Heisenberg manifold

Let Nil(3) denote the 3-dimensional *Heisenberg group*. This is a two-step nilpotent Lie group which can be seen as the subgroup of 3×3 matrices given by

$$\mathrm{Nil}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; (x,y,z) \in \mathbb{R}^3 \right\} \subset GL(3,\mathbb{R}).$$

We denote the corresponding Lie algebra by

$$\mathcal{L}(\mathrm{Nil}(3)) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} ; (x,y,z) \in \mathbb{R}^3 \right\}.$$

Using the exponential map, $\exp : \mathcal{L}(\mathrm{Nil}(3)) \to \mathrm{Nil}(3)$, and the Campbell-Hausdorff formula,

$$\exp\big(A\big)\exp\big(B\big) = \exp\big(A + B + \frac{1}{2}[A,B]\big), \quad \forall A, B \in \mathcal{L}(\mathrm{Nil}(3)),$$

we can view Nil(3) as $\mathbb{R}^3$ equiped with the group structure $\star$ given by

$$(x,y,z) \star (x',y',z') = \Big(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\Big),$$

with neutral element $0 = (0,0,0)$ and inverse $\check{p}$ of $p = (a,b,c)$ given by $\check{p} = (-a,-b,-c)$. The left-multiplication by p in Nil(3), $L_p : q \mapsto p \star q$, has tangent map

$$T_q L_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}b & \frac{1}{2}a & 1 \end{pmatrix} \tag{1}$$

in the canonical coordinates $\{x, y, z\}$ of $\mathbb{R}^3$ (they are often referred to as *exponential coordinates*). Let $\{\partial_x, \partial_y, \partial_z\}$ denote the canonical vector fields in $\mathbb{R}^3$. It follows from the expression (1) that the vector fields

$$\begin{cases} X(x,y,z) &= T_0 L_{(x,y,z)}(\partial_x) &= \partial_x - \frac{y}{2}\,\partial_z, \\ Y(x,y,z) &= T_0 L_{(x,y,z)}(\partial_y) &= \partial_y + \frac{x}{2}\,\partial_z, \\ Z(x,y,z) &= T_0 L_{(x,y,z)}(\partial_z) &= \partial_z, \end{cases} \tag{2}$$

form a basis of left-invariant vector fields in Nil(3).

The metric $\hat{g}$ on Nil(3). From now on, we fix the left-invariant metric $\hat{g}$ on Nil(3) to be such that the family $\{X, Y, Z\}$ is an orthonormal frame. In the coordinates $\{x, y, z\}$, this metric is given by

$$\hat{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \big(\mathrm{d}z + \frac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y)\big)^2.$$

The following properties are well-known and can be found for example in [8], Section 1. Equiped with the left-invariant metric $\hat{g}$, the Heisenberg group Nil(3) is a homogeneous Riemannian manifold whose group of isometries has dimension 4. A basis of Killing vector fields on $(\mathrm{Nil}(3), \hat{g})$ is given by

$$\begin{cases} \xi &= X + yZ, \\ \eta &= Y - xZ, \\ \zeta &= Z, \\ \rho &= yX - xY + \frac{1}{2}(x^2 + y^2)Z. \end{cases}$$

The first three vector fields ξ, η and ζ correspond to the one-parameter subgroups of isometries generated by right-invariant vector fields in Nil(3), while the vector field ρ corresponds to the one-parameter subgroup of isometries defined by

$$\psi_\theta\big((x,y,z)\big) = \big(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z\big), (x,y,z) \in \mathsf{R}^3, \theta \in \mathsf{R}, \tag{3}$$

in the representation $(\mathbb{R}^3, \star)$ of Nil(3). We call them *rotations around the z-axis*. Notice that the z-axis is precisely the center of Nil(3).

2.2 Surfaces of revolution in Nil(3)

We say that a surface M in Nil(3) is a *surface of revolution* if M is invariant under the action of the one-parameter subgroup $\{\psi_\theta, \theta \in \mathbb{R}\}$ given by (3). We will consider surfaces of revolution whose generating curves are graphs $t \to \big(f(t), t\big)$ above the z-axis in the 2-plane $\{x, z\}$, where f is a positive function, and where t varies in some interval $I \subset \mathbb{R}$. They are given by a map

$$\mathcal{F}(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t), \tag{4}$$

for $t \in I \subset \mathbb{R}$ and $\theta \in [0, 2\pi]$.

Catenoids, *i.e.* minimal surfaces of revolution, in Nil(3) are described in [7, 8], using the methods of equivariant differential geometry. They come in a one-parameter family of complete minimal surfaces, $\{\mathcal{C}_a, a > 0\}$. For the sake of completeness and for later purposes, we will derive the differential equations satisfied by the generating curve of a catenoid using a *flux formula* which we now state.

Proposition 2.1. Let $(M^n, g) \looparrowright (\widehat{M}^{n+1}, \hat{g})$ be an isometric immersion with Riemannian measure μ_g and normalized mean curvature vector $\vec{H}$. Let Ω be a relatively compact smooth domain in M. Let ν_{int} denote the unit normal to $\partial\Omega$ in M, pointing inwards, and σ_g the Riemannian measure on $\partial\Omega$ induced by g. Then, for any Killing vector field $\mathcal{K}$ on $\widehat{M}^{n+1}$, we have

$$\int_{\partial\Omega} \hat{g}(\mathcal{K}, \nu_{int})\, \mathrm{d}\sigma_g = -n \int_{\Omega} \hat{g}(\mathcal{K}, \vec{H})\, \mathrm{d}\mu_g.$$

Proof. Let κ be the restriction to M of the 1-form dual to $\mathcal{K}$, i.e. $\kappa = \hat{g}(\mathcal{K}, .)|_M$. Recall that $\mathcal{K}$ is a Killing field if and only if, for any vector field X on $\widehat{M}$, $\hat{g}(\widehat{D}_X\mathcal{K}, X) = 0$ ([9], Proposition 3.2, p. 237). A straightforward computation shows that the divergence $\delta_g\kappa$ of the 1-form κ, for the induced metric g on M, is given by

$$\delta_g\kappa = -n\hat{g}(\mathcal{K}, \vec{H}).$$

The proposition follows from the divergence theorem. ■

Let $M = \mathcal{F}(I \times [0, 2\pi])$ be a minimal surface of revolution in Nil(3), given by an immersion $\mathcal{F}(t, \theta)$ as in (4), with $t \in I \subset \mathbb{R}$, $\theta \in [0, 2\pi]$. We can make a coherent choice of a unit vector field ν tangent to M and orthogonal to the circles $C_t = \mathcal{F}(\{t\} \times [0, 2\pi])$ in such a way that Proposition 2.1 gives

$$\int_{C_t} \hat{g}(\mathcal{K}, \nu)\, \mathrm{d}\sigma_{C_t} = \int_{C_{t_0}} \hat{g}(\mathcal{K}, \nu)\, \mathrm{d}\sigma_{C_{t_0}}, \tag{5}$$

for all $t_0, t \in I$ and for any Killing vector field $\mathcal{K}$ in Nil(3).

Proposition 2.2. The generating curve of a minimal surface of revolution of the form (4) in Nil(3) satisfies the first order differential equation

$$f\left(4 + f^2 f_t^2 + 4f_t^2\right)^{-1/2} = C \quad \text{(a constant)}, \tag{6}$$

and the second order differential equation

$$f(4 + f^2)\, f_{tt} = 4\,(1 + f_t^2), \tag{7}$$

textupwhere f_t and f_{tt} denote respectively the first and second derivatives of the function f with respect to the variable t.

Proof. According to [8] Theorem 3, we already know that minimal surfaces of revolution do exist in Nil(3). Equation (6) is established by applying Proposition 2.1 with the Killing field $\mathcal{K} = Z$. The constant C can then be interpreted in terms of a flux. The vectors $\mathcal{F}_t$ and $\mathcal{F}_\theta$ are tangent to the surface. Using (2), they can be expressed in the orthonormal frame $\{X, Y, Z\}$ at $\mathcal{F}(t, \theta)$ as

$$\begin{cases} \mathcal{F}_t &= f_t \cos\theta\, X + f_t \sin\theta\, Y + Z, \\ \mathcal{F}_\theta &= -f \sin\theta\, X + f \cos\theta\, Y - \frac{1}{2} f^2\, Z. \end{cases} \tag{8}$$

The Riemannian measure σ_{C_t} is given by

$$\mathrm{d}\sigma_{C_t} = \sqrt{\hat{g}(\mathcal{F}_\theta, \mathcal{F}_\theta)}\, \mathrm{d}\theta = f\sqrt{1 + \frac{1}{4} f^2}\, \mathrm{d}\theta.$$

Up to sign, the vector ν is characterized by the facts that it is unitary, tangent to the surface – hence a linear combination of $\mathcal{F}_t$ and $\mathcal{F}_\theta$ – and orthogonal to $\mathcal{F}_\theta$. Consider the vector $n = \mathcal{F}_t + \alpha\mathcal{F}_\theta$ with α such that $\hat{g}(n, \mathcal{F}_\theta) = 0$. Choose $\nu = \hat{g}(n,n)^{-1/2}n$. The expression $\hat{g}(Z,\nu)$ which appears in (5) when we choose $\mathcal{K} = Z$, is the Z-component of ν. A straightforward computation gives that $\alpha = 2(4+f^2)^{-1}$, $\hat{g}(n,Z) = 4(4+f^2)^{-1}$ and $\hat{g}(n,n) = f_t^2 + 4(4+f^2)^{-1}$. It follows that

$$\hat{g}(Z,\nu) = 4(4+f^2)^{-1}\big(f_t^2 + \frac{4}{4+f^2}\big)^{-1/2}.$$

Using (5),we obtain that the quantity (a flux)

$$f(t)\big[4+f^2(t)\,f_t^2(t) + 4f_t^2(t)\big]^{-1/2}$$

is independent of t. Equation (6) follows. Taking the derivative of (6) and using the fact that $f_t \not\equiv 0$ (see [8]), we obtain Equation (7). ■

Remark. The above equations can also be derived directly from [8] (using the computations in the proof of their Theorem 3) or by minimizing the area of a rotational domain, in the spirit of the calculus of variations.

2.3 Qualitative analysis of Equation (7)

Given $a > 0$, consider the *Cauchy problem*,

$$\begin{cases} f(f^2+4)f_{tt} &= 4(1+f_t^2), \\ f(0) &= a, \\ f_t(0) &= 0, \end{cases} \tag{9}$$

where the subscript t means that we take the derivative with respect to t. Recall that this differential equation admits a first integral and, more precisely, that

$$\frac{(f^2+4)(1+f_t^2)}{f^2} = \frac{a^2+4}{a^2}. \tag{10}$$

A simple analysis shows that (9) admits a maximal solution $f(a,t)$ which is an even function of t on some interval $(-A_a, A_a)$. Furthermore, the function

$$f(a,\cdot) : [0, A_a) \to [a, \infty)$$

is an increasing function and we can introduce its inverse function

$$\phi(a,\cdot) : [a,\infty) \to [0, A_a).$$

Using (10), we infer that ϕ is given by the integral

$$\phi(a,\tau) = \frac{a}{2}\int_1^{\tau/a} \sqrt{\frac{a^2v^2+4}{v^2-1}}\,\mathrm{d}v. \tag{11}$$

It follows that

$$\phi(a,\tau) \sim \frac{a}{2}\tau, \quad \text{when} \quad \tau \to \infty.$$

Finally, we conclude that the Cauchy problem (9) admits a global solution $f(a,\cdot) : \mathbb{R} \to [a,\infty)$ which satisfies

$$\begin{cases} f(a,t) = f(a,-t), \\ f(a,t) \sim \frac{2}{a}|t|, \quad \text{and} \\ f_t(a,t) \sim \frac{2}{a}\mathrm{sgn}(t), \quad \text{when} \quad |t| \to \infty. \end{cases}$$

2.4 The Jacobi operator of minimal surfaces

In this section, we recall some classical definitions and facts about the Jacobi operator of minimal surfaces. Let $M^2 \looparrowright \widehat{M}^3$ be an orientable minimal surface immersed into an oriented Riemannian manifold $(\widehat{M},\hat{g})$. Let N_M be a unit normal field along M, A_M the second fundamental form of the immersion with respect to the normal N_M, and let $\widehat{\mathrm{Ric}}$ be the Ricci curvature of $\widehat{M}$. The second variation of the volume functional gives rise to the *Jacobi operator* J_M of M (see [10])

$$J_M := -\Delta_M - (|A_M|^2 + \widehat{\mathrm{Ric}}(N_M)), \tag{12}$$

where Δ_M is the non-positive Laplacian on M for the induced metric.

Given a relatively compact regular domain Ω on the surface M, we let $\mathrm{Ind}(\Omega)$ denote the number of negative eigenvalues of J_M for the Dirichlet problem in Ω. The *Morse index* of M is defined to be the supremum

$$\mathrm{Ind}(M) := \sup\{\mathrm{Ind}(\Omega); \Omega \Subset M\} \leq \infty,$$

taken over all relatively compact regular domains. Let $\lambda_1(\Omega)$ be the least eigenvalue of the operator J_M with the Dirichlet boundary conditions in Ω. We call a relatively compact regular domain Ω *stable* if $\lambda_1(\Omega) > 0$, *unstable* if $\lambda_1(\Omega) < 0$, and *stable-unstable* if $\lambda_1(\Omega) = 0$. More generally, we say that a domain Ω (not necessarily relatively compact) is *r-stable* if any relatively compact subdomain is stable. In the following proposition, we collect classical results which will be used later on.

Proposition 2.1. *Given a minimal immersion $M^2 \looparrowright \widehat{M}^3$, the following properties hold.*

(i) Let Ω be a stable-unstable relatively compact domain. Then, any smaller domain is stable while any larger domain is unstable.

(ii) We refer to the solutions of the equation $J_M(u) = 0$ as Jacobi functions *on M. Let $X_a : M^2 \looparrowright (\widehat{M}^3, \hat{g})$ be a one-parameter family of oriented minimal immersions, with variation field $V_a = \frac{\partial X_a}{\partial a}$ and with unit normal N_a. Then, the function $\hat{g}(V_a, N_a)$ is a Jacobi function on M.*

(iii) Let Ω be a relatively compact domain on a minimal submanifold M. If there exists a positive function u on Ω such that $J_M(u) \geq 0$, then Ω is stable or stable-unstable.

Proof. *Assertion (i)* follows from the min-max characterization of eigenvalues and the maximum principle. *Assertion (ii)* appears in [1] (Theorem 2.7 and its proof) in a more general framework. For *Assertion (iii)*, see the proof of Theorem 1 in [6]. ■

3 Stable domains of revolution on the catenoids

We consider a catenoid $\mathcal{C}$ given by the map,

$$\mathcal{F} : \mathbb{R} \times [0, 2\pi] \to \mathcal{C} \looparrowright \mathrm{Nil}(3),$$

$$\mathcal{F}(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t),$$

where f is a global solution of (7). It follows from (8) that the first fundamental form induced by $\mathcal{F}$ is given by

$$g_{\mathcal{F}} = \begin{pmatrix} 1+f_t^2 & -\frac{1}{2}f^2 \\ -\frac{1}{2}f^2 & f^2(1+\frac{1}{4}f^2) \end{pmatrix}.$$

For later purposes, we introduce the functions

$$G = f^2(1+\frac{1}{4}f^2) \quad \text{and} \quad D = \sqrt{\mathrm{Det}(g_{\mathcal{F}})} = f\big(1+f_t^2+\frac{1}{4}f^2f_t^2\big)^{1/2}. \qquad (13)$$

Let N be a unit normal field to $\mathcal{F}$. Writing $N = \alpha X + \beta Y + \gamma Z$, we can choose N to be

$$\begin{cases} \alpha &= W(-\cos\theta - \frac{1}{2}ff_t\sin\theta), \\ \beta &= W(-\sin\theta + \frac{1}{2}ff_t\cos\theta), \\ \gamma &= Wf_t, \quad \text{where} \\ W &= \big(1+f_t^2+\frac{1}{4}f^2f_t^2\big)^{-1/2}. \end{cases} \qquad (14)$$

3.1 Jacobi functions coming from ambient Killing fields.

Since the set $\{\xi, \eta, \zeta, \rho\}$ is a basis of Killing vector fields, it follows from Proposition 2.1(ii) that the functions

$$\begin{cases} v_\xi &= \hat{g}(\xi, N) &= W(-\cos\theta + \frac{1}{2}ff_t\sin\theta), \\ v_\eta &= \hat{g}(\eta, N) &= W(-\sin\theta - \frac{1}{2}ff_t\cos\theta), \\ v_\zeta &= \hat{g}(\zeta, N) &= Wf_t, \end{cases} \qquad (15)$$

are Jacobi functions on the surface $\mathcal{F}$ (note that $v_\rho = \hat{g}(\rho, N) = 0$).

Remark. The Jacobi functions v_ξ, v_η and v_ζ are linearly independent.

3.2 A Jacobi function coming from the variation of the family

We now consider the one-parameter family of catenoids $\{\mathcal{C}_a, a > 0\}$, associated with the family of maps

$$\mathcal{F}(a,t,\theta) = \big(f(a,t)\cos\theta, f(a,t)\sin\theta, t\big), \quad a > 0, \qquad (16)$$

where $f(a,\cdot)$ is the unique global solution of the Cauchy problem (9). The variational field of this family is given by

$$\mathcal{F}_a(a,t,\theta) = f_a(a,t)\cos\theta\, X + f_a(a,t)\sin\theta\, Y, \tag{17}$$

where $f_a(a,t) := \frac{\partial f}{\partial a}(a,t)$. By Proposition 2.1(ii), this yields another Jacobi function on $\mathcal{C}_a$, namely, $e(a,\cdot) = -\hat{g}(\mathcal{F}_a, N)$. More precisely,

$$e(a,t) = \big(W f_a\big)(a,t), \tag{18}$$

where the function W is given by the last line in (14). We note that $e(a,\cdot)$ does not depend on θ and is an even function of t. Furthermore, since $f(a,0) = a$ and $f_t(a,0) = 0$, $\forall a > 0$, we have $e(a,0) = 1$, $\forall a > 0$.

The rotationally invariant stable domains of the catenoids $\mathcal{C}_a$ are described in the following theorem.

Theorem 3.1. Let $\mathcal{C}_a$ be a catenoid in Nil(3). Then

(i) The upper (resp. the lower) half catenoid $\mathcal{C}_{a,+} = \mathcal{C}_a \cap \{z > 0\}$ (resp. $\mathcal{C}_{a,-} = \mathcal{C}_a \cap \{z < 0\}$) is r-stable.

(ii) The function $e(a,\cdot)$ is even and has exactly one zero $z(a)$ on $(0,\infty)$. The domain $\mathcal{F}(a,[-z(a),z(a)],[0,2\pi])$ is a stable-unstable domain in $\mathcal{C}_a$.

(iii) Given any $t_1 > 0$, there exists some $t_2 > 0$ such that the domain $\mathcal{D}_a(-t_1,t_2) = \mathcal{F}(a,[-t_1,t_2],[0,2\pi])$ is stable-unstable. This implies in particular that both $\mathcal{C}_{a,+}$ and $\mathcal{C}_{a,-}$ are maximal r-stable rotationally invariant domains (*i.e.* in some sense, stable-unstable).

Proof. *Assertion (i).* It follows from Section 2.3 that the Jacobi function v_ζ is positive on $(0,+\infty)$ and negative on $(-\infty,0)$. The assertion follows from Proposition 2.1(iii).

Assertion (ii). We already know that $e(a,\cdot)$ is an even function of t and that $e(a,0) = 1$ for all $a > 0$. *Claim 1.* The function $e(a,\cdot)$ has at most one zero in $(0,+\infty)$. If not, $e(a,\cdot)$ would have two consecutive positive zeroes, $0 < z_1(a) < z_2(a)$ and the domain $\mathcal{F}(a,[z_1(a),z_2(a)],[0,2\pi])$ would be stable-unstable. According to Proposition 2.1(i), this would contradict the r-stability of $\mathcal{C}_{a,+}$ in Assertion (i). *Claim 2.* The function $e(a,\cdot)$ has at least one zero in $(0,+\infty)$. Indeed, $e(a,\cdot)$ has the sign of $f_a(a,t)$. Using the function ϕ defined by (11), we have

$$\phi(a,f(a,t)) \equiv t \text{ and } \phi_a\big(a,f(a,t)\big) + f_a(a,t)\,\phi_\tau\big(a,f(a,t)\big) \equiv 0$$

for all $a,t > 0$. Since ϕ_τ is positive, it suffices to look at the sign of ϕ_a. We find that

$$\phi_a(a,\tau) = \int_1^{\tau/a} \frac{a^2v^2+2}{\sqrt{(a^2v^2+4)(v^2-1)}}\,\mathrm{d}v - \frac{\tau}{2}\sqrt{\frac{\tau^2+4}{\tau^2-a^2}} \tag{19}$$

and we easily conclude that $\phi_a(a,\tau)$ is positive when τ is large enough. It follows that $e(a,t)$ is negative for t large enough so that it must vanish at least once in $(0,+\infty)$.

Assertion (iii). Fix some $t_1 > 0$ and consider the function

$$w(a,t_1,t) = v(a,t_1)e(a,t) + e(a,t_1)v(a,t),$$

where we have written $v(a,t)$ instead of $v_\zeta(a,t)$ for short. This is a Jacobi function on $\mathcal{C}_a$, which vanishes at $t = -t_1$. Note that $w(a,t_1,0) = v(a,t_1) > 0$ because $e(a,0) = 1$ and $v(a,t) > 0$ for any $t > 0$. As in the proof of Assertion (ii), Claim 1, we see that $w(a,t_1,\cdot)$ can vanish at most once in $(-\infty,0)$ and $(0,\infty)$. It follows that $w(a,t_1,\cdot)$ has exactly one zero in $(-\infty,0)$ – namely $-t_1$ – and that it vanishes in $(0,\infty)$ if and only if it takes some negative value near infinity. Recall that

$$v(a,t) = \frac{f_t}{\sqrt{1+f_t^2+\frac{1}{4}f^2f_t^2}}(a,t). \tag{20}$$

As in the proof of Assertion (ii), Claim 2, we use the functional equations $\phi\big(a,f(a,t)\big) \equiv t$ and $\phi_\tau\big(a,f(a,t)\big)\,f_t(a,t) \equiv 1$ for all $t > 0$. Plugging these relations into (20), we find that

$$v(a,t) = \tilde{v}\big(a,f(a,t)\big), \quad \forall t > 0,$$

where $\tilde{v}(a,\tau) = \big(1 + \frac{\tau^2}{4} + \phi_\tau^2(a,\tau)\big)^{-1/2}$. Similar computations yield the relation

$$e(a,t) = \tilde{e}\big(a, f(a,t)\big), \quad \forall t > 0,$$

where $\tilde{e}(a,\tau) = -\phi_a(a,\tau)\tilde{v}(a,\tau)$. Define $\tau_1 := f(a,t_1)$ and

$$\tilde{w}(a,\tau_1,\tau) = \tilde{v}(a,\tau_1)\tilde{e}(a,\tau) + \tilde{e}(a,\tau_1)\tilde{v}(a,\tau),$$

so that $w(a,t_1,t) = \tilde{w}\big(a,\tau_1, f(a,t)\big)$. Then,

$$\tilde{w}(a,\tau_1,\tau) = -\tilde{v}(a,\tau)\tilde{v}(a,\tau_1)\big(\phi_a(a,\tau) + \phi_a(a,\tau_1)\big).$$

Using (19), we conclude that w is negative when τ approches infinity, for any given $a, t_1 > 0$. This proves the existence of a positive t_2 such that the domain $\mathcal{D}_a(-t_1,t_2)$ ist stable-unstable. The last assertion follows immediately. ■

Remarks. (i) Consider the family of curves $\Gamma_a : t \mapsto \big(f(a,t),t\big)$. This family admits an envelope $\mathcal{E}$ and the values $\pm z(a)$ correspond to the points at which the curve Γ_a is tangent to $\mathcal{E}$. (ii) Using (14) and Section 2.3, we can see that the Gauss map of the catenoid $\mathcal{C}_a$ covers a closed symmetric strip about the equator of the unit sphere in the Lie algebra $\mathcal{L}(\mathrm{Nil}(3))$. This strip, whose width depends on a, is strictly contained in the sphere minus the south and north poles. Each point of the open strip is covered exactly twice, except the points of the equator which are covered once (look at the variations of the Z-component γ of the vector N).

4 The index of the catenoids $\mathcal{C}_a$ in Nil(3)

In this section, we study the Morse index of the catenoids $\mathcal{C}_a$. It turns out that the representation $\mathcal{F}$ given by (4), with the function f satisfying (7), is not well-adapted to Fourier analysis on $\mathcal{C}_a$ because the vectors $\mathcal{F}_t$ and $\mathcal{F}_\theta$ are not orthogonal. To avoid this problem, we introduce a perturbed representation,

$$\widetilde{\mathcal{F}}(t,\theta) := \mathcal{F}(t,\theta+\varphi(t)) = \Big(f(t)\cos(\theta+\varphi(t)), f(t)\sin(\theta+\varphi(t)), t\Big).$$

The tangent vectors are given by

$$\begin{cases} \widetilde{\mathcal{F}}_t(t,\theta) & = & \mathcal{F}_t(t,\theta+\varphi(t)) + \varphi_t(t)\mathcal{F}_\theta(t,\theta+\varphi(t)),\\ \widetilde{\mathcal{F}}_\theta(t,\theta) & = & \mathcal{F}_\theta(t,\theta+\varphi(t)). \end{cases}$$

It follows that the representation $\widetilde{\mathcal{F}}$ is orthogonal – *i.e.* the vectors $\widetilde{\mathcal{F}}_t$ and $\widetilde{\mathcal{F}}_\theta$ are orthogonal – if and only if the function φ satisfies the differential equation

$$\varphi_t = \frac{2}{4+f^2}. \tag{21}$$

From now on, we choose φ to be the solution of (21) such that $\varphi(0)=0$.

Note that in the above expressions, we have omitted the dependence on the parameter a. The unit normal vector to $\mathcal{C}_a$ at the point $\widetilde{\mathcal{F}}(t,\theta)$ is $\widetilde{N}(t,\theta) = N(t,\theta+\varphi(t))$. In the representation $\widetilde{\mathcal{F}}$, the Riemannian metric induced by the immersion $\mathcal{C}_a \looparrowright \mathrm{Nil}(3)$ is of the form $D^2G^{-1}\mathrm{d}t^2 + G\mathrm{d}\theta^2$, with the functions D, G as in (13). It follows that the Laplacian on $\mathcal{C}_a$ is given, in the representation $\widetilde{\mathcal{F}}$, by the expression

$$\widetilde{\Delta} = \frac{1}{D}\,\partial_t\Big(\frac{G}{D}\partial_t\Big) + \frac{1}{G}\,\partial^2_{\theta\theta}.$$

We introduce the operator

$$\widetilde{L} = -\frac{1}{D}\,\partial_t\Big(\frac{G}{D}\partial_t\Big),$$

and the function

$$\widetilde{V} = \big(\widehat{\mathrm{Ric}}(\widetilde{N}) + |\widetilde{A}|^2\big),$$

which only depend on the variable t (and the parameter a). In the parametrization $\widetilde{\mathcal{F}}$, the Jacobi operator (12) of the immersion $\mathcal{C}_a \looparrowright \mathrm{Nil}(3)$ is given by the expression

$$\widetilde{J} = \widetilde{L} - \widetilde{V} - \frac{1}{G}\,\partial^2_{\theta\theta}.$$

We have the following lemma.

Lemma 4.1. With the above notations, the function $\widetilde{V}$ on the catenoid $\mathcal{C}_a$ is given by,

$$\widetilde{V} = \frac{2a^2}{f^4} + \frac{2(a^2+4)}{(4+f^2)^2}.$$

Furthermore, the function $G\widetilde{V}$ is equal to $\frac{a^2}{2}\,\frac{4+f^2}{f^2} + \frac{a^2+4}{2}\,\frac{f^2}{4+f^2}$ and satisfies the inequalities

$$(a^2+2)\sqrt{1-\frac{4}{(a^2+2)^2}} = a\sqrt{a^2+4} \le (G\widetilde{V})(a,t) \le a^2+2,$$

for all $a > 0$ and all $t \in \mathbb{R}$.

Proof. For the catenoid $\mathcal{C}_a$, the function f satisfies the differential equations (10) and (7) and we have $W = \frac{a}{f}$, where the function W is defined in (14). The Z-component γ of the unit normal $\widetilde{N}$ is a Jacobi function which only depends on t, hence $\widetilde{L}(\gamma) = \widetilde{V}\gamma$. Using (10) and (7) again, we can compute $\widetilde{L}(\gamma)$ and derive the formulas for $\widetilde{V}$ on the catenoid $\mathcal{C}_a$. The second assertion follows easily. ∎

Let $\widetilde{v}_\xi$ and $\widetilde{v}_\eta$ be the expressions of the Jacobi functions associated with the Killing fields ξ and η in the parametrization $\widetilde{\mathcal{F}}$. It follows from (15) that

$$\widetilde{v}_\xi(t,\theta) = \hat{g}\Big(\xi(\widetilde{\mathcal{F}}(t,\theta)), \widetilde{N}(t,\theta)\Big) = W\Big(-\cos(\theta+\varphi) + \frac{1}{2} f f_t \sin(\theta+\varphi)\Big),$$

and similarly for $\widetilde{v}_\eta$ (we have omitted the dependence on a). We introduce the smooth function $\psi(a,t)$ such that

$$\begin{cases} \cos\psi &=& (1+\frac{1}{4}f^2 f_t^2)^{-1/2}, \\ \sin\psi &=& \frac{1}{2} f f_t (1+\frac{1}{4}f^2 f_t^2)^{-1/2}, \\ \psi(a,0) &=& 0. \end{cases}$$

It follows immediately that

$$\begin{cases} \widetilde{v}_\xi(a,t,\theta) &=& -W_1(a,t)\cos\big(\theta+\varphi(a,t)+\psi(a,t)\big), \\ \widetilde{v}_\eta(a,t,\theta) &=& -W_1(a,t)\sin\big(\theta+\varphi(a,t)+\psi(a,t)\big), \quad \text{where} \\ W_1 &=& W(1+\frac{1}{4}f^2 f_t^2)^{1/2} = \frac{1}{f}\sqrt{\frac{4a^2+f^4}{4+f^2}}. \end{cases}$$

With the above notations, we have the following lemma.

Lemma 4.2. Let $\omega := \varphi + \psi$, a function of the variable t and the parameter a. Then,

(i) The functions

$$\begin{cases} w_1(a,t,\theta) := W_1(a,t)\cos(\omega(a,t))\cos\theta, \\ w_2(a,t,\theta) := W_1(a,t)\cos(\omega(a,t))\sin\theta, \\ w_3(a,t,\theta) := W_1(a,t)\sin(\omega(a,t))\cos\theta, \\ w_4(a,t,\theta) := W_1(a,t)\sin(\omega(a,t))\sin\theta, \end{cases} \tag{22}$$

are bounded Jacobi functions on $\mathcal{C}_a$, $\widetilde{J}(w_i) = 0$, *for* $1 \leq i \leq 4$.

(ii) The function $\omega(a,\cdot)$ is an odd function of t, satisfying $\omega(a,0) = 0$ and $\omega_t = 4f^2(f^4 + 4a^2)^{-1}$.

(iii) Let $\Omega(a) := \lim_{t\to+\infty} \omega(a,t)$. Then

$$\Omega(a) = 2a \int_a^\infty \frac{u^2\sqrt{u^2+4}}{(u^4+4a^2)\sqrt{u^2-a^2}}\, du.$$

(iv) For all $a > 0$, we have $\frac{\pi}{2} < \Omega(a) \leq \pi$ and the lower and upper bounds are achieved as limits when a tends respectively to zero and infinity.

Proof. *Assertion (i)* follows from the equalities $\widetilde{v}_\xi = -w_1 + w_4$, $\widetilde{v}_\eta = -w_2 - w_3$, and the fact that the operator $\widetilde{J}$ separates variables. *Assertion (ii)*. The computation of ω_t is straightforward. To prove *Assertion (iii)*, we use the fact that f_t is positive for positive t and can be computed from (10), namely,

$$f_t = \frac{2\sqrt{f^2-a^2}}{a\sqrt{f^2+4}}.$$

We write

$$\omega_t = \frac{2af^2\sqrt{f^2+4}}{(f^4+4a^2)\sqrt{f^2-a^2}}\, f_t$$

for $t > 0$, and we compute the integral $\int_0^t \omega_\tau \, \mathrm{d}\tau$ by making the change of variables $u = f(t)$. *Assertion (iv).* Assume by contradiction that $\Omega(a_0) > \pi$ for some a_0. There would then exist a value t_0 such that $\omega(a_0, t_0) = \pi$. The function w_3 in (22) would then vanish on the circles $\widetilde{\mathcal{F}}(a_0, \{0\}, [0, 2\pi])$ and $\widetilde{\mathcal{F}}(a_0, \{t_0\}, [0, 2\pi])$. Because this function is a Jacobi function, this would contradict Assertion (i) in Theorem 3.1. The fact that $\frac{\pi}{2} < \Omega(a)$ follows from a direct estimate of the integral, [11]. Indeed, making the change of variables $u = av$, we get $\Omega(a) = \Omega_1(4/a^2)$ where

$$\Omega_1(b) = 2\int_1^\infty \frac{v^2\sqrt{v^2+b}}{(v^4+b)\sqrt{v^2-1}} \, \mathrm{d}v = \int_1^\infty \frac{\sqrt{u}}{\sqrt{u-1}} \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u > I(b),$$

where

$$I(b) = \int_1^\infty \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u$$

and we claim that $I(b) > \frac{\pi}{2}$. To prove this last assertion, we consider two cases, $0 \le b \le 1$ and $b > 1$.

- We have $I(0) = 2$ and for $0 < b \le 1$,

$$I(b) > \int_1^\infty \frac{\sqrt{u}}{u^2+1} \, \mathrm{d}u \ge 2\int_1^\infty \frac{1}{v^2+1} \, \mathrm{d}v = \frac{\pi}{2}.$$

- When $b \ge 1$, we can write

$$I(b) = \int_1^b \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u + \int_b^\infty \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u$$

and estimate the integrals on the right-hand side separately.

$$\begin{aligned} \int_b^\infty \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u &> \int_b^\infty \frac{\sqrt{u}}{u^2+b} \, \mathrm{d}u = 2\int_{\sqrt{b}}^\infty \frac{v^2}{v^4+b} \, \mathrm{d}v \\ &\ge 2\int_{\sqrt{b}}^\infty \frac{1}{v^2+1} \, \mathrm{d}v = \pi - 2\arctan\sqrt{b}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_1^b \frac{\sqrt{u+b}}{u^2+b} \, \mathrm{d}u &\ge \sqrt{b}\int_1^b \frac{1}{u^2+b} \, \mathrm{d}u = \int_{\frac{1}{\sqrt{b}}}^{\sqrt{b}} \frac{dv}{v^2+1} \\ &= \arctan(\sqrt{b}) - \arctan(\tfrac{1}{\sqrt{b}}) \\ &= 2\arctan(\sqrt{b}) - \tfrac{\pi}{2}. \end{aligned}$$

It follows that $I(b) > \frac{\pi}{2}$ and hence that $\Omega(a) > \frac{\pi}{2}$. Recall that $\Omega(a) \leq \pi$ for geometric reasons. Clearly, when b tends to zero, $\Omega_1(b)$ tends to π, and hence $\Omega(a)$ tends to π when a tends to infinity. Making the change of variable $u = \sqrt{b}\, v$, one can show that $I(b)$ tends to $\frac{\pi}{2}$ when b tends to infinity. On the other hand, it is easy to see that $\Omega_1(b) - I(b)$ tends to 0 when b tends to infinity. It follows that $\Omega(a)$ tends to $\frac{\pi}{2}$ when a tends to zero. This finishes the proof of the lemma. ∎

Lemma 4.3. For $k \in \mathbb{N}$, consider the operator $\widetilde{L_k} := \widetilde{L} + \frac{k^2}{G} - \widetilde{V}$ in $L^2([-r,r], D\,\mathrm{d}t)$, with Dirichlet boundary conditions. Then,

(i) For any $r > 0$, the operator $\widetilde{L_k}$ has at most one negative eigenvalue (with multiplicity one).

(ii) For all $k \geq \sqrt{a^2+2}$ and $r > 0$, the operator $\widetilde{L_k}$ is positive.

Proof. *Assertion (i).* Recall that the eigenvalues of a Sturm-Liouville problem with Dirichlet boundary conditions are always simple. If $\widetilde{L_k}$ had at least two negative eigenvalues, we would have an eigenfunction v of $\widetilde{L_k}$ associated with a negative eigenvalue and having one zero in $(-r,r)$. The function $v\cos(k\theta)$ would be an eigenfunction of the Jacobi operator $\widetilde{J}$ with negative eigenvalue, vanishing on the boundary of an annulus contained in $\mathcal{C}_{a,+}$ or in $\mathcal{C}_{a,-}$. This would contradict Assertion (i) in Theorem 3.1. *Assertion (ii).* By Lemma 4.1, $G\widetilde{V} \leq a^2 + 2$ and the second assertion follows from the positivity of the operator $\widetilde{L}$ in $L^2([-r,r], D\,\mathrm{d}t)$. ∎

Theorem 4.1. *Consider the catenoids $\{\mathcal{C}_a, a > 0\}$ in* Nil(3)*.*

(i) For all $a > 0$, the catenoid $\mathcal{C}_a$ has finite Morse index at least equal to 3 *and at most equal to* $1+2[\sqrt{a^2+2}]$*, where $[x]$ is the integer part of x. In particular, the index of $\mathcal{C}_a$ is equal to* 3 *for a close to zero.*

(ii) When a tends to infinity, the index of $\mathcal{C}_a$ grows at least like $\sqrt{3}\,a$. In particular, it tends to infinity when a tends to infinity.

Proof. Fourier analysis and Lemma 4.3(i) show that the Morse index of $\mathcal{C}_a$ is equal to 1 plus twice the number of positive integers k such that the operator $\widetilde{L}_k$ has a negative eigenvalue. *Assertion (i).* The fact that the index of $\mathcal{C}_a$ is at most $1 + 2[\sqrt{a^2+2}]$ follows from Lemma 4.3(ii). By Lemma 4.2(iv), for any $a > 0$, $\Omega(a) > \pi/2$. Since $\omega(0) = 0$, there exists some $r_a > 0$ such that $\omega(a, r_a) = \frac{\pi}{2}$. The functions w_1, w_2 of Lemma 4.2 (i) are Jacobi functions which vanish on the boundary of the domain $\mathcal{F}(a, (-r_a, r_a), [0, 2\pi])$. It follows easily that the index of the operator $\widetilde{L}_1$ is equal to 1 and hence the index of the catenoid $\mathcal{C}_a$ is at least 3. *Assertion (ii).* To determine whether the index of $\widetilde{L}_k$ is 1 or 0, consider the associated quadratic form on functions $\phi \in C^1_0(\mathbb{R})$,

$$Q_k(\phi) = \int_{-\infty}^{\infty} \{\frac{G}{D}\phi_t^2 + (k^2 - G\widetilde{V})\frac{D}{G}\phi^2\}\, dt.$$

Write $\phi(t) = \psi\big(s(t)\big)$, with

$$s_t = \frac{D}{G} = \frac{4}{a(4+f^2)}, \quad s(0) = 0.$$

The function s is a diffeomorphism from $\mathbb{R}$ onto $\big(-S(a), S(a)\big)$, where

$$S(a) = 2\int_0^{\infty} \frac{f_t\, dt}{\sqrt{(4+f^2)(f^2-a^2)}} = \frac{2}{a}\int_1^{\infty} \frac{du}{\sqrt{(u^2+\frac{4}{a^2})(u^2-1)}}. \qquad (23)$$

It follows that

$$Q_k(\phi) = \int_{-S(a)}^{S(a)} \{\psi_s^2 + \big(k^2 - U(s)\big)\psi^2\}\, ds,$$

where the function U is defined by $U\big(s(t)\big) = (G\widetilde{V})(t)$. Choose the function ψ to be $\psi_0(s) = \cos\big(\frac{\pi s}{2S(a)}\big)$ and let ϕ_0 be the corresponding function. Using Lemma 4.1, one finds that $Q_k(\phi_0) < 0$, *i.e.* that the index of $\widetilde{L}_k$ is 1, as soon as

$$k^2 < (a^2+2)\sqrt{1 - \big(\frac{2}{a^2+2}\big)^2} - \big(\frac{\pi}{2S(a)}\big)^2. \qquad (24)$$

By (23), $S(a) = \frac{\pi}{a} - \frac{4}{a^3}J(a)$, where the function $J(a)$ is given by

$$J(a) = 2\int_1^\infty \frac{dv}{v(v + \sqrt{v^2 + \frac{4}{a^2}})\sqrt{(v^2-1)(v^2 + \frac{4}{a^2})}}.$$

This function tends to $\frac{\pi}{4}$ when a tends to infinity and hence the right-hand side of (24) is equivalent to $\frac{3a^2}{4}$ when a tends to infinity. This proves the second assertion. ■

Remarks.

(i) Given $a > 0$, there is a simple criterion to decide whether the operator $\widetilde{L}_k$ has a negative eigenvalue in the interval $[-r, r]$ (with Dirichlet boundary conditions). Let u_k be the solution of the Cauchy problem $\widetilde{L}_k(u) = 0$, $u(0) = 1$ and $u_t(0) = 0$. If u_k has a zero in the interval $(0, r)$, then $\widetilde{L}_k$ has a negative eigenvalue in $[-r, r]$; if u_k does not vanish in the interval $(0, r)$, then $\widetilde{L}_k(u) \geq 0$ in $[-r, r]$.

(ii) Using the fact that the metric $\hat{g}$ on Nil(3) is left-invariant, one can easily express the associated Levi-Civita connexion and curvature tensors in the orthonormal basis $\{X, Y, Z\}$ of left-invariant vector fields. In particular, given a unit vector $N = \alpha X + \beta Y + \gamma Z$, we find the following formula for the Ricci curvature,

$$\widehat{\mathrm{Ric}}(N, N) = -\frac{1}{2} + \gamma^2.$$

(iii) Using the preceding remark, we can write the Jacobi operator on an orientable minimal surface in Nil(3) as

$$J = -\Delta + \frac{1}{2} - \gamma^2 - |A|^2,$$

where γ is the Z-component of the unit normal to the surface. Using the fact that the scalar curvature of Nil(3) is $-\frac{1}{4}$, we also have the formula

$$J = -\Delta + \frac{1}{4} + K_M - \frac{1}{2}|A|^2,$$

where K_M is the Gauss curvature of the surface M.

(iv) Using Lemma 4.1 and the second remark, we deduce the following expression for the second fundamental form of the catenoid $\mathcal{C}_a$ in Nil(3),

$$|A|^2 = \frac{1}{2} - \frac{4}{f^2} + \frac{4(a^2+4)}{f^2(f^2+4)} + \frac{2(a^2+4)}{(f^2+4)^2}.$$

This shows that the norm squared of the second fundamental form tends to $\frac{1}{2}$ uniformly at infinity. This is in contrast with the situation in $\mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^3$.

5 Catenoids in higher dimensions

In this section, we study the rotationally symmetric stable domains on the higher dimensional catenoids. Let Nil$(2n+1)$ be the $(2n+1)$-dimensional Heisenberg group. As in Section 2, we use the exponential coordinates and choose the left-invariant metric $\hat{g}$ to be such that the left-invariant vector fields $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z\}$ form an orthonormal basis, where

$$\begin{cases} X_i(x,y,z) &= \partial_{x_i} - \frac{1}{2} y_i \partial_z, \ 1 \le i \le n, \\ Y_i(x,y,z) &= \partial_{y_i} + \frac{1}{2} x_i \partial_z, \ 1 \le i \le n, \\ Z(x,y,z) &= \partial_z. \end{cases}$$

We look for hypersurfaces of revolution of the form

$$\mathcal{F} : \begin{cases} \mathsf{R} \times S^{2n-1} \to \mathrm{Nil}(2n+1), \\ (t,\theta) \mapsto \mathcal{F}(t,\theta) = \big(f(t)\theta, t\big), \end{cases}$$

where f is a positive function of t. If follows from [7, 8] that the hypersurface $\mathcal{F}$ is minimal if and only if f satisfies the second order differential equation,

$$f(4+f^2) f_{tt} = 4(2n-1)(1+f_t^2) + (2n-2) f^2 f_t^2.$$

As in Section 2.3, one can show that for $a > 0$, there is a unique maximal solution $f(a,t)$ such that $f(a,0) = a$ and $f_t(a,0) = 0$. This is an even function of t defined on the interval $(-T(a), T(a))$, where $T(a)$ is finite when $n \geq 2$. As in dimension 3 ($n = 1$), the above differential equation admits a first integral,

$$f^{2n-1}\left(1 + f_t^2 + f^2 f_t^2\right)^{-1/2} \equiv a^{2n-1}.$$

As in (14), we let $W := \left(1 + f_t^2 + f^2 f_t^2\right)^{-1/2}$. We also use the following notations,

$$\begin{cases} \mathcal{C}_a & = & \mathcal{F}\Big(a, \big(-T(a), T(a)\big), S^{2n-1}\Big), \\ \mathcal{C}_{a,+} & = & \mathcal{F}\Big(a, \big(0, T(a)\big), S^{2n-1}\Big), \\ \mathcal{C}_{a,-} & = & \mathcal{F}\Big(a, \big(-T(a), 0\big), S^{2n-1}\Big), \\ \mathcal{D}_a(r,s) & = & \mathcal{F}\Big(a, (r,s), S^{2n-1}\Big). \end{cases}$$

We can now state the following result.

Theorem 5.1. *Assume that $n \geq 2$ and $a > 0$.*

(i) The half-catenoids $\mathcal{C}_{a,\pm}$ are r-stable.

(ii) There exists some $z(a) > 0$ such that the domain $\mathcal{D}_a(-z(a), z(a))$ is stable-unstable. In particular, the catenoid $\mathcal{C}_a$ has index at least 1.

(iii) There exists some $\ell(a) > 0$ such that the domain $\mathcal{D}_a(-\ell(a), T(a))$ is r-stable.

(iv) For any $r > \ell(a)$, there exists some $s > 0$ such that the domain $\mathcal{D}_a(-r, s)$ is stable-unstable.

Proof. The proof relies on the expressions of two explicit Jacobi functions on $\mathcal{C}_a$, namely the Jacobi functions $v(a,t) = \hat{g}(N, Z)$, and $e(a,t) = -\hat{g}(\mathcal{F}_a, N)$, where N is a unit normal to $\mathcal{C}_a$, and $\mathcal{F}_a$ is the variation field along $\mathcal{F}$ when the parameter a varies. As in dimension 2, we have $v(a,t) = W(a,t) f_t(a,t)$ and Assertion (i) follows immediately from the fact that $f_t(a,t) > 0$ for $t > 0$.

To prove the other Assertions, notice that $e(a,t)$ is an even function of t which can be studied using the inverse function $\phi(a,\tau)$ of the function $f(a,\cdot):[0,\infty)\to[a,T(a))$. It turns out that

$$\phi(a,\tau)=\frac{a^{2n-1}}{2}\int_a^{\tau}\sqrt{\frac{u^2+4}{u^{4n-2}-a^{4n-2}}}\,\mathrm{d}u.$$

This formula shows that $\phi(a,\tau)$ has a finite limit $T(a)$ when τ tends to infinity and that its derivative $\phi_a(a,\tau)$ has a positive finite limit when τ tends to infinity. We now use the same method as in the proof of Theorem 3.1. Assertion (ii), follows from the fact that $e(a,0)=1$ and that $e(a,t)$ takes negative values near infinity. For the proofs of Assertions (iii) and (iv), we use the fact that in higher dimensions ($n\geq 2$), both $\phi(a,\tau)$ and $\phi_a(a,\tau)$ have finite limits at infinity, so that the higher dimensional case differs from the case in which $n=1$. ■

Remark. Theorem 3.1(iii) tells us that the half-catenoids $\mathcal{C}_{a,\pm}$ in Nil(3) are stable-unstable, *i.e.* that they satisfy the *Lindeloef's property* as defined in [2, 3]. Theorem 5.1(iii) and (iv) tell us that catenoids in Nil($2n$+1), $n\geq 2$, do not satisfy Lindeloef's property. As for catenoids in $\mathbb{R}^{n+2}$ and $\mathbb{H}^n\times\mathsf{R}$, $n\geq 2$, this is related to the fact that these catenoids have finite height.

References

[1] J. L. Barbosa, J. Gomes and A. da Silveira, *Foliation of* 3*-dimensional space forms by surfaces with constant mean curvature,* Bol. Soc. Bras. Mat. **38** (1987), 1-12.

[2] P. Bérard and R. Sa Earp, *Minimal hypersurfaces in* $\mathbb{H}^n\times\mathbb{R}$*, total curvature and index*, arXiv:0808.3838v3.

[3] P. Bérard and R. Sa Earp, *Lindelöf's theorem for hyperbolic catenoids,* Proc. Amer. Math. Soc. **138** (2010), 3657–3669.

[4] B. Daniel, The Gauss map of minimal surfaces in the Heisenberg group, Int. Math. Res. Not. IMRN **3** (2011), 674–695.

[5] I. Fernández and P. Mira, *Holomorphic quadratic differential and the Bernstein problem,* Trans. Amer. Math. Soc. **361** (2009), 5737–5752.

[6] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds with non-negative scalar curvature,* Comm. Pure Applied Math. **33** (1980), 199–211.

[7] C. Figueroa, *Geometria das subvariedades do grupo de Heisenberg,* PhD Thesis UNICAMP, Campinas (Brazil) 1996.

[8] C. Figueroa, F. Mercuri and R. Pedrosa, *Invariant surfaces of the Heisenberg groups,* Ann. Mat. Pura Appl. **177** (1999),173–194.

[9] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. 1.* Interscience Publishers, New York-London, 1963.

[10] B. Lawson Jr., *Lectures on minimal submanifolds. Vol I,* Math. Lectures Series, vol 9, Publish or Perish Inc., Wilmington, Del. 1980.

[11] Y. Lima, *Private communication,* 2010.

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