



Characterizations of linear Weingarten spacelike hypersurfaces in locally symmetric Lorentz spaces

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Abstract

We deal with complete linear Weingarten hypersurfaces immersed in a locally symmetric Lorentz space, whose sectional curvatures are supposed to obey some standard controls. In this setting, under suitable boundedness on the norm of the traceless part of the second fundamental form, we are able to show that such a hypersurface must be either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

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1 Introduction and statements of the main results

In 1970, Calabi [8] proved that the only complete maximal surfaces (that is, with zero mean curvature) in the 3-dimensional Lorentz-Minkowski space \mathbb{L}^3 are the spacelike planes. Equivalently, he showed that the only entire maximal graphs in \mathbb{L}^3 are the spacelike planes. This result has been the origin of a wide productive branch of research in differential geometry. Later, this result has been generalized to general dimension by Cheng and Yau [13] and, afterwards, also to complete spacelike hypersurfaces of constant mean curvature in \mathbb{L}^{n+1} under some additional geometric assumptions (see, for instance, [1], [3], [21] and [23]).

Many authors have studied similar problems in other Lorentzian ambient spaces. As for the case of the de Sitter space \mathbb{S}_1^{n+1} , which is the standard simply connected Lorentz space form of positive constant sectional curvature 1, Goddard [16] conjectured that every complete spacelike hypersurface with constant mean curvature H in \mathbb{S}_1^{n+1} should be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. For instance, in [2] Akutagawa showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Later on, Montiel [19] solved Goddard's problem in the compact case proving that the only closed spacelike hypersurfaces in \mathbb{S}_1^{n+1} with constant mean curvature are the totally umbilical hypersurfaces.

Another Goddard-like problem is to characterize spacelike hypersurfaces immersed in a Lorentz space with constant scalar curvature. An interesting result due to Cheng and Ishikawa [12] states that the totally umbilical round spheres are the only compact spacelike hypersurfaces in \mathbb{S}_1^{n+1} with constant normalized scalar curvature $R < 1$. More recently, many other authors, such as Aledo and Alías [4], Brasil, Colares and Palmas [7], Carmargo, Chaves and Sousa Jr. [9], Caminha [10], and Hu, Scherfner and

Zhai [17], have also worked on related problems.

It is natural to study the geometry of spacelike hypersurfaces immersed in more general Lorentz spaces, since they have important meaning in the relativity theory and are of substantial interest from geometric and mathematical cosmology points of view. In such direction, for constants c_1 and c_2 , Choi et al. [15, 22] introduced the class of $(n + 1)$ -dimensional Lorentz spaces L_1^{n+1} which satisfy the following two conditions (here, K denotes the sectional curvature of L_1^{n+1}):

$$K(u, v) = -\frac{c_1}{n}, \quad (1)$$

for any spacelike vector u and timelike vector v ; and

$$K(u, v) \geq c_2, \quad (2)$$

for any spacelike vectors u and v .

We observe that Lorentz space forms $L_1^{n+1}(c)$ satisfy conditions (1) and (2) for $-\frac{c_1}{n} = c_2 = c$. Moreover, there are several examples of Lorentz spaces which are not Lorentz space forms and satisfy (1) and (2). For instance, semi-Riemannian product manifolds $\mathbb{H}_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, where $c_1 > 0$, and $\mathbb{R}_1^k \times \mathbb{S}^{n+1-k}$. In particular, $\mathbb{R}_1^1 \times \mathbb{S}^n$ is a so-called *Einstein Static Universe*. Also the so-called *Robertson-Walker spacetime* $N(c, f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of \mathbb{R}_1^1 , f is a positive smooth function defined on the interval I and $N^3(c)$ is a 3-dimensional Riemannian manifold of constant curvature c . $N(c, f)$ also satisfies conditions (1) and (2) for an appropriate choice of the function f (for more details, see [15] and [22]).

Here, our purpose is to study the geometry of complete *linear Weingarten* spacelike hypersurfaces, that is, complete spacelike hypersurfaces whose mean curvature H and normalized scalar curvature R satisfy

$$R = aH + b,$$

for some $a, b \in \mathbb{R}$. In this setting, by exploring the ellipticity of a suitable Cheng-Yau modified operator (see Section 3), we are able to establish

characterization theorems concerning to such spacelike hypersurfaces immersed in a locally symmetric Lorentz space L_1^{n+1} , which is supposed to obey conditions (1) and (2). We recall that a Lorentz space L_1^{n+1} is said *locally symmetric* if all the covariant derivative components $\bar{R}_{ABCD;E}$ of the curvature tensor of L_1^{n+1} vanish identically.

In order to state our result, we will need some basic facts. Denoting by \bar{R}_{AB} the components of the Ricci tensor of L_1^{n+1} satisfying condition (1), the scalar curvature \bar{R} of L_1^{n+1} is given by

$$\bar{R} = \sum_{A=1}^{n+1} \varepsilon_A \bar{R}_{AA} = \sum_{i,j=1}^n \bar{R}_{ijji} - 2 \sum_{i=1}^n \bar{R}_{(n+1)ii(n+1)} = \sum_{i,j=1}^n \bar{R}_{ijji} + 2c_1.$$

Moreover, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant. Consequently, $\sum_{i,j} \bar{R}_{ijji}$ is a constant naturally attached to a locally symmetric Lorentz space satisfying condition (1).

Now, we are in position to present our results. In what follows, for the sake of simplicity of notation, $\bar{\mathcal{R}} = \frac{1}{n(n-1)} \sum_{i,j} \bar{R}_{ijji}$ and Φ stands for the traceless part of the second fundamental form of the spacelike hypersurface M^n .

Theorem 1.1. Let L_1^{n+1} be a locally symmetric Lorentz space satisfying conditions (1) and (2), with $n \geq 3$ and $c = \frac{c_1}{n} + 2c_2 > 0$. Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1} , such that $R = aH + b$ with $b < \bar{\mathcal{R}}$. Suppose that $\bar{\mathcal{R}} - c < R < \bar{\mathcal{R}} - \frac{2c}{n}$. If H can attain the maximum on M^n and

$$\sup_M |\Phi|^2 \leq \frac{(n-1)(\bar{\mathcal{R}} - R - c)^2}{(n-2) \left(\bar{\mathcal{R}} - R - \frac{2c}{n} \right)}, \quad (3)$$

then either

i. $|\Phi| \equiv 0$ and M^n is totally umbilical;

ii. or $|\Phi|^2 \equiv \frac{(n-1)(\bar{\mathcal{R}} - R - c)^2}{(n-2) \left(\bar{\mathcal{R}} - R - \frac{2c}{n} \right)}$ and M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Related to the compact case, when the ambient space is a locally symmetric Einstein spacetime, we obtain the following:

Theorem 1.2. Let L_1^{n+1} be a locally symmetric Einstein spacetime satisfying conditions (1) and (2), with $n \geq 3$ and $c = \frac{c_1}{n} + 2c_2 > 0$. Let M^n be a compact linear Weingarten spacelike hypersurface immersed in L_1^{n+1} , such that $R = aH + b$ with $(n-1)a^2 + 4n(\overline{\mathcal{R}} - b) \geq 0$ and $b \neq \overline{\mathcal{R}}$. Suppose that $\overline{\mathcal{R}} - c < R < \overline{\mathcal{R}} - \frac{2c}{n}$. If

$$\sup_M |\Phi|^2 < \frac{(n-1)(\overline{\mathcal{R}} - R - c)^2}{(n-2)\left(\overline{\mathcal{R}} - R - \frac{2c}{n}\right)}, \quad (4)$$

then $|\Phi| \equiv 0$ and M^n is totally umbilical.

Proceeding, we also get the following:

Theorem 1.3. Let L_1^{n+1} be a locally symmetric Lorentz space satisfying conditions (1) and (2), with $n \geq 3$ and $c = \frac{c_1}{n} + 2c_2$. Let M^n be a complete linear Weingarten spacelike hypersurface immersed in L_1^{n+1} , such that $R = aH + b$ with $b < \overline{\mathcal{R}}$. Suppose that either $R < \overline{\mathcal{R}} - c$, if $c > 0$, or $R < \overline{\mathcal{R}} - \frac{2c}{n}$, if $c \leq 0$. If H can attain the maximum on M^n and

$$\inf_M |\Phi|^2 \geq \frac{(n-1)(\overline{\mathcal{R}} - R - c)^2}{(n-2)\left(\overline{\mathcal{R}} - R - \frac{2c}{n}\right)}, \quad (5)$$

then $|\Phi|^2 \equiv \frac{(n-1)(\overline{\mathcal{R}} - R - c)^2}{(n-2)\left(\overline{\mathcal{R}} - R - \frac{2c}{n}\right)}$ and M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

The proofs of Theorems 1.1, 1.2 and 1.3 are given in Section 4.

2 Preliminaries

We recall that an $(n+1)$ -dimensional Lorentz space L_1^{n+1} is a semi-Riemannian manifold of index 1 and that a hypersurface M^n immersed in

L_1^{n+1} is said to be *spacelike* if the metric on M^n induced from that of the ambient space L_1^{n+1} is positive definite. In this setting, we choose a local field of semi-Riemannian orthonormal frame $\{e_A\}_{1 \leq A \leq n+1}$ in L_1^{n+1} , with dual coframe $\{\omega_A\}_{1 \leq A \leq n+1}$, such that, at each point of M^n , e_1, \dots, e_n are tangent to M^n and e_{n+1} is normal to M^n . We will use the following convention for the indices:

$$1 \leq A, B, C, \dots \leq n+1, \quad 1 \leq i, j, k, \dots \leq n.$$

Denoting by $\{\omega_{AB}\}$ the connection forms of L_1^{n+1} , we have that the structure equations of L_1^{n+1} are given by:

$$d\omega_A = - \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \varepsilon_i = 1, \varepsilon_{n+1} = -1, \quad (6)$$

$$d\omega_{AB} = - \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D. \quad (7)$$

Here, \bar{R}_{ABCD} , \bar{R}_{CD} and \bar{R} denote respectively the semi-Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Lorentz space L_1^{n+1} . In this setting, we have

$$\bar{R}_{CD} = \sum_B \varepsilon_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \varepsilon_A \bar{R}_{AA}.$$

Moreover, the components $\bar{R}_{ABCD;E}$ of the covariant derivative of the semi-Riemannian curvature tensor of L_1^{n+1} are defined by

$$\begin{aligned} \sum_E \varepsilon_E \bar{R}_{ABCD;E} \omega_E &= d\bar{R}_{ABCD} - \sum_E \varepsilon_E (\bar{R}_{EBCD} \omega_{EA} \\ &\quad + \bar{R}_{AECD} \omega_{EB} + \bar{R}_{ABED} \omega_{EC} + \bar{R}_{ABCE} \omega_{ED}). \end{aligned}$$

Next, we restrict all the tensors to the spacelike hypersurface M^n in L_1^{n+1} . First of all, $\omega_{n+1} = 0$ on M^n , so $\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0$. Consequently, by *Cartan's Lemma* [11], there are h_{ij} such that

$$\omega_{(n+1)i} = \sum_j h_{ij} \omega_j \quad \text{and} \quad h_{ij} = h_{ji}. \quad (8)$$

This gives the second fundamental form of M^n , $h = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}$, and its square length $S = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n} \sum_i h_{ii}$.

The connection forms $\{\omega_{ij}\}$ of M^n are characterized by the structure equations of M^n :

$$d\omega_i = - \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad (9)$$

$$d\omega_{ij} = - \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad (10)$$

where R_{ijkl} are the components of the curvature tensor of M^n .

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}). \quad (11)$$

The components R_{ij} of the Ricci tensor and the scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \bar{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj} \quad (12)$$

and

$$n(n-1)R = \sum_{j,k} \bar{R}_{kjjk} - n^2H^2 + S. \quad (13)$$

The first covariant derivatives h_{ijk} of h_{ij} satisfy

$$\sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ik} \omega_{kj} - \sum_k h_{jk} \omega_{ki}. \quad (14)$$

Then, by exterior differentiation of (8), we obtain the *Codazzi equation*

$$h_{ijk} - h_{ikj} = \bar{R}_{(n+1)ijk}. \quad (15)$$

Similarly, the second covariant derivatives h_{ijkl} of h_{ij} are given by

$$\sum_l h_{ijkl} \omega_l = dh_{ijk} - \sum_l h_{lj} \omega_{li} - \sum_l h_{il} \omega_{lj} - \sum_l h_{ijl} \omega_{lk}. \quad (16)$$

By exterior differentiation of (14), we can get the following *Ricci formula*

$$h_{ijkl} - h_{ijlk} = - \sum_m h_{im} R_{mjkl} - \sum_m h_{jm} R_{mikl}. \quad (17)$$

Restricting the covariant derivative $\bar{R}_{ABCD;E}$ of \bar{R}_{ABCD} on M^n , then $\bar{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned} \bar{R}_{(n+1)ijk;l} &= \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k} h_{jl} \\ &\quad + \bar{R}_{(n+1)ij(n+1)} h_{kl} + \sum_m \bar{R}_{mijk} h_{ml}, \end{aligned} \quad (18)$$

where $\bar{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\begin{aligned} \sum_l \bar{R}_{(n+1)ijkl} \omega_l &= d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk} \omega_{li} \\ &\quad - \sum_l \bar{R}_{(n+1)ilk} \omega_{lj} - \sum_l \bar{R}_{(n+1)ijl} \omega_{lk}. \end{aligned}$$

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From (15), (17) and (18), after a straightforward computation we obtain

$$\begin{aligned} \Delta h_{ij} &= (nH)_{ij} - nH \sum_l h_{il} h_{lj} + S h_{ij} \\ &\quad + \sum_k (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) \\ &\quad - \sum_k (h_{kk} \bar{R}_{(n+1)ij(n+1)} + h_{ij} \bar{R}_{(n+1)k(n+1)k}) \\ &\quad - \sum_{k,l} (2h_{kl} \bar{R}_{lij} + h_{jl} \bar{R}_{lkik} + h_{il} \bar{R}_{lkjk}). \end{aligned} \quad (19)$$

Since $\Delta S = 2 \left(\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} \Delta h_{ij} \right)$, from (19) we get

$$\begin{aligned}
 \frac{1}{2}\Delta S &= S^2 + \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} (nH)_{ij} h_{ij} & (20) \\
 &+ \sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} \\
 &- \left(\sum_{i,j} nH h_{ij} \bar{R}_{(n+1)ij(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right) \\
 &- 2 \sum_{i,j,k,l} (h_{kl} h_{ij} \bar{R}_{lijk} + h_{il} h_{ij} \bar{R}_{lkjk}) - nH \sum_{i,j,l} h_{il} h_{lj} h_{ij}.
 \end{aligned}$$

Now, let $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH \delta_{ij} - h_{ij}.$$

Following Cheng-Yau [14], we introduce a operator \square associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH \delta_{ij} - h_{ij}) f_{ij}. \quad (21)$$

Setting $f = nH$ in (21) and taking a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (13) we obtain the following

$$\begin{aligned}
 \square(nH) &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} & (22) \\
 &= \frac{1}{2}\Delta S - n^2 |\nabla H|^2 - \sum_i \lambda_i (nH)_{ii} \\
 &\quad + \frac{1}{2}\Delta \left(\sum_{i,j} \bar{R}_{ijji} - n(n-1)R \right).
 \end{aligned}$$

3 Some auxiliary results

In order to prove our results, we will need some auxiliary lemmas. The first one is a classic algebraic lemma due to M. Okumura in [20], and

completed with the equality case proved in [5] by H. Alencar and M. do Carmo.

Lemma 3.1. *Let μ_1, \dots, μ_n be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, with $\beta \geq 0$. Then,*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3, \quad (23)$$

and equality holds if, and only if, at least $(n-1)$ of the numbers μ_i are equal.

Now, we present our second auxiliary lemma. Following the steps of the proof of Lemma 2.1 of [18], we get

Lemma 3.2. *Let M^n be a linear Weingarten spacelike hypersurface immersed in a locally symmetric Lorentz space L_1^{n+1} satisfying condition (1), such that $R = aH + b$ with $b \neq \bar{\mathcal{R}}$. Suppose that*

$$(n-1)a^2 + 4n(\bar{\mathcal{R}} - b) \geq 0. \quad (24)$$

Then,

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2. \quad (25)$$

Moreover, if the inequality (24) is strict and the equality holds in (25) on M^n , then H is constant on M^n .

Proof. Since we are supposing that $R = aH + b$ and the term $\bar{\mathcal{R}}$ is constant, from equation (13) we get

$$2 \sum_{i,j} h_{ij} h_{ijk} = (2n^2 H + n(n-1)a) (H)_k.$$

Thus,

$$4 \sum_k \left(\sum_{i,j} h_{ij} h_{ijk} \right)^2 = (2n^2 H + n(n-1)a)^2 |\nabla H|^2.$$

Consequently, using Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}
 4S \sum_{i,j,k} h_{ijk}^2 &= 4 \left(\sum_{i,j} h_{ij}^2 \right) \left(\sum_{i,j,k} h_{ijk}^2 \right) \\
 &\geq 4 \sum_k \left(\sum_{i,j} h_{ij} h_{ijk} \right)^2 \\
 &= (2n^2H + n(n-1)a)^2 |\nabla H|^2.
 \end{aligned} \tag{26}$$

On the other hand, since $R = aH + b$, using again equation (13) we easily verify that

$$\begin{aligned}
 (2n^2H + n(n-1)a)^2 &= 4n^3(n-1)\overline{\mathcal{R}} - 4n^3(n-1)b \\
 &\quad + n^2(n-1)^2a^2 + 4n^2S.
 \end{aligned} \tag{27}$$

Consequently, from (24), (26) and (27), we get

$$S \sum_{i,j,k} h_{ijk}^2 \geq n^2S |\nabla H|^2.$$

Therefore, we obtain either $S = 0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2 |\nabla H|^2$ or $\sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2$. Moreover, if the inequality (24) is strict, from (27) we get that

$$(2n^2H + n(n-1)a)^2 > 4n^2S.$$

Consequently, if $\sum_{i,j,k} h_{ijk}^2 = n^2 |\nabla H|^2$ holds on M^n , from (26) we conclude that $\nabla H = 0$ on M^n and, hence, H is constant on M^n . ■

Now, we consider the Cheng-Yau's modified operator

$$L = \square + \frac{n-1}{2} a \Delta. \tag{28}$$

Related to such operator, we have the following sufficient criterion of ellipticity.

Lemma 3.3. *Let M^n be a linear Weingarten spacelike hypersurface immersed in a locally symmetric Lorentz space L_1^{n+1} satisfying condition (1), such that $R = aH + b$ with $b < \overline{\mathcal{R}}$. Then, H does not vanish on M^n and L is elliptic.*

Proof. From equation (13), since $R = aH + b$ with $b < \bar{\mathcal{R}}$, we easily see that H cannot vanish on M^n and, by choosing the appropriate Gauss mapping, we may assume that $H > 0$ on M^n .

Let us consider the case that $a = 0$. Since $R = b < \bar{\mathcal{R}}$, from equation (13) if we choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, we have that $\sum_{i < j} \lambda_i \lambda_j > 0$. Consequently,

$$n^2 H^2 = \sum_i \lambda_i^2 + 2 \sum_{i < j} \lambda_i \lambda_j > \lambda_i^2$$

for every $i = 1, \dots, n$ and, hence, we have that $nH - \lambda_i > 0$ for every i . Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From equation (13) we get that

$$a = \frac{1}{n(n-1)H} (S - n^2 H^2 + n(n-1)(\bar{\mathcal{R}} - b)).$$

Consequently, for every $i = 1, \dots, n$, with a straightforward algebraic computation we verify that

$$\begin{aligned} nH - \lambda_i + \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH} (S - n^2 H^2 + n(n-1)(\bar{\mathcal{R}} - b)) \\ &= \frac{1}{2nH} \left(\sum_{j \neq i} \lambda_j^2 + \left(\sum_{j \neq i} \lambda_j \right)^2 + n(n-1)(\bar{\mathcal{R}} - b) \right). \end{aligned}$$

Therefore, since $b < \bar{\mathcal{R}}$, we also conclude in this case that L is elliptic. ■

4 Proofs of Theorems 1.1, 1.2 and 1.3

In what follows, we present some computations which are common for the proofs of Theorems 1.1, 1.2 and 1.3. At this point, we assume that M^n is a complete linear Weingarten spacelike hypersurface immersed in a locally symmetric Lorentzian space L_1^{n+1} , $n \geq 3$, satisfying conditions (1) and (2), such that $R = aH + b$ with $(n-1)a^2 + 4n(\bar{\mathcal{R}} - b) \geq 0$.

Initially, we observe that the local symmetry of L_1^{n+1} implies that

$$\sum_{i,j,k} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} = 0.$$

Consequently, if we choose a (local) orthonormal frame $\{e_i\}_{1 \leq i \leq n}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, taking into account equations (20) and (22), we get from (28) that

$$\begin{aligned} L(nH) &= \sum_{i,j,k} h_{ijk}^2 - n^2 |\nabla H|^2 + S^2 - nH \sum_i \lambda_i^3 \\ &\quad - 2 \sum_{i,k} (\lambda_i \lambda_k \bar{R}_{kiii} + \lambda_i^2 \bar{R}_{ikik}) \\ &\quad - \left(\sum_i nH \lambda_i \bar{R}_{(n+1)ii(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right). \end{aligned} \quad (29)$$

Thus, from Lemma 3.2 we have

$$\begin{aligned} L(nH) &\geq S^2 - nH \sum_i \lambda_i^3 - 2 \sum_{i,k} (\lambda_i \lambda_k \bar{R}_{kiii} + \lambda_i^2 \bar{R}_{ikik}) \\ &\quad - \left(\sum_i nH \lambda_i \bar{R}_{(n+1)ii(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right). \end{aligned} \quad (30)$$

Now, set $\Phi_{ij} = h_{ij} - H \delta_{ij}$. We will consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij} \omega_i \otimes \omega_j. \quad (31)$$

It is easy to check that Φ is traceless and, for this reason, it is called *traceless part* of the second fundamental form of M^n . Moreover, if $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ is the square of the length of Φ , then

$$|\Phi|^2 = S - nH^2. \quad (32)$$

If we take a (local) frame field $\{e_i\}_{1 \leq i \leq n}$ at $p \in M^n$, such that

$$h_{ij} = \lambda_i \delta_{ij} \quad \text{and} \quad \Phi_{ij} = \mu_i \delta_{ij},$$

it is straightforward to check the following algebraic relations:

$$\left\{ \begin{array}{l} \sum_i \mu_i = 0, \\ \sum_i \mu_i^2 = |\Phi|^2, \\ \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \end{array} \right.$$

Consequently, by applying Lemma 3.1 to the real numbers μ_1, \dots, μ_n , we get

$$\begin{aligned} S^2 - nH \sum_i \lambda_i^3 &= (|\Phi|^2 + nH^2)^2 - n^2 H^4 \\ &\quad - 3nH^2 |\Phi|^2 - nH \sum_i \mu_i^3 \\ &\geq |\Phi|^4 - nH^2 |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|^3. \end{aligned} \quad (33)$$

Using curvature conditions (1) and (2), we get

$$- \left(\sum_{i,j} nH \lambda_i \bar{R}_{(n+1)ii(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \right) = c_1 (S - nH^2) \quad (34)$$

and

$$\begin{aligned} -2 \sum_{i,j,k,l} (\lambda_i \lambda_k \bar{R}_{kiii} + \lambda_i^2 \bar{R}_{ikik}) &\geq c_2 \sum_{i,k} (\lambda_i - \lambda_k)^2 \\ &= 2nc_2 (S - nH^2). \end{aligned} \quad (35)$$

Hence, setting $c = \frac{c_1}{n} + 2c_2$, from (30), (32), (33), (34) and (35) we obtain that

$$\begin{aligned} L(nH) &\geq |\Phi|^2 \left(nc + S - 2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi| \right) \\ &= |\Phi|^2 \left(nc + |\Phi|^2 - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi| \right). \end{aligned} \quad (36)$$

On the other hand, from (13) and (32) we have

$$H^2 = \frac{1}{n(n-1)} |\Phi|^2 + \bar{\mathcal{R}} - R, \quad (37)$$

and, since we can assume that $H > 0$ on M^n ,

$$H = \frac{1}{\sqrt{n(n-1)}} \sqrt{|\Phi|^2 + n(n-1)(\bar{\mathcal{R}} - R)}. \quad (38)$$

Substituting (37) and (38) into (36), we finally get

$$L(nH) \geq \frac{1}{n-1} |\Phi|^2 P_R(|\Phi|), \quad (39)$$

where

$$P_R(x) = (n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(\overline{\mathcal{R}} - R)} + n(n-1)(c + R - \overline{\mathcal{R}}). \quad (40)$$

We observed that $P_R(x) = 0$ if, and only if,

$$x^2 = \frac{(n(n-1)(c + R - \overline{\mathcal{R}}))^2}{n(n-2)(n(n-1)(\overline{\mathcal{R}} - R - \frac{2c}{n}))}. \quad (41)$$

At this point, we proceed with the proofs of Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1.

Since we are supposing that $c > 0$ and $\overline{\mathcal{R}} - c < R < \overline{\mathcal{R}} - \frac{2c}{n}$, from (40) and (41) we easily verify that $P_R(0) = n(n-1)(c + R - \overline{\mathcal{R}}) > 0$ and the function $P_R(x)$ is strictly decreasing for $x \geq 0$, with $P_R(\tilde{x}) = 0$ at

$$\tilde{x} = \frac{n(n-1)(c + R - \overline{\mathcal{R}})}{\sqrt{n(n-2)}\sqrt{n(n-1)(\overline{\mathcal{R}} - R - \frac{2c}{n})}} > 0.$$

Thus, hypothesis (3) ensures that $0 \leq |\Phi| \leq \tilde{x}$ and $P_R(|\Phi|) \geq 0$. Hence, from (39) we have

$$L(nH) \geq \frac{1}{n-1} |\Phi|^2 P_R(|\Phi|) \geq 0. \quad (42)$$

Since we are supposing $b < \overline{\mathcal{R}}$, Lemma 3.3 guarantees that L is elliptic. Consequently, since we are also assuming that H attains its maximum on M^n , from (42) we can apply Hopf's strong maximum principle in order to conclude that H is constant on M^n . Hence, taking into account equation (29), we get

$$\sum_{i,j,k} h_{ijk}^2 = n^2 |\nabla H|^2 = 0,$$

and it follows that λ_i is constant for every $i \in \{1, \dots, n\}$.

If $|\Phi| < \tilde{x}$, then from (42) we have that $|\Phi| = 0$ and, hence, M^n is totally umbilical. If $|\Phi| = \tilde{x}$, since the equality holds in (23) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric spacelike hypersurface with two distinct principal curvatures one of which is simple. ■

Proof of Theorem 1.2.

From (21) we have that

$$\square f = \text{trace}(P_1 \circ \nabla^2 f),$$

where, denoting by I the identity in the algebra of smooth vector fields on M^n , $P_1 = nHI - h$ and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the hessian of f . Thus, by using the standard notation $\langle \cdot, \cdot \rangle$ for the (induced) metric of M^n , we get

$$\square f = \sum_i \langle P_1(\nabla_{e_i} \nabla f), e_i \rangle,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M^n . Consequently, we have that

$$\begin{aligned} \text{div}(P_1(\nabla f)) &= \sum_i \langle (\nabla_{e_i} P_1)(\nabla f), e_i \rangle + \sum_i \langle P_1(\nabla_{e_i} \nabla f), e_i \rangle \quad (43) \\ &= \langle \text{div} P_1, \nabla f \rangle + \square f, \end{aligned}$$

where

$$\text{div} P_1 := \text{trace}(\nabla P_1) = \sum_i (\nabla_{e_i} P_1)(e_i).$$

On the other hand, since we are assuming that L_1^{n+1} is an Einstein space-time, there exist a constant λ such that $\overline{\text{Ric}} = \lambda \langle \cdot, \cdot \rangle$, where $\overline{\text{Ric}}$ denotes the Ricci tensor of L_1^{n+1} . Thus, denoting by \overline{R} the curvature tensor of L_1^{n+1} , from Lemma 3.1 of [6] we have

$$\langle \text{div} P_1, \nabla f \rangle = \sum_i \langle \overline{R}(N, e_i) e_i, \nabla f \rangle = -\overline{\text{Ric}}(N, \nabla f) = -\lambda \langle N, \nabla f \rangle = 0,$$

where N stands for the Gauss mapping of M^n . Hence, from (43), we conclude that

$$\square f = \operatorname{div}(P_1(\nabla f)). \quad (44)$$

Consequently, from (28) and (44), we have that

$$L(nH) = \operatorname{div}(P(\nabla H)), \quad (45)$$

where $P = nP_1 + \frac{n(n-1)}{2}aI$.

Now, by applying the divergence theorem, from (42) we get

$$0 = \int_M L(nH) \geq \int_M \left\{ \frac{1}{n-1} |\Phi|^2 P_R(|\Phi|) \right\} dM \geq 0. \quad (46)$$

Consequently, taking into account hypothesis (4), from (46) we conclude that $|\Phi| = 0$ on M^n and, hence, M^n is totally umbilical. \blacksquare

Proof of Theorem 1.3.

First, from our restrictions on R , we note that

- (a) $R < \overline{\mathcal{R}} - c < \overline{\mathcal{R}} - \frac{2c}{n}$, when $c > 0$;
- (b) $R < \overline{\mathcal{R}} - \frac{2c}{n} \leq \overline{\mathcal{R}} - c$, when $c \leq 0$.

In both of these case, from (40) and (41) we have that

$$P_R(0) = -n(n-1)(\overline{\mathcal{R}} - R - c) < 0$$

and the function $P_R(x)$ is strictly increasing for $x \geq 0$, with $P_R(\hat{x}) = 0$ at

$$\hat{x} = \frac{n(n-1)(\overline{\mathcal{R}} - R - c)}{\sqrt{n(n-2)}\sqrt{n(n-1)(\overline{\mathcal{R}} - R - \frac{2c}{n})}} > 0.$$

Thus, hypothesis (5) guarantees that $|\Phi| \geq \hat{x} > 0$ and $P_R(|\Phi|) \geq 0$. Hence, from (39) we have

$$L(H) \geq \frac{1}{n-1} |\Phi|^2 P_R(|\Phi|) \geq 0. \quad (47)$$

In a similar way of the proof of Theorem 1.1, since we are supposing that $b < \overline{\mathcal{R}}$, we can apply Hopf's strong maximum principle to guarantee that

H is constant on M^n and that λ_i is constant for every $i \in \{1, \dots, n\}$. Since $|\Phi| > 0$, from (47) we obtain that $L(H) \geq 0$ if, and only if, $|\Phi| = \hat{x}$. Hence, Lemma 3.1 assures that M^n is an isoparametric spacelike hypersurface with two distinct principal curvatures one of which is simple. ■

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