Matemática Contemporânea, Vol. 43, 1–16
http://doi.org/10.21711/231766362012/rmc431

On symmetries of singular implicit ODEs

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Abstract

We study implicit ODEs, cubic in derivative, with infinitesimal symmetry at singular points. Cartan showed that even at regular points the existence of nontrivial symmetry imposes restrictions on the ODE. Namely, this algebra has the maximal possible dimension 3 iff the web of solutions is flat. For cubic ODEs with flat 3-web of solutions we establish sufficient conditions for the existence of nontrivial symmetries at singular points and show that under natural assumptions such a symmetry is semi-simple, i.e. is a scaling is some coordinates. We use this symmetry to find first integrals of the ODE.

1 Introduction

Consider an implicit ODE, cubic in derivative. Its solutions form 3 foliations in the plane, i.e. a planar 3-web. We suppose that the web directions are well defined everywhere. In suitable coordinates one can write such an equation in the monic form:

$$p^3 + a(x,y)p^2 + b(x,y)p + c(x,y) = 0. \tag{1}$$

2000 AMS Subject Classification: 53A60 (primary), 32S65 (secondary).
Key Words and Phrases: hexagonal 3-web, infinitesimal symmetries, Chern connection, implicit ODE.
* Partially Supported by MCT/CNPq/MEC/CAPES - Grant 552758/2011-6.

In this paper we study the complex analytic case locally, i.e., a, b, c are germs of holomorphic function at $(\mathbb{C}^2, 0)$ and the equivalence relation is induced by the group of germs of biholomorphisms $\mathrm{Diff}(\mathbb{C}^2, 0)$.

A point is called **regular** if for each pair of foliations the leaves are transverse to each other. Blaschke discovered (see [4]) that generically even a regular 3-web germ is not equivalent to the web germ of 3 families of parallel lines: a generic 3-web has a non-vanishing curvature 2-form.

Definition 1.1. A 3-web is flat, or hexagonal if at each regular points it is diffeomorphic (biholomorphic) to the 3-web germ of 3 pencils of parallel straight lines.

We call a web germ at $q_0 \in \mathbb{C}^2$ **singular** if at least two web directions coincide at q_0. Singular hexagonal web germs are not necessarily equivalent, unlike regular ones which have no local invariants by definition.

Curvature 2-form of a 3-web is defined as the derivative $d(\gamma)$ of the Chern connection 1-form γ (see [5] and Section 2). Thus, for hexagonal 3-webs, this form is closed. But it is not exact in general: on the **discriminant curve** of the web, which is the locus of singular points, the Chern connection form usually has a pole. For instance, for the Clairaut equation $p^3 + px - y = 0$ we have

$$\gamma = \frac{6x^2dx + 27ydy}{4x^3 + 27y^2},$$

whereas the EDO $p^3 + 2xp + y = 0$ has the zero connection 1-form. (See the 3-webs of solutions to these equations in Figure 1.)

Observe that the above two equations are invariant under the flow of the vector field $X = 2x\partial_x + 3y\partial_y$. We say that a web has an infinitesimal symmetry

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{2}$$

if the local flow of the vector field X respects its web of solutions. Cartan proved (see [6]) that at a regular point a 3-web either does not have infinitesimal symmetries (generic case), or has one-dimensional symmetry

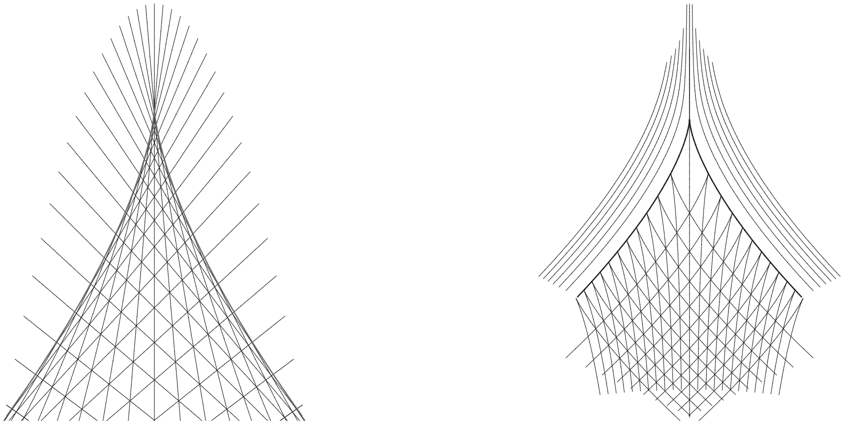

Figure 1: Solutions of $p^3 + px - y = 0$ and $p^3 + 2xp + y = 0$. (The y-axis is horizontal.)

algebra (then in suitable coordinates it can be defined by the form $dx \cdot dy \cdot (dy + u(x+y)dx) = 0$ with the symmetry $\partial_y - \partial_x$), or has a three-dimensional symmetry algebra (then it is equivalent to the web defined by the form $dx \cdot dy \cdot (dy + dx) = 0$ with the symmetry algebra generated by $\{\partial_x,\ \partial_y, x\partial_x + y\partial_y\}$). In the last case, when the symmetry algebra has the largest possible dimension 3, the 3-web is hexagonal. Note that not all symmetries survive at a singular point; in the above examples the dimension of the symmetry algebra drops to 2 at a generic point of the discriminant curve and to 1 at the cusp point. The condition to have at least one-dimensional symmetry at a singular point is not trivial. The following equation has a flat 3-web of solutions but does not admit non-trivial symmetries at $(0,0)$

$$p^3 - 2x^2y(1+x^2)p + 8x^3y^2 = 0.$$

In this paper we study flat (or hexagonal) web germs admitting an infinitesimal symmetry at the singular point. The classical Lie approach would lead to a system of linear singular PDEs for ξ, η. To avoid a difficult problem of existence of regular solutions to this system, we look into the local monodromy group of the cubic equation (1). (This group permutes the roots on going around the discriminant curve.) The results of this study can be summarized as follows.

1) If this group is the largest possible (i.e. S_3), and the symmetry operator vanishes at the singular point then the symmetry is scaling in suitable coordinates:

$$E = w_x x \frac{\partial}{\partial x} + w_y y \frac{\partial}{\partial y}, \qquad w_x, w_y = \text{const}. \tag{3}$$

2) For the case of holomorphic (i.e. locally exact) connection form there is the existence theorem. Namely the algebra is at least 2-dimensional for a point with a double root and with the local monodromy Z_2; and 1-dimensional for a point with a triple root and the local monodromy S_3.

3) If the connection form is holomorphic and the symmetry operator vanishes at the singular point then this symmetry also is equivalent to some scaling.

4) In the above cases the first integrals of the foliations can be chosen algebraic integer over the ring of holomorphic function germs.

Studying singular points of implicit ODEs was initiated by Thom in [14]. For a generic quadratic ODE, normal forms were established by Davydov in [7]. For cubic ODEs the classification problem becomes more complicated: the obstacle is the curvature. Moreover, Nakai showed that the topological and analytic classifications are in fact the same in this case (see [12]). Even the zero curvature condition will not compress the class of ODEs to guarantee a sensible classification (see [1] for discussion and a partial classification result).

The principal motivation for the study of the above defined class of ODEs, possessing a symmetry and holomorphic connection form, is a relation to Frobenius 3-folds (see [2] and [3]). The approach to webs based on implicit ODEs turned out a useful tool for studying abelian relations and singularities of webs (see [8, 9]). Infinitesimal symmetries were used for constructing families of so-called exceptional webs in [11].

2 Chern connection, first integrals, and abelian relations

Following Blaschke's approach based on differential forms [5], we present a formula for the Chern connection form of a 3-web, formed by solutions of an implicit cubic ODE.

Let p_1, p_2, p_3 be the roots of (1) at a point (x, y) outside the discriminant curve. One-forms vanishing on the solutions can be chosen as follows

$$\sigma_1 = (p_2-p_3)(dy-p_1dx), \ \ \sigma_2 = (p_3-p_1)(dy-p_2dx), \ \ \sigma_3 = (p_1-p_2)(dy-p_3dx).$$

They are normalized to satisfy the condition $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The Chern connection form is defined as

$$\gamma := h_2\sigma_1 - h_1\sigma_2 = h_3\sigma_2 - h_2\sigma_3 = h_1\sigma_3 - h_3\sigma_1,$$

where h_i are determined by $d\sigma_i = h_i\Omega$ with

$$\Omega = \sigma_1 \wedge \sigma_2 = \sigma_2 \wedge \sigma_3 = \sigma_3 \wedge \sigma_1 = (p_1 - p_2)(p_2 - p_3)(p_3 - p_1)dy \wedge dx.$$

The web is flat iff the connection form is closed: $d(\gamma) = 0$. This implies $d\sigma_i = \gamma \wedge \sigma_i$. Defining

$$dk = -\gamma k, \tag{4}$$

we introduce first integrals u_i of the foliations (at least locally at regular points) by

$$du_1 = k\sigma_1, \quad du_2 = k\sigma_2, \quad du_3 = k\sigma_3. \tag{5}$$

Remark. Let η_1, η_2, η_3 be germs of differential forms in $(\mathbb{C}^2, q_0)$ satisfying the conditions:

- the forms are closed: $d(\eta_i) = 0, \quad i = 1, 2, 3,$
- the forms define the web: $\eta_i \wedge \sigma_i = 0, \quad i = 1, 2, 3,$
- the forms sum up to zero: $\eta_1 + \eta_2 + \eta_3 = 0,$

then these forms are proportional to $k\sigma_i$: $\eta_i = ck\sigma_i, \quad i = 1,2,3,\ c = const$. One says that the space of abelian relations is one-dimensional for a hexagonal 3-web. In what follows these first integrals are called **abelian**.

To simplify the final formulas we bring equation (1) to the form

$$p^3 + A(x,y)p + B(x,y) = 0, \tag{6}$$

killing the coefficient by p^2 by a coordinate transform

$$y = f(\tilde{x},\tilde{y}),\ x = \tilde{x}, \ \text{ satisfying } \ 3f_{\tilde{x}} + a(\tilde{x}, f) = 0. \tag{7}$$

By a coordinate transform $y = F(X,Y), x = G(X,Y)$ the forms σ_i are multiplied by the factor $\frac{(G_X F_Y - G_Y F_X)^2}{G_X^3 + aG_X G_Y^2 - G_Y^3 b}$, where $a(X,Y) = A(G,F)$, and $b(X,Y) = B(G,F)$.

Lemma 2.1. *Let $k(x,y)$ be a function not vanishing at $(0,0)$; then the following system of PDEs*

$$\begin{aligned} k(G,F)(G_X F_Y - G_Y F_X)^2 &= G_X^3 + aG_X G_Y^2 - G_Y^3 b, \\ (3F_Y^2 + aG_Y^2)F_X + 2aF_Y G_X G_Y + 3bG_X G_Y^2 &= 0 \end{aligned} \tag{8}$$

has a solution germ at $(0,0)$ satisfying $(G_X F_Y - G_Y F_X) \neq 0$, $F(0,0) = G(0,0) = 0$.

Proof: One easily checks the local solvability of the above system via the Cauchy-Kovalevskaya Theorem; locally the above system can be represented in Kovalevskaya form with respect to F_X, G_X by adjusting Cauchy data. □

Lemma 2.2. *Suppose the Chern connection form is exact $\gamma = d(f)$, where the function f is defined on some neighborhood U of a point on the discriminant curve. Then one can choose new local coordinates to keep the coefficient by p^2 to be zero and simultaneously to ensure $k \equiv 1$.*

Proof: From (4) one has $k = exp(-f) \neq 0$. Now choose F, G to satisfy system (8) and $(G_X F_Y - G_Y F_X) \neq 0$. The second equation of (8) ensures that the coefficient by p^2 remains zero. □

Computing the Chern connection form in terms of roots p_i and using the Viete formulas one gets

$$\gamma = \frac{(2A^2Ax - 4A^2By + 6ABAy + 9BBx)}{4A^3 + 27B^2}dx + \frac{(4A^2Ay + 6ABx + 18BBy - 9BAx)}{4A^3 + 27B^2}dy. \quad (9)$$

Notice that this form can have a pole on the **discriminant curve**

$$\Delta := \{(x, y) : 4A^3(x, y) + 27B^2(x, y) = 0\}.$$

3 Infinitesimal symmetries

Pick up a point q_0 on the discriminant curve and select some connected neighborhood U of this point. At a point $q \in U \setminus \Delta$, equation (6) implicitly defines function germs p_1, p_2, p_3. Analytical continuation of these germs along all closed paths in U passing through q generates a subgroup of the group S_3 permuting the roots p_i. We call this subgroup a **local monodromy group** of (6) at q_0.

Notice that equation (6) defines an analytic set germ $\mathcal{A}$ in $(\mathbb{C}^5, 0)$ by

$$p_1 + p_2 + p_3 = 0, \quad p_1p_2 + p_2p_3 + p_3p_1 = A(x, y), \quad p_1p_2p_3 = -B(x, y). \quad (10)$$

We will need the following representation of functions holomorphic on $\mathcal{A}$.

Lemma 3.1. *Suppose that equation (6) is irreducible over the ring of holomorphic function germs $\mathcal{O}_0$ on $(\mathbb{C}^2, 0)$ and the local monodromy group of (6) acts on the roots as the permutation group S_3. Then each holomorphic function germ F on the analytic set germ $\mathcal{A}$ can be represented in the form*

$$F = F_0(x, y) + p_1F_1(x, y) + p_2F_2(x, y) + p_1p_2F_3(x, y) + p_2^2F_4(x, y) + p_1p_2^2F_5(x, y),$$

where F_i, $i = 0, ..., 5$ are holomorphic function germs on $(\mathbb{C}^2, 0)$. Moreover, this representation is unique.

Proof: The existence of the representation follows from Malgrange's Preparation Theorem. In fact, the identities

$$p_1^2 = -p_1p_2 - p_2^2 - A, \quad p_1^3 = -p_1A - B, \quad p_1^2p_2 = -p_1p_2^2 + B, \quad p_2^3 = -p_2A - B$$

imply $\langle p_1, p_2\rangle^4 \subset \langle A, B\rangle$ and $\mathcal{O}_2(p_1,p_2)/\langle A,B\rangle = \mathbb{C}\{1, p_1, p_2, p_1p_2, p_2^2, p_1p_2^2\}$. To prove the uniqueness one applies all the permutations of S_3 to the representation of the zero function germ, normalize the results using the above identities and shows that all F_i are zero function germs. □

At each regular point one can choose any pair of the abelian first integrals (5) as local coordinates. The symmetry algebra at this point is generated by the 3 vector fields $\partial_{u_1},\ \partial_{u_2},\ u_1\partial_{u_1} + u_2\partial_{u_2}$. If an operator X is a symmetry then $X(u_i) = \varphi_i(u_i)$ for some function germs φ_i. As the space of abelian relations for a hexagonal 3-web is one-dimensional and the symmetry X maps abelian relations into abelian relations, the functions φ_i are linear:

$$X(u_i) = Cu_i + c_i. \tag{11}$$

Lemma 3.2. *Suppose that X is a symmetry of equation (6) and the local monodromy group is S_3. Then $C \neq 0$ in the equality (11).*

Proof: Consider a point $q_0 = (x_0, y_0) \notin \Delta$. Suppose $C = 0$; then at least two of the constants c_i, say c_1 and c_2, do not vanish. Indeed, the corresponding first integrals are functionally independent at q_0 and a non-trivial symmetry operator cannot have 2 independent invariants. Equations (5) imply $c_1 = k(p_2 - p_3)(\eta - p_1\xi)$, $c_2 = k(p_3 - p_1)(\eta - p_2\xi)$, where $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$. Excluding the function k gives $c_2(p_2 - p_3)(\eta - p_1\xi) = c_1(p_3 - p_1)(\eta - p_2\xi)$. Rewriting this as $c_2\xi A + \eta(2c_1 + c_2)p_1 + \eta(2c_2 + c_1)p_2 - \xi(2c_1 + c_2)p_1p_2 - \xi(c_1 - c_2)p_2^2 = 0$ and applying Lemma 3.1 we get $X \equiv 0$. □

Theorem 3.1. Suppose that ODE (6) has a flat web of solutions and admits a symmetry operator (2) such that $C \neq 0$ in the equality (11). Then one can choose germs $I_i,\ i = 1,2,3$ of first integrals of (6) to satisfy

$I_i = k^2U_i$, where $dk = -k\gamma$, γ is the Chern connection form, and U_i are the roots of the following cubic equation:

$$U^3 - 2\alpha U^2 + \alpha^2 U + \beta = 0, \quad \text{where} \tag{12}$$

$$\alpha = \xi^2A^2 - 3\eta^2A - 9\xi\eta B, \quad \beta = (4A^3 + 27B^2)(\eta^3 + \xi^2\eta A + \xi^3 B)^2. \tag{13}$$

Proof: Let $q = (x_0, y_0) \notin \Delta$. Consider the germs of abelian integrals (5) at q. Normalizing X and adjusting integration constant we have $u_i = X(u_i) = k\sigma_i(X)$, i.e.

$$u_1 = k(p_2 - p_3)(\eta - p_1\xi), \ u_2 = k(p_3 - p_1)(\eta - p_2\xi), \ u_3 = k(p_1 - p_2)(\eta - p_3\xi).$$

Note that $I_i := u_i^2$ are also first integrals. Calculating elementary symmetric function of $\frac{I_i}{k^2}$ one arrives at (12). □

Remark. Lie discovered (see [10]) that an explicit ODE in differentials $M(x,y)dx + N(x,y)dy = 0$ with an infinitesimal symmetry X has the integrating factor $\mu = \frac{1}{\xi M + \eta N}$, i.e. $d(\mu M dx + \mu N dy) = 0$. The above Theorem gives an analog of this Lie result for implicit cubic ODEs.

If X is a symmetry of equation (6) then the Lie derivative of the connection form γ vanishes. Therefore $\mathcal{L}_X(\gamma) = i_X(d(\gamma)) + d(i_X(\gamma)) = d(\gamma(X)) = 0$ since the connection form is closed. Thus $\gamma(X)$ is constant:

$$\gamma(X) = c. \tag{14}$$

Theorem 3.2. Suppose that ODE (6) has a flat web of solutions, $C \neq 0$ in the equality (11), and a symmetry X of (6) vanishes at the singular point $(0,0) \in \Delta$. Then the equation is equivalent to a weighted homogeneous ODE and the symmetry operator X to some scaling.

Proof: Choose a point $q = (x_0, y_0) \notin \Delta$. The condition $C \neq 0$ allows one to normalize the symmetry operator X and the first integrals u_i to satisfy $X(u_i) = u_i$. Let us calculate the action of the symmetry operator X on the functions α and β defined by (13): $X(\alpha) = X(\frac{u_1^2 + u_2^2 + u_3^3}{k^2}) =$

$\frac{2u_1X(u_1)+2u_2X(u_2)+2u_3X(u_3)}{k^2}-2\frac{u_1^2+u_2^2+u_3^3}{k^3}X(k)=2(1-\frac{X(k)}{k})\alpha$. Since $\frac{X(k)}{k}=-(\gamma(X))=-c$ we have $X(\alpha)=2(1+c)\alpha$. Similarly $X(\beta)=6(1+c)\beta$. One can choose a coordinate system to make the functions α and β functionally independent at q. (This is a slightly modified version of Lemma 2.1.) This condition is equivalent to $c\neq -1$ in the formula (14).

The operator X vanishes at $(0,0)$ hence we can apply the following results of K.Saito (see [13]).

1. In suitable coordinates the operator X can be written as a sum $X=X_s+X_n$ of a scaling operator X_s (semi-simple in Saito's terminology) and a commuting with X_s nilpotent operator $X_n=n_1(x,y)\partial_x+n_2(x,y)\partial_y$) (i.e. all eigenvalues of the matrix

$$\begin{pmatrix} \frac{\partial n_1}{\partial x} & \frac{\partial n_1}{\partial y} \\ \frac{\partial n_2}{\partial x} & \frac{\partial n_2}{\partial y} \end{pmatrix}$$

   are zeros at $(0,0)$).

2. Moreover, the following two conditions are equivalent:

   a) $X(f)=\lambda f$,

   b) $X_s(f)=\lambda f$, $X_n(f)=0$,
   where f is a function germ and λ is a complex number.

Thus we have from the condition b): $X_n(\alpha)=X_n(\beta)=0$. As the functions α and β are functionally independent we get $X_n=0$. □

Theorem 3.3. Suppose that ODE (6) has a flat web of solutions and admits a symmetry X at the singular point $(0,0)\in\Delta$, the operator X vanishes at this point, and the local monodromy group of (6) is S_3. Then the equation is equivalent to a weighted homogeneous ODE and the symmetry operator X to a scaling.

Proof: The claim follows from Lemma 3.2 and Theorem 3.2. □

Remark. Unfortunately, Lemma 3.2 is not true if the local monodromy group is smaller than S_3. It is not difficult to find counter-examples.

4 ODEs with exact Chern connection form.

Lemma 4.1. *Let $q_0 = (x_0, y_0) \in \Delta$. Suppose that the Chern connection form is exact $\gamma = d(f)$, where the function f is defined on some connected neighborhood U of q_0. Then the abelian first integrals u_i are algebraic integers over the ring of holomorphic function germs $\mathcal{O}_{q_0}$.*

Proof: Define $U_\Delta := U \backslash \Delta$. Then the connection form γ is exact on U_Δ. Let $q \in U_\Delta$ be some point outside the discriminant curve and V a simply connected neighborhood of q contained in U_Δ, i.e. $q \in V \subset U_\Delta$. Select a path $\alpha : [0,1] \mapsto U$ connecting q_0 and q: $\alpha(0) = q_0$, $\alpha(1) = q$ and satisfying $\alpha((0,1]) \in U_\Delta$.

Define functions $u_1, u_2, u_3 : V \mapsto \mathbb{C}$ by equations (5), where $k = \exp(-f)$ and $p_1, p_2, p_3 : V \mapsto \mathbb{C}$ are functions implicitly defined by equation (6). Then u_1, u_2, u_3 are well-defined up to a choice of the initial values $u_1(q),\ u_2(q),\ u_3(q)$. Let us fix them by

$$\begin{aligned} u_1(q) &= \textstyle\int_\alpha k(p_2 - p_3)(dy - p_1 dx), \\ u_2(q) &= \textstyle\int_\alpha k(p_3 - p_1)(dy - p_2 dx), \\ u_3(q) &= \textstyle\int_\alpha k(p_1 - p_2)(dy - p_3 dx). \end{aligned}$$

The analytical continuation of u_i along all the paths contained in U_Δ gives multivalued functions $\tilde{u}_i$ on U_Δ. Due to the choice of initial conditions one has

$$\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 0.$$

Moreover, these initial conditions also imply that the functions

$$f := \tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2, \quad h := \tilde{u}_1^2 \tilde{u}_2^2 \tilde{u}_3^2$$

are one-valued on U_Δ. In fact, the analytic continuation along each closed path in U_Δ induces a permutation of roots p_1, p_2, p_3. This permutation generates an action on the differentials $d(u_1), d(u_2), d(u_3)$. On the other hand, due to the choice of the initial values of u_i the action on $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ coincides with the action on the differentials. Moreover, being bounded, the functions f, h are holomorphic on the whole neighborhood U by the

Riemann theorem. Therefore each of the functions $\tilde{u}_i$ is integer over the ring $\mathcal{O}(U)$ of functions analytical on U as satisfying the following equation

$$u^6 - fu^4 + \frac{f^2}{4}u^2 - h = 0$$

Differentiating the function f and using (5) one shows that the functions $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are well defined meromorphic functions on the germ of analytic set $\mathcal{A}$ determined by equations (10). Further, being integer also over $\mathcal{O}(\mathcal{A})$ these functions are in fact holomorphic on $\mathcal{O}(\mathcal{A})$. □

According to the classical Lie results the components of our symmetry operator X satisfy a system of linear PDEs. In a neighborhood of a regular point the space of solutions to this system is 3-dimensional. When there exists a solution that can be extended to a neighborhood of a point on the discriminant curve Δ? A sufficient condition gives the following theorem.

Theorem 4.1. Let $q_0 = (x_0, y_0) \in \Delta$. Suppose that the Chern connection form is exact $\gamma = d(f)$, where the function f is defined on some neighborhood U of q_0. Then the dimension of the symmetry algebra of equation (6) at q_0 is

- at least 1, if the root is triple and the local monodromy group is S_3,
- at least 2, if the root is double and the local monodromy group is Z_2.

Proof: Define the first integrals as in Lemma 4.1.

• *Triple root.* Each holomorphic function germ on $\mathcal{A}$ can be written in the normal form given by Lemma 3.1. Using the symmetry properties of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ under the permutations of the roots one gets

$$\tilde{u}_1 = (p_2 - p_3)(M(x,y) - p_1 L(x,y)), \quad \tilde{u}_2 = (p_3 - p_1)(M(x,y) - p_2 L(x,y)),$$

where M and L are holomorphic on U. Define

$$\xi = \tfrac{1}{k(p_1 - p_2)} \left(\tfrac{\tilde{u}_2}{p_3 - p_1} - \tfrac{\tilde{u}_1}{p_2 - p_3} \right), \quad \eta = \tfrac{1}{k(p_1 - p_2)} \left(\tfrac{p_1 \tilde{u}_2}{p_3 - p_1} - \tfrac{p_2 \tilde{u}_1}{p_2 - p_3} \right). \quad (15)$$

It is immediate that the functions $\xi = \frac{L}{k}$ and $\eta = \frac{M}{k}$ are well defined on U. Therefore the vector field $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is a symmetry of our ODE, as its action on the first integrals satisfies $X(u_1) = u_1$, $X(u_2) = u_2$ due to the equalities (15) and (5).

• *Double root.* Suppose that p_1, p_2 satisfy an irredicible quadratic equation at 0 and $p_1 = p_2 \neq p_3$. Then $a := p_3$ is a holomorphic function germ on (U, q_0) and $a(q_0) \neq 0$ since $p_1 + p_2 + p_3 = 0$. The function germ $2p_3^2 + p_1p_2 = 2a^2 + p_1p_2$ is also holomorphic. Moreover, it does not vanish at q_0 since $2a^2 + p_1p_2|_{q_0} = \frac{9}{4}a^2(q_0)$. Then the vector field

$$X_1 = \frac{\partial_x + p_3\partial_y}{k(2p_3^2 + p_1p_2)}$$

is an infinitesimal symmetry. Indeed, its action on the first integrals is the following: $X_1(u_1) = -1$, $X_1(u_2) = 1$. The second symmetry X_2 is defined by the same formula as for the case of triple root. To check that the vector field (15) is well defined on (U, q_0) write

$$\tilde{u_1}(p) = R(x,y) + p_1S(x,y)$$

instead of the normal form given by Lemma 3.1, observe that $\tilde{u}_2(p) = aSR - R + p_1S$ due to the permutation symmetry properties, and substitute these expressions into (15). One immediately checks that this vector field satisfies $X_2(u_1) = u_1$, $X_2(u_2) = u_2$. On some neighborhood $V \subset U$ of a point $q \neq q_0$, one can rewrite the symmetry operators as $X_1 = \partial_{u_2} - \partial_{u_1}$, $X_2 = u_1\partial_{u_1} + u_2\partial_{u_2}$, i.e. they are linearly independent. □

Proposition 4.1. If an infinitesimal symmetry of equation (1) vanishes at $(0,0)$ and the Chern connection form is exact, then the equation is equivalent to a weighted homogeneous one and the symmetry to a scaling.

Proof: In fact, choosing the first integrals as in Lemma 4.1 we have $c_1 = c_2 = c_3 = 0$ in formula (11). For example, $X(u_1)|_0 = C\, u_1|_0 + c_1 = c_1$. On the other hand $X(u_1) = k(p_2 - p_3)(\eta - p_1\xi) = 0$. Whence $c_1 = 0$, $C \neq 0$ and the equation is weighted homogeneous due to Theorem 3.2. □

Note that if the symmetry algebra is 3-dimensional and the Chern connection form is exact, then the roots of equation (1) are simple. In fact, there are two symmetries X_1, X_2 satisfying $X_i(u_j) = \delta_{ij}$. The function k in equations (5) for abelian first integrals can be reduced to $k = 1$ (Lemma 2.2).

Corollary 4.1. Suppose that equation (1) has a non-trivial symmetry algebra at $(0, 0)$ and the Chern connection form of the web germ of solutions is exact.

- If the symmetry algebra is 3-dimensional, then the equation has simple roots at $(0, 0)$.
- If the symmetry algebra is 2-dimensional, then the equation has a double root at $(0, 0)$.
- If the symmetry algebra is 1-dimensional, then the equation has a triple root..

5 Concluding remarks

The results obtained in this paper distinguish the ODEs with scaling symmetries and holomorphic Chern connection form. In fact, such ODEs can be effectively classified. We present the corresponding classification results elsewhere.

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