AVD-total-colouring of complete equipartite graphs

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Abstract

An AVD-total-colouring of a simple graph $G$ is a mapping $\phi : V(G) \cup E(G) \to C$, with $C$ a set of colours, such that: (i) for each adjacent or incident elements $x, y \in V(G) \cup E(G)$, $\phi(x) \neq \phi(y)$; (ii) and for each pair of adjacent vertices $x, y \in V(G)$, sets $\{\phi(x)\} \cup \{\phi(xv) : xv \in E(G)\}$ and $\{\phi(y)\} \cup \{\phi(yv) : yv \in E(G)\}$ are distincts. The AVD-total-chromatic number, $\chi''_a(G)$, is the smallest number of colours for which $G$ admits an AVD-total-colouring. In 2005, Zhang et al. conjectured that $\chi''_a(G) \leq \Delta(G) + 3$ for any simple graph $G$. In this article this conjecture is verified for complete equipartite graphs $G$ and it is also shown that $\chi''_a(G) = \Delta(G) + 2$, if $G$ has even order.

1 Introduction

Let $G := (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is the order of $G$. We denote an edge $e \in E(G)$ by $uv$ when $u$ and $v$ are its ends. An element of $G$ is a

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vertex or an edge of $G$. As usual, we denote by $d(v)$ the degree of a vertex $v \in V(G)$, and by $\Delta(G)$ the maximum degree of $G$.

Let $S := V(G) \cup E(G)$ and let $C$ be a set of colours. A total-colouring of $G$ is a mapping $\phi : S \to C$, such that for each adjacent or incident elements $x, y \in S$, we have $\phi(x) \neq \phi(y)$. If $|C| = k$, then mapping $\phi$ is called a $k$-total-colouring of $G$. If $S = V(G)$, then $\phi$ is a vertex-colouring of $G$, and if $S = E(G)$, then $\phi$ is an edge-colouring of $G$.

The chromatic number of $G$, $\chi(G)$, is the smallest number of colours for which $G$ admits a vertex-colouring. Similarly, we define the chromatic index of $G$, $\chi'(G)$, as the smallest number of colours for which $G$ admits an edge-colouring; and the total-chromatic number of $G$, $\chi''(G)$, as the smallest number of colours for which $G$ admits a total-colouring.

Let $\phi$ be a total-colouring of $G$ and let $C(u) := \{\phi(u)\} \cup \{\phi(uv) : uv \in E(G)\}$ be the set of colours that occurs in a vertex $u \in V(G)$. Two vertices $u$ and $v$ are distinguishable when $C(u) \neq C(v)$. If this property is true for every pair of adjacent vertices, then $\phi$ is an adjacent-vertex-distinguishing-total-colouring (AVD-total-colouring). The AVD-total-chromatic number, $\chi''_a(G)$, is the smallest number of colours for which $G$ admits an AVD-total-colouring. If $\phi$ uses $k$ colours, then it is called a $k$-AVD-total-colouring.

A vertex-distinguishing-proper-edge-colouring is an edge-colouring of $G$ that requires $C(u) \neq C(v)$ for each $u, v \in V(G)$. This colouring was first examined by Burris and Schelp [3], and further investigated by many others, including Bazgan et al. [2] and Balister et al. [1]. The motivation for studying vertex-distinguishing-proper-edge-colourings came from irregular networks. In these networks, it is necessary to associate positive integer weights to the edges in such a way that the sum of weights of the edges incident with each vertex form a set of distinct numbers [2]. Zhang et al. [9] considered edge-colourings in which only adjacent vertices were distinguishable. After that, around 2005, they studied the problem of distinguishable vertices in the context of total-colourings [10], giving rise to AVD-total-colourings. In their seminal article, Zhang et al. determined the AVD-total-chromatic number for some classes of graphs and, based on
their results, the authors posed the following conjecture:

**Conjecture 1.1 (AVD-total-colouring conjecture).** If $G$ is a simple graph, then $\chi''_a(G) \leq \Delta(G) + 3$.

This conjecture has been verified for some classes of graphs, which include complete graphs, complete bipartite graphs, trees [10], hypercubes [4], graphs with $\Delta(G) = 3$ [6], outerplanar graphs [8], indifference graphs [7], and Halin graphs [5]. In this work, we consider the class of complete equipartite graphs. We prove that the AVD-total-colouring conjecture holds for this class and we determine the AVD-total-chromatic number for complete equipartite graphs of even order.

2 Main results

A subset of $V(G) \cup E(G)$ is independent if its elements are pairwise nonadjacent and nonincident. For positive integers $r$ and $n$, a complete equipartite graph, $K_{r(n)}$, is a simple graph whose vertex set can be partitioned into $r$ independent sets (parts) of cardinality $n$, where any two vertices that belong to different parts are joined by an edge. In this note, we verify the AVD-total-colouring conjecture for complete equipartite graphs. We consider graphs $K_{r(n)}$ with $r \geq 2$ and $n \geq 2$ since the results when $r < 2$ or $n < 2$ are known [10]. Moreover, we also determine the AVD-total-chromatic number for even order complete equipartite graphs.

A canonical labelling of $K_{r(n)}$ is a labelling of the vertices of $K_{r(n)}$, such that for each part $j$, $1 \leq j \leq r$, each vertex in the part receives a distinct label $u^j_i$, where $1 \leq i \leq n$. For $r \geq 2$, we define the canonical decomposition $[\mathcal{K}, \mathcal{B}]$ of $K_{r(n)}$ as the union of edge-disjoint subgraphs. This decomposition is described in the following.

Let $K_{r(n)}$ be a complete equipartite graph endowed with canonical labelling. Considering $G[S]$ denotes the subgraph induced by set $S \subseteq V(G)$, note that subgraphs $K^i_r := G[\{u^i_1, \ldots, u^i_r\}]$, $1 \leq i \leq n$, are isomorphic to the complete graph $K_r$. Thus, $K_{r(n)}$ has $n$ disjoint copies of $K_r$ as induced
subgraphs. Figure 1 illustrates $K_{4(2)}$ endowed with canonical labelling and two induced subgraphs $K_4^1$ and $K_4^2$ isomorphic to $K_4$.

![](image1.png)

(a) $K_{4(2)}$ endowed with canonical labelling. The four parts of $K_{4(2)}$ are identified by $P1$, $P2$, $P3$, and $P4$.

(b) Two disjoint induced subgraphs of $K_{4(2)}$ isomorphic to $K_4$.

Figure 1: $K_{4(2)}$ and its induced subgraphs $K_4^1$ and $K_4^2$ that are isomorphic to $K_4$.

The subgraph induced by edges joining vertices of $K_r^i$ to vertices of $K_r^j$, is a bipartite graph, denoted by $B_{ij} = G[V(K_r^i), V(K_r^j)]$, $1 \leq i < j \leq n$. Moreover, $B_{ij}$ is an $(r-1)$-regular graph. In fact, edges $u_x^i u_x^j$ ($1 \leq x \leq r$) do not exist since vertices $u_x^i$ and $u_x^j$ are in the same part of $K_r(n)$. Figure 2 illustrates a $K_{4(2)}$ endowed with canonical labelling and its unique bipartite subgraph, $B_{12}$, induced by the edges joining vertices from $K_4^1$ to vertices of $K_4^2$.

Using the above notation, we define the canonical decomposition $[\mathcal{K}, \mathcal{B}]$ of $K_r(n)$ as:

$\mathcal{K} := \bigcup_{1 \leq i \leq n} K_r^i$, and $\mathcal{B} := \bigcup_{1 \leq i < j \leq n} B_{ij}$.

The previous definition implies that $K_r(n) \cong (\mathcal{K} \cup \mathcal{B})$. Also, note that $\mathcal{K}$ is a disconnected graph composed by exactly $n$ components $K_r^i$, each one isomorphic to a complete graph $K_r$. 
Let $G_R$ be the underlying simple graph obtained from $[\mathcal{K}, \mathcal{B}]$ by shrinking each $K_r^i$ into a vertex $v_i$. Graph $G_R$ is called the representative graph of $K_{r(n)}$ since the previous decomposition can be represented by $G_R$ in the following way: each vertex $v_i \in V(G_R)$ represents a component $K_r^i \subseteq \mathcal{K}$ and each edge $v_iv_j \in E(G_R)$ represents a bipartite graph $B_{ij} \subseteq \mathcal{B}$. Note that $G_R \cong K_n$. For example, observe that the representative graph of $K_{4(2)}$ is the complete graph $K_2$. Figure 3 illustrates the canonical decomposition of $K_{4(3)}$ and its representative graph.

Now, we are ready to establish our main result.

**Theorem 2.1.** Let $G := K_{r(n)}$ be a complete equipartite graph with $r \geq 2$ and $n \geq 2$. If $G$ has even order, then $\chi''_a(G) = \Delta(G) + 2$; otherwise, $\chi''_a(G) \leq \Delta(G) + 3$.

**Proof.** (Sketch)

Initially, note that $\chi''_a(G) \geq \Delta(G) + 2$ since $G$ has two adjacent vertices of maximum degree. Therefore, to prove Theorem 2.1, it is enough to build a $(\Delta(G) + 2)$-AVD-total-colouring for $G$ of even order and a $(\Delta(G) + 3)$-AVD-total-colouring for $G$ of odd order.

In order to build the required colouring, we decompose $K_{r(n)}$ into the
A. G. Luiz, C. N. Campos and C. P. de Mello

(a) $K_{4(3)}$ endowed with canonical labelling. (b) The subgraph $\mathcal{K}$ composed by 3 componentes isomorphic to $K_4$.

(c) A scheme showing the canonical decomposition of $K_{4(3)}$. Each square represents a component $K_i^j \subseteq \mathcal{K}$. Thick lines joining squares represent the edges of bipartite graphs $B_{12}$, $B_{13}$, and $B_{23}$.

(d) The representative graph $G_R$ of $K_{4(3)}$. Note that $G_R \cong K_3$.

Figure 3: Canonical decomposition of $K_{4(3)}$ and its representative graph $G_R$.

canonical decomposition $[\mathcal{K}, \mathcal{B}]$ and consider four cases depending on the parity of $n$ and $r$. In each case, using the representative graph $G_R$, we assign suitable edge-colourings to subgraph $\mathcal{B}$ and an AVD-total-colouring to subgraph $\mathcal{K}$ in such a way that the result is an AVD-total-colouring of $K_r(n)$.

As an illustration, for the case $n$ and $r$ even, the components of subgraph $\mathcal{K}$ receive an AVD-total-colouring with $r + 1$ colours, while subgraph $\mathcal{B}$ receives an edge-colouring with $(n - 1)(r - 1)$ new colours. The result is a $(\Delta(K_r(n)) + 2)$-AVD-total-colouring of $K_r(n)$. For example, using
the canonical decomposition \([\mathcal{K}, \mathcal{B}]\) of \(K_{4(2)}\), Figure 4(a) shows a 5-AVD-total-colouring of subgraph \(\mathcal{K}\) and Figure 4(b) shows a 3-edge-colouring of subgraph \(\mathcal{B}\). The result is an 8-AVD-total-colouring of \(K_{4(2)}\).

![Figure 4: A canonical decomposition of \(K_{4(2)}\) showing its 8-AVD-total colouring.](image)

(a) A 5-AVD-total-colouring of subgraph \(\mathcal{K}\) using colours 1, 2, 3, 4, 5.

(b) The subgraph \(\mathcal{B}\) endowed with a 3-edge-colouring using colours 6, 7, 8.

### 3 Concluding Remarks

According to Theorem 2.1, the AVD-total-colouring conjecture holds for complete equipartite graphs. Although the conjecture holds for \(K_{r(n)}\) of odd order, the AVD-total-chromatic number is not determined for this case. Nevertheless, we have obtained \((\Delta(K_{r(n)}) + 2)\)-AVD-total-colourings for some equipartite graphs of odd order. Based on these findings, we pose the following conjecture.

**Conjecture 3.1.** If \(K_{r(n)}\) has odd order, then \(\chi''_a(K_{r(n)}) = \Delta(K_{r(n)}) + 2\).

It is well known that the restriction of an AVD-total-colouring to a proper subgraph \(H\) of \(G\) is not necessarily an AVD-total-colouring of \(H\). However, we observe that the restriction of our AVD-total-colouring of \(K_{r(n)}\), with \(r\) and \(n\) even, to certain subgraphs of \(K_{r(n)}\) is an AVD-total colouring for these subgraphs. Therefore, an extension of this work could
be the study of the conditions under which the restriction of AVD-total-colourings of complete equipartite graphs to their proper subgraphs results in AVD-total colourings for these subgraphs.

References


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