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DV-model that can be rooted

Marisa Gutierrez **Silvia B Tondato**

Abstract

A graph G is a directed (respectively rooted directed) path graph or a DV (RDV) graph if it admits a DV-model (RDV-model), i.e a clique tree T whose edges can be directed (directed from a root towards the leaves) such that for every vertex of G the subtree induced by the cliques that contain it is a directed subpath of T. An interval graph is the intersection graph of a family of subpaths of a path. Clearly, a path can be rooted on any of its leaves. In this work, we prove that if an interval graph has a DV-model with at least three leaves then there is a DV-model with exactly three leaves which can be rooted.

1 Introduction.

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [3] states that a graph G is chordal if and only if there is a tree T, called *clique tree*, whose vertices are the cliques of the graph and for every vertex x of G the cliques that contain x induce a subtree in the tree which we will denote by T_x. Note that G is the intersection graph of the subtrees $(T_x)_{x \in V(G)}$. Clique trees are also called *models* of the graph.

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A graph G is a *directed (respectively rooted directed) path graph* or a DV (RDV) graph if it admits a *DV-model* (*RDV-model*), i.e a clique tree T whose edges can be directed (directed from a root towards the leaves) such that T_x is a directed subpath of T for every $x \in V(G)$ [6]. We say that a DV-model T of a DV graph G can be rooted if T can be rooted on a vertex such that it becomes in RDV-model of G. It is clear that a graph is an *interval graph* if there is a clique tree which is a path. By definition we have the following inclusions between the different considered classes (and these inclusions are strict): interval $\subset$ rooted path $\subset$ directed path $\subset$ chordal.

Lekkerkerler and Boland [4] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple.

An *asteroidal triple* in a graph G is a set of three non-adjacent vertices (Figure 1: $\{3, 7, 13\}$) such that for any two of them there is a path between them that does not intersect the neighborhood of the third. An *asteroidal quadruple* is a set of four non-adjacent vertices such that any three of them is an asteroidal triple(Figure 1: $\{3, 7, 13, 17\}$).

Cameron, Hoáng and Lévêque [1] gave a conjecture which propose a characterization of RDV graphs in terms of forbidden asteroids: G is a DV non RDV graph if and only if there is an asteroidal quadruple a_1, a_2, a_3, a_4 such that a_1, a_2 and a_3, a_4 are linked by a special paths which force the direction in any DV-model between the cliques that contain to a_1, a_2 (a_3, a_4). Unfortunately, the special paths described are not the only ones who force the direction in any DV-model between those cliques. Gutierrez, Lévêque and Tondato [5] proved that every DV non RDV graph has an asteroidal quadruple. It is certainly too difficult to characterizing RDV graphs by forbidden induced subgraphs as there are too many (families of) graphs to exclude. Surprisingly the minimal forbidden families known have a DV-model with minimum number of leaves(*leafage*) equal to four what motivates the study of this subclass. In this context it can be proved that there is an asteroidal quadruple with special paths between its vertices. Then, a natural question arises: is there any interval graph that has

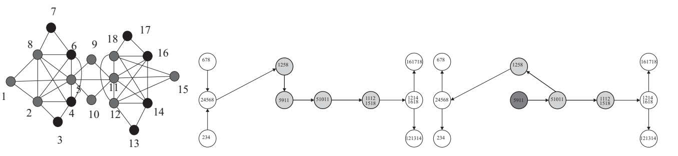

Figure 1: A DV graph, a DV-model and a RDV-model

a model than can be rooted with exactly three leaves ?
In Figure 1 we show a graph, a DV-model non RDV-model and a RDV-model obtained by changing an interval model of an induced interval subgraph of the graph (induced by the vertices highlighted in grey) by a RDV-model with three leaves.

In this paper, we prove the following result:

Theorem 1.1. Let G be an interval graph and there is a DV-model with at least three leaves then there is a DV-model with exactly three leaves.

The paper is organized as follows: in Section 2, we give some definitions and notation. In Section 3, we present properties about DV-model of interval graphs and we give a proof of Theorem 1.1.

2 Definitions and notation

A *clique* in a graph G is a maximal set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all cliques* of G. We use capital letters to refer to elements of $\boldsymbol{C}(G)$.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex is *simplicial* if its (closed) neighborhood is a clique.

Let T be a clique tree. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\cup_{X \in V(T')} X$.

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of

G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G+G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and whose edges are $E(G) \cup E(G')$.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimal subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X,Y]$ is the subpath of T between X and Y. Let $T[X,Y) = T[X,Y] \setminus Y$, $T(X,Y] = T[X,Y] \setminus X$ and $T(X,Y) = T[X,Y] \setminus \{X,Y\}$. A vertex $X \in V(T(Y,Z))$ has a *vertex crossing in* $T[Y,Z]$ if $X' \cap X'' \neq \emptyset$ where X' and X'' are the two neighbors of X in $T[Y,Z]$.

Let G be a chordal graph and T a clique tree. We will denote by $ln(T)$, the numbers of leaves of T. The *label* of an edge AB of T is defined as $lab(AB) = A \cap B$. We say that $X \in V(T)$ *dominates* $e \in E(T)$ if $lab(e) \subseteq X$ and $e' \in E(T)$ *dominates* $e \in E(T)$ if $lab(e) \subseteq lab(e')$. Recall that the labels of edges correspond to minimal separators of G. The *multiplicity* of e is the number of edges having its label and the *multiplicity* of a minimal separator S in a graph G is $c-1$ where c is the number of connected components of $G \setminus S$ having a vertex complete to S. It is clear that both multiplicity share. We will say that a graph has *multiplicity one* if it does not have minimal separators of multiplicity at least two. Observe that the graph has multiplicity one in every clique tree there are not two edges with the same label.

On the other hand, for each edge e of a clique tree, in every clique tree there is $\widetilde{e}$ such that $lab(e) = lab(\widetilde{e})$, we will say that e and $\widetilde{e}$ are *equivalents.*

Let T be a DV-model of G, we say that a leaf H of T is *good in* T if there is no edge of $T \setminus H$ dominated by H, otherwise H is *bad in* T.

Observation 1: Let T be a DV-model of a graph G having minimal separators of multiplicity at least two. Clearly, T has at least two edges $e_1 = A_1B_1, e_2 = A_2B_2$ with $B_1, A_2 \in V(T[A_1, B_2])$ such that $lab(e_1) = lab(e_2)$. Let T_1, T_2, T_3 be the connected components of $T - \{e_1, e_2\}$ being T_1 which contains A_1, T_2 which contains B_1, A_2 and T_3 which contains B_2. The vertices of $lab(e_1)$ are twins in G_{T_2}, they have the same neighbors in G_{T_2}. Let T_2' and T_{13} be DV-models of G_{T_2} and $G_{T_1+A_1B_2+T_3}$ respectively. Obviously, if $x \in lab(e_1)$, $T'_{2x} = T_2'[X,Y]$. Also, there is CD in T_{13} an equivalent edge to A_1B_2. Thus $T_{13} - CD + CX + T_2' + YD$ is a DV-model of G. In case that: G is an interval graph, T_{13} is an interval model, T_2 has at least three leaves and e_1, e_2 are chosen minimizing the distance then A_1 is always a leaf in every DV-model of $G_{T_2+e_1}$ and there is at least a vertex Z in T_2 such that $Z \cap lab(e_1) = \emptyset$. Hence, if T_2'' is an interval model of $G_{T_2+e_1}$ then X or Y (not both) is a leaf of $T_2'' - A_1$. Therefore $T_{13} - CD + CX + T_2'' - A_1 + YD$ is a DV-model of G with exactly three leaves.

3 DV-models for interval graphs

Lemma 3.1. *Let G be an interval graph and $A, B \in \mathbf{C}(G)$ such that $G \backslash A$ and $G \backslash B$ are connected graphs. If T is a DV-model of G with $ln(T) > 2$ then every leaf $H \neq A, B$ is bad and it has a dominated edge in $T[A, B]$.*

Lemma 3.2. *Let G be an interval graph with multiplicity one and $A, B \in \mathbf{C}(G)$ such that $G \backslash A$ and $G \backslash B$ are connected graphs. If T is a DV-model of G with $ln(T) > 2$, H is a bad leaf in T and e is a dominated edge by H in $T[A, B]$ then i) none edge of $T(e, H]$ can be dominated by a leaf different from H. ii) all edges of T dominated by H are in $T[A, B, H]$. iii) there is at most two leaves of T, H_i $i = 1, 2$ such that $T[A, B] \cap T[X, H_i] = \{X\}$ being X a vertex in $T[A, B]$. iv) if X is a vertex of T without a vertex crossing by X in $T[A, B]$ satisfying $T[A, B] \cap T[H, X] = \{X\}$ then every vertex of $T(X, e]$ has degree two.*

Proof. All proofs will be by way of contradiction. i) Suppose that $\exists$ e' an edge of $T(e, H]$ dominated by a leaf $H' \neq H$. As T is a DV-model then $e \in E(T[H', H])$. Clearly e, e', H appear in this order in $T[H', H]$ then $lab(e) = lab(e')$, a contradiction.
ii) Suppose that $\exists$ e' an edge dominated by H that is not in $T[A, B, H]$ then there is H', other leaf of T, and $X' \in V(T[A, B])$ such that $T[A, B] \cap T[H', X'] = \{X'\}$ and $e' \in E(T[H', X'])$, By Lemma 3.1, there is an edge $e'' \in E(T[A, B])$ dominated by H'. Clearly $e'' \in E(T[X', X])$. So there is

an edge e' in $T(e'', H']$ dominated by H contradicting i).
iii) Suppose that $\exists\ H_i,\ i = 1,2,3$. By Lemma 3.1, there are edges e_i in $T[A,B]$, $i = 1,2,3$ dominated by H_i, $i = 1,2,3$ respectively. Since T is a DV-model of G, $\exists\ i \in \{1,2,3\}\ \exists\ j \neq i \in \{1,2,3\}$ such that $e_i \in E(T[H_j, e_j])$ contradicting i).
iv) Suppose that there is X_1 a vertex of $T(X,e]$ with degree at least 3 and let H_1 be a leaf of T such that $T[A,B] \cap T[X_1,H_1] = \{X_1\}$. By Lemma 3.1, there is e_1 an edge of $T[A,B]$ dominated by H_1. As X does not have a vertex crossing in $T[A,B]$, $e_1 \in E(T[X,e])$ or $e \in E(T[e_1,X])$, contradicting i) or that T is a DV-model of G. ■

Lemma 3.3. *Let G be an interval graph and $A, B \in \mathbf{C}(G)$ such that $G \setminus A$ and $G \setminus B$ are connected graphs. If G has a DV-model with at least three leaves then there is a DV-model with exactly three leaves that can be rooted on the bad leaf.*

Proof. By Observations 1, it is sufficient to prove this for graphs with multiplicity one. As $G \setminus A$ and $G \setminus B$ are connected graphs in every DV-model of G, A and B are always leaves. Let T be a DV-model of G with at least three leaves and H a bad leaf in T. Let X be a vertex of T such that $T[A,B] \cap T[H,X] = \{X\}$. If there is a vertex crossing by X in $T[A,B]$ then we will built T' a DV-model of G which does not have a vertex crossing by X in $T'[A,B]$ as follows: by Lemma 3.1, $\exists\ e \in E(T[A,B])$ a dominated edge by H. In order to fix ideas suppose $e \in E(T[A,X])$ and $e = A_1B_1$ with $B_1 \in V(T[A_1,X])$. Then there is a vertex $y \in lab(e) \cap H$ that is a vertex crossing by X in $T[A,H]$. As G is a DV-model, it has not a 3-sun as induced subgraph ([2]) then there is no vertex crossing by X in $T[H,B]$. Let $T' = T - e + A_1H$. Thus T' is a DV-model of G without a vertex crossing by X in $T'[A,B]$. By the before exposed, we assume that T does not have a vertex crossing by some vertex of degree at least three of T in $T[A,B]$. If T has exactly three leaves then it can be rooted on the bad leaf. Let T be a *DV*-model of G such that there is $X \in V(T[A,B])$

without vertex crossing by its in $T[A,B]$ and maximizing $|V(T[A,B])|$. Let H be a leaf of T such that $T[A,B] \cap T[X,H] = \{X\}$.

Claim 1: Let X' be a vertex of T and H' a leaf of T such that $T[A,B] \cap T[X',H'] = \{H'\}$. i) If e is an edge of $T[X',B]$ ($T[X',A]$) dominated by H' which maximally farthest from H' then every vertex of $T(e,B]$ ($T(e,A]$) has degree 2. ii) $ln(T) = 3$.

All proofs will be by way of contradiction. i) Suppose that $T(e,B]$ has a vertex of degree at least three. Let $e = A'B'$ with $B' \in V(T[B,A'])$. Let T_1 be the connected component of $T - e$ containing B. By Lemma 3.2ii) and by the choice of e, none edge of T_1 is dominated by e. Let $T' = T_1 + e$. By the before exposed, A' is good in T' and as $G \setminus B$ is a connected graph, B is good in T'. Hence any model of $G_{T'}$ has A' and B as leaves. Let T_0 be an interval model of $G_{T'}$. Thus it is possible to built a DV-model of G, $T'' = T - T' + T_0$ that holds the hypothesis but with $|V(T''[A,B])|$ bigger than $|V(T[A,B])|$ a contradiction.

ii) Take X' minimizing the distance between X and X' in $T[A,B]$. Observe that X' may be X in this case $H \neq H'$. In order to fix ideas suppose that X', X appear in this order in $T[A,B]$. Let e and e' be edges dominated by H and H' respectively and maximally farthest from X and X' respectively. By Lemma 3.2iv) $e \notin T[A,X']$ by Claim 1i) $e \notin T[X',X]$. As there is no vertex crossing by X in $T[A,B]$, $e' \notin T[X,B]$ also by Claim 1i) $e' \notin T[X',X]$. Let $e = A'B'$ and $e' = A''B''$ with $B' \in V(T[A',B])$ and $A'' \in V(T[B'',A])$. In case $X \neq X'$, let C be a vertex of $T[X,X']$ incident in X' and T_1 the connected component of $T - CX'$ that contains A. As T is a DV-model and $e'' \in T_1[A,H']$, CX' does not dominate any edge of $T_1[A,H']$. Hence if CX' dominates an edge in T_1 then by Claim 1i) it must be in $T_1[H'',X'']$ being H'' a leaf of T_1 and $X'' \in T[B'',X']$. As T is a DV-model, H'' must have all dominated edges in $T[X',X]$ contradicting Claim 1i). Hence C is good in $T_1 + CX'$ and it is a leaf of every model of $G_{T_1+CX'}$. Let T_1' be an interval model of $G_{T_1+CX'}$. Thus $T - (T_1 + CX') + T_1'$ is a DV-model of G with three leaves, a contradiction.

In case $X = X'$. Take e and e' the nearest X. We will prove that

$B'' = X$ and $A' = X$. Suppose $B'' \neq X$. By the existence of e, B'' does not dominate edges in $T[H, B]$. As T maximizes $|V(T[A, B])|$, B'' does not dominate edges in $T[X, H']$. Hence, there is an asteroidal triple between the simplicial vertices of H', B and a vertex of $B'' - C$ being $CB'' \in E(T[B'', X])$, a contradiction. Therefore $B'' = X$, analogously $A' = X$. On the other hand, if $T[X, H] \cap T[X, H'] = \{X\}$, as T is a DV-model, there is no vertex crossing by X in $T[H', B]$. Then $T' = T - e' + A''H'$ is a DV-model of G satisfying hypothesis and the number of vertices in $T'[A, B]$ is greater than $T[A, B]$ a contradiction. If $T[X, H] \cap T[X, H'] = T[X, X''] \neq \{X\}$, there is no vertex crossing by X'' in $T[H, H']$. Thus $T' = T - \{e, e'\} + A''H' + B'H$ contradicts the election of T. Hence $ln(T) = 3$. ■

Proof of Theorem 1.1. Suppose there is a DV-model with at least three leaves. By Observation 1, it is sufficient to prove the theorem for graph with multiplicity one. If there are two cliques A and B in G such that $G \setminus A$ and $G \setminus B$ are connected graphs by Lemma 3.3, we are done. Suppose that there are not two cliques in these conditions. Let T be a DV-model of G with at least three leaves. If T has three leaves, we are done. Suppose that T has at least four leaves. Let A be a leaf of T such that $G \setminus A$ is not a connected graph. Clearly there is at least an edge in $T - A$ dominated by A. Let $e = A_1A'_1$ be a dominated edge by A maximality farthest from A and $A_1 \in V(T[A, A'_1])$. Let T_1 and T'_1 be the connected components of $T - e$ that contains A_1 and A'_1 respectively. As G has multiplicity one A'_1 is always a leaf in every clique tree of G_{T_1+e}. By the election of e, A_1 is always a leaf in every clique tree of $G_{T'_1+e}$. If there is a vertex of degree at least three in T_1, let $\overline{T}_1$ be an interval model of G_{T_1+e} with $A'_1X \in E(\overline{T}_1)$ and $\overline{T}'_1$ an interval model of $G_{T'_1+e}$ with $A_1Y \in E(\overline{T}'_1)$. Thus $\overline{T}_1 - A'_1 + XY + \overline{T}'_1 - A_1$ is a DV-model of G with exactly three leaves by Observation 1.

Suppose there is no vertex of degree at least three in T_1. If there is a clique $B \neq A_1$ in $G_{T'_1+e}$ such that $G_{T'_1+e} \setminus B$ is a connected graph by Lemma 3.3, there is a $\overline{T}'_1$ a DV-model of $G_{T'_1+e}$ with three leaves being

A_1 one of them. Thus $T_1 + \overline{T}'_1$ is a DV-model of G with exactly three leaves. If there is no clique $B \neq A_1$ in $G_{T'_1+e}$ such that $G_{T'_1+e} \setminus B$ is a connected graph we complete the prove applying the reasoning of the second paragraph of the proof to $T'_1 + e$, $G_{T'_1+e}$ and B instead of T, G and A. □

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Marisa Gutierrez
Departamento de
Matemática,
FCE-UNLP
Argentina
marisa@mate.unlp.edu.ar

Silvia B Tondato
Departamento de
Matemática,
FCE-UNLP
Argentina
tondato@mate.unlp.edu.ar