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Strong reducibility of Solitaire Clobber played on Cartesian product of graphs

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Abstract

Solitaire Clobber is a game played on a graph G by placing a stone, black or white, on each vertex of the graph. We pick up a stone and clobber another one of opposite color located on an adjacent vertex; the clobbered stone is removed from the graph and it is replaced by the picked one. The goal is to find a succession of moves that minimizes the number of remaining stones, when no move is possible. In 2008, Dorbec et al. proposed a more restrictive question related to determining the color and the location of the remaining stones. A graph G is strongly 1-reducible if, for any vertex v of G, for any arrangement of stones on G such that $G \setminus v$ is non-monochromatic, and for any color c, there exists a succession of moves that yields a single stone of color c on v. This question was studied by those authors for multiple cartesian product of cliques (Hamming graphs). In this paper, we show a generalization of this result by proving that if G and H are two strongly 1-reducible connected graphs (both graphs with at least seven vertices), then the cartesian product $G \square H$ is strongly 1-reducible.

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1 Introduction

Albert et al. [1] introduced a combinatorial game called Clobber which is played as follows. Black and White stones are placed on a subset of squares of an n x m checkerboard. The two players, Black and White, move alternately by picking up one of their own stones and clobbering an opponent's stone on a horizontally or vertically adjacent square. The clobbered stone is removed from the board and it is replaced by the picked one. The game ends when one player, on their turn, is unable to move, and then this player loses.

We investigate Solitaire Clobber, a 1-player version of the game, which was proposed by Demaine et al. [4]. We play Solitaire Clobber on a graph G by placing a stone, black or white, on each vertex of the graph. A move consists of picking a stone and clobbering another one of opposite color located on an adjacent vertex. The clobbered stone is removed from the graph and it is replaced by the picked one. Now, the vertex with no stone is deleted from the graph (together with their induced incident edges). The goal is to find a succession of moves that minimizes the number of remaining stones, when no further move is possible.

Let $G = (V, E)$ be a simple graph where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. A configuration Φ of a graph G is a mapping from V to $\Phi : V \rightarrow \{\bullet, \circ\}$. We say that (G, Φ) is k-reducible (for a positive integer k) if there exists a succession of moves that leaves k stones on the graph.

Many authors have been studied the Clobber game. For example, one of the contributions of Blondel et al. [3] was the proof that Clobber is equivalent to an optimization problem on a set of words. This helped Duchêne et al. [6] to present some new results on Solitaire Clobber, mostly regarding the complexity aspect of the game. More results on this subject can be found in the survey by Beaudou et al [2].

It was shown by Itaí et al. [7] that the HAMILTONIAN PATH problem is NP-complete on graphs in general, and for grid graphs in particular. Since there is a reduction from the HAMILTONIAN PATH problem for grid graphs to the problem of deciding 1-reducibility of a given pair (G, Φ), SOLITAIRE CLOBBER is NP-hard for grid graphs.

We need some few definitions before introducing one more complex question about Solitaire Clobber proposed by Dorbec et al. [5].

Given a configuration Φ of G, if $\Phi(v) = \circ$ (resp. $\bullet$), for all $v \in V(G)$, then Φ is called *monochromatic*; otherwise, Φ is *non-monochromatic.*

The cartesian product of G and H, written $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ with (u, v) adjacent to (u', v') if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Graph $G\Box H$ is composed by $|V(G)| = n_G$ vertical copies of H, called H_j, and $|V(H)| = n_H$ horizontal copies of G, called G_i, $1 \leq i \leq n_H$, $1 \leq j \leq n_G$. We say that vertex $(i, j) \in V(G\Box H)$ is located in line G_i and column H_j.

In 2008, Dorbec et al. [5] posed the following question: for any vertex v of G, for any configuration of G (provided $G \setminus v$ is non-monochromatic), for any color c (black or white), does there exist a way to play that yields a single stone of color c on v? If the answer is yes, then the graph G is strongly 1-reducible. The authors have proved that all cliques of size $n \geq 3$ are strongly 1-reducible. Also, they have shown the result below concerning the cartesian product of a clique and a strongly 1-reducible graph G.

Theorem 1.1. *[5] Let G be a strongly 1-reducible graph containing at least four vertices. Then for any positive integer n, $K_n\Box G$ is strongly 1-reducible.* ■

In our previous work [9], we have proved that powers of paths P_n^r and powers of cycles C_n^r, $r \geq 3$ are strongly 1-reducible. In this work we show a generalization of Theorem 1.1 [5] by proving the following result:

Theorem 1.2. *Let G and H be two strongly 1-reducible connected graphs, with $|V(G)| \geq |V(H)| \geq 7$, then $G\Box H$ is strongly 1-reducible.*

Next section presents notation and our main results. Due to space limit, some proofs will be omitted. We conclude with some remarks about the small cases.

2 Main Results

A color c is *rare* on G if there exists exactly one vertex $v \in V(G)$ such that $\Phi(v) = c$. We can also say that vertex v is rare in Φ. In this case, we have a *quasi-monochromatic* configuration, or more precisely, a *quasi-monochromatic* configuration on vertex v. A *non-quasi-monochromatic* configuration Φ of G is a configuration with no rare color on G. We remark that a *non-quasi-monochromatic configuration* Φ *on vertex* v is a configuration in which the rare color is not in v, that is, Φ can be quasi-monochromatic with rare color in $v' \neq v$, $v' \in V(G)$.

We say that (G, Φ) is $(1, v, c)$-*reducible* if (G, Φ) is 1-reducible on v with color c. Therefore G is *strongly* 1-*reducible* if (G, Φ) is $(1, v, c)$-reducible for all v and for all c such that $G \setminus v$ is non-monochromatic.

Let G' be a subgraph of $G \Box H$. We call $\Phi_{G'}$ the restriction of Φ to the vertices of $V(G')$. If $c = \circ$ (resp. $\bullet$), then $\bar{c} = \bullet$ (resp. $\circ$). From now on, $|V(H)| \geq |V(G)| \geq 7$ and, without loss of generality, we consider that vertex v, on which we have to leave the last stone, is in line G_1 and column H_1, i.e., $v = (1, 1)$.

Lemma 2.1. Let G and H be two strongly 1-reducible connected graphs of order at least seven. Let Φ of $G \Box H$ be non-monochromatic. If there exist $i, i' \in \{1, \ldots, n_H\}$, such that for all $u \in V(G_i - (i, 1))$, $\Phi(u) = \bullet$ and, for all $u' \in V(G_{i'} - (i', 1))$, $\Phi(u') = \circ$, then $G \Box H$ is $(1, v, c^*)$-reducible.

Proof. Let G and H be two strongly 1-reducible graphs with at least seven vertices. Assume that v is the vertex on which we leave the last stone. Let $c^* \in \{\bullet, \circ\}$ be the color of vertex v in the end of the game. We consider any arrangement of stones such that for all $u \in V(G_i - (i, 1))$, $\Phi(u) = \bullet$ and, for all $u' \in V(G_{i'} - (i', 1))$, $\Phi(u') = \circ$, $i, i' \in \{1, \ldots, n_H\}$. We play on H_j columns, $j \neq 1$, clobbering and reducing them to a unique line $G_{i''}$, and then we play on $G_{i''}$ reducing it to one stone of color c' on vertex $(i'', 1)$. Finally, we play on column H_1 which contains v. We have three different cases.

If $\Phi(i, 1) = \bullet$ and $\Phi(i', 1) = \circ$ then let $i'' \in \{1, \ldots, n_H\}$ such that $i'' \neq i$, $i'' \neq i'$ and w.l.o.g. let $c' = \bullet$. For all $j > 1$, (H_j, Φ_{H_j}) is $(1, (i'', j), c)$-reducible for any chosen $c \in \{\circ, \bullet\}$ such that Φ restricted to $V(G_{i''} - (i'', 1))$ is non-monochromatic. Thus $(G_{i''}, \Phi'_{G_{i''}})$ is $(1, (i'', 1), c')$-reducible. Hence, (H_1, Φ'_{H_1}) is $(1, v, c^*)$-reducible. This case is depicted in Figure 1.

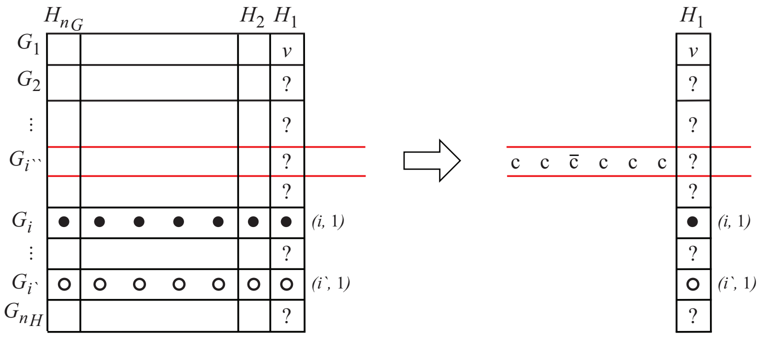

Figure 1: Case $\Phi(i, 1) = \bullet$ and $\Phi(i', 1) = \circ$.

Otherwise suppose w.l.o.g. $\Phi(i, 1) = \circ$ and $\Phi(i', 1) = \circ$. If there exists $l \in \{1, \ldots, n_H\} - \{1, i, i'\}$ such that $\Phi(l, 1) = \circ$ then we set $i'' = l$ and $c' = \bullet$, and we proceed as in the previous case.

Else, for any $l \in \{1, \ldots, n_H\} - \{1, i, i'\}$, we have $\Phi(l, 1) = \bullet$. Let $i'' = 1$ and $c' = \circ$. Now, for all $j > 2$, (H_j, Φ_{H_j}) is $(1, (i'', j), \bullet)$-reducible and (H_2, Φ_{H_2}) is $(1, (i'', j), \circ)$-reducible. Thus $(G_{i''}, \Phi'_{G_{i''}})$ is $(1, (i'', j), c')$-reducible. Hence the choice of i'' and c' guarantee that (H_1, Φ'_{H_1}), is $(1, v, c^*)$-reducible. ■

Corollary 2.1. Let G and H be two strongly 1-reducible connected graphs of order at least seven. Let Φ of $G \square H$ be non-monochromatic. If there exist $j, j' \in \{1, \ldots, n_G\}$, such that for all $u \in V(H_j - (1, j))$, $\Phi(u) = \bullet$ and, for all $u' \in V(H_{j'} - (1, j'))$, $\Phi(u') = \circ$, then $G \square H$ is

$(1, v, c^*)$-reducible. ■

Thanks to Lemma 2.1, one may assume that, for any pair (i, i'), if for all $u \in V(G_i - (i,1))$, $\Phi(u) = c_1$ and, for all $u' \in V(G_{i'} - (i',1))$, $\Phi(u') = c_2$, then $c_1 = c_2$, with $i, i' \in \{1, \dots, n_H\}$, c_1, $c_2 \in \{\bullet, \circ\}$. Similarly for H_j.

If for all $u \in V(G_i - (i,1))$, $\Phi(u) = c_1$, and for all $u' \in V(H_j - (1,j))$, $\Phi(u') = c_1$, then Φ of $G \setminus v$ is monochromatic (contradiction), $i \in \{1, \dots, n_H\}$, $j \in \{1, \dots, n_G\}$, $c_1 \in \{\bullet, \circ\}$. Therefore, one may assume that there exists $i \in \{1, \dots, n_H\}$ and u, $u' \in V(G_i - (i,1))$ such that $\Phi(u) = \circ$ and $\Phi(u') = \bullet$. Thus, there exists $i \in \{1, \dots, n_H\}$ such that (G_i, Φ_{G_i}) is $(1, (i,1), c)$-reducible for any $c \in \{\bullet, \circ\}$.

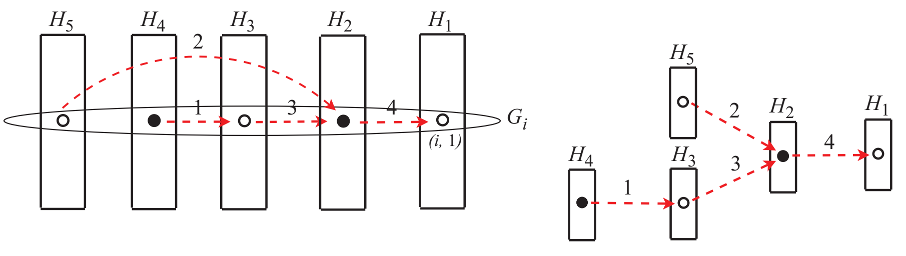

Figure 2: Example of a sequence M of moves in (G_i, Φ_{G_i}).

Let Φ_{G_i} be non-monochromatic and non-quasi-monochromatic on vertex $(i, 1)$. Then (G_i, Φ_{G_i}) is $(1, (i,1), c)$-reducible, for all $c \in \{\bullet, \circ\}$. Let M be a sequence of moves in (G_i, Φ_{G_i}) corresponding to a $(1, (i,1), c)$-reduction. Observe that M induces a spanning tree T of G_i rooted on $(i, 1)$. See Figure 2.

The idea is to clobber the columns according to sequence M and to the maximal quasi-stars of T, which we define next.

We define a *star* s with center m as a digraph with vertex set $V(s) = \{i_1, \dots, i_p, m, o\}$ with arcs $\overrightarrow{i_j m}$ and $\overrightarrow{mo}$, $1 \leq j \leq p$. A model of star is depicted in Figure 3. A *quasi-star* is either a star or a star minus vertex o.

We can partition the edges of T into stars s_1, s_2, $\dots$, s_{t-1} plus a quasi-

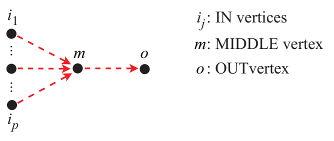

Figure 3: A star s.

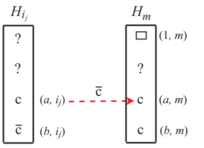

Figure 4: Playing from H_{i_j} to H_m.

star s_t. Let s_j be the maximal quasi-star with center m in T such that m is a vertex where each vertex i_j of arc $\overrightarrow{i_j m}$ is a leaf in the subgraph induced by $T - \bigcup_{e \in s_\ell} e$, $\ell < j$. Observe that any arc (move) of T belongs to exactly one star of $s_1, \ldots, s_t$ and that each vertex i_j is associated with column H_{i_j}. Moreover, all s_j $(j < t)$ are stars and s_t is a quasi-star which is not a star.

Now, for induction purposes, let $S_k = \{s_k, \ldots, s_t\}$, $1 < k \leq t$, be the set of stars plus the quasi-star s_t. Let T^k be the subgraph of T induced by vertices belonging to $\bigcup_{j=k}^{t} s_j$ and let G^k be the subgraph induced by the vertices of T^k. Let Φ^k be an assignment of $V(G^k)$ to $\{\bullet, \circ\}$ which satisfies the following property (P).

(P): If configuration $\Phi^k_{H_j}$ is monochromatic or quasi-monochromatic on (i, j), then $\Phi^k_{H_j} = \Phi_{H_j}$.

This property states that if, in an inductive step k, $\Phi^k_{H_j}$ is monochromatic then this is its original configuration Φ_{H_j}, which means that our strategy never turns a configuration of a column H_j monochromatic or quasi-monochromatic on (i, j). In addition, we observe that each time we modify a configuration of a column then it turns non-monochromatic and non-quasi-monochromatic on (i, j).

We consider vertex v the vertex on which we leave the last stone and let $c^* \in \{\bullet, \circ\}$ be the color of vertex v in the end of the game. In the next Claim we show how to play on the maximal quasi-stars, according to the columns represented by each vertex i_j. We show strategies to clobber columns H_{i_j} in such a way that the last one, H_m or H_o, is non-monochromatic and non-quasi-monochromatic on v.

Claim 2.1. $(G^k \Box H, \Phi^k_H)$ is $(1, v, c^*)$-reducible.

Sketch of Proof. The proof works by induction on k. Let s_k be a quasi-star in T^k. Let $\{i_1, \ldots, i_p, m\}$ be the vertices of s_k (if $t > 1$, then s_k contains also a vertex o). We consider the following cases.

Case 1. First assume that each $\Phi^k_{H_{i_j}}$ is non-monochromatic, $i_j \in \{1, \ldots, p\}$.

Play, step by step, from H_{i_j} to H_m in such a way that the resulting labeling of H_m is non-monochromatic and non-quasi-monochromatic on vertex $(1, m)$. See Figure 4. For instance, choose two vertices (a, m) and (b, m) distinct from $(1, m)$ and having the same color c, with c in $\{\bullet, \circ\}$. These vertices exist due to the fact that $|V(H)| \geq 7$.

By hypothesis, $(H_{i_j}, \Phi^k_{H_{i_j}})$ is either $(1, (a, i_j), \bar{c})$-reducible or $(1, (b, i_j), \bar{c})$-reducible. Suppose that $(H_{i_j}, \Phi^k_{H_{i_j}})$ is $(1, (a, i_j), \bar{c})$-reducible. Then, do the corresponding moves. Move from (a, i_j) to (a, m). We get a configuration of H_m that is non-monochromatic and non-quasi-monochromatic on $(1, m)$. One can iterate this process for all $i_j \in \{2, \ldots, p\}$. Let Φ'_{H_m} be the resulting configuration.

Now, to achieve the moves of s_1, we consider the next cases $t > 1$ or $t = 1$.

Case 1a. If $t = 1$, then $v = (1, m)$. Since (H_m, Φ'_{H_m}) is $(1, (1, m), c^*)$-reducible, this completes the proof in this case.

Case 1b. If $t > 1$, then (again) let (a, o) and (b, o) be two vertices of H_o, distinct from $(1, o)$ and having the same color c. Without loss of generality, one may assume that (H_m, Φ'_{H_m}) is $(1, (a, m), \bar{c})$-reducible. Then do the corresponding moves. Move from (a, m) to (a, o). We get a labeling of H_o satisfying (P). Conclude by applying the induction hypothesis on $s_{k+1}, ..., s_t$, with $\Phi^{k+1} = \Phi^k$, except for (a, m).

Case 2. Assume now that there is some $r \leq p$ such that $\Phi^k_{H_{i_j}}$ is monochromatic $\bullet$, for all $i_j \leq r$.

We define $((x,y),c) \rightarrow ((x',y'),\bar{c})$ as a move from vertex (x,y) with color c to vertex (x',y') with color $\bar{c}$. Our goal is to clobber these r monochromatic columns H_{i_j} by producing a sequence that reduces (H_m, Φ_{H_m}) with r moves $\circ \rightarrow \bullet$ or $\bullet \rightarrow \circ \rightarrow \circ$. We proceed by guaranteeing a sufficient number of black stones in H_m, by using the techniques below and by applying the induction hypothesis.

A move $((a,m),\circ) \rightarrow ((b,m),\bullet)$ in H_m, i.e. $\circ \rightarrow \bullet$ for short, is played as follows. We refer to Figure 5. Move $((a,m),\circ) \rightarrow ((a,i_j),\bullet)$ with $i_j \leq r$. Now consider the $(1,(b,i_j),\circ)$-reduction of $(H_{i_j}, \Phi_{H_{i_j}})$. Move $((b,i_j),\circ) \rightarrow ((b,m),\bullet)$.

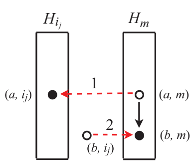

Figure 5: A move $\circ \rightarrow \bullet$.

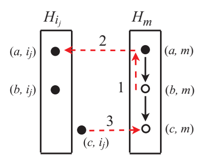

Figure 6: A move $\bullet \rightarrow \circ \rightarrow \circ$.

A move $((a,m),\bullet) \rightarrow ((b,m),\circ) \rightarrow ((c,m),\circ)$ in H_m, i.e. $\bullet \rightarrow \circ \rightarrow \circ$ for short, is played as follows. We refer to Figure 6. Move $((b,m),\circ) \rightarrow ((a,m),\bullet)$ and then $((a,m),\circ) \rightarrow ((a,i_j),\bullet)$ with $i_j \leq r$. Now consider the $(1,(c,i_j),\bullet)$-reduction of $(H_{i_j}, \Phi_{H_{i_j}})$. Move $((c,i_j),\bullet) \rightarrow ((c,m),\circ)$.

Similarly to *Case 1*, we consider the following cases: *Case 2a* $(t > 1)$ and *Case 2b* $(t = 1)$. Finally, we conclude this sketch of the proof by showing the last technique used in case *Case 2b*.

A move $((a,m),\bullet) \rightarrow ((b,m),\circ)$ in H_m, i.e. $\bullet \rightarrow \circ$ for short, is defined if there exists some $i_{j'} > r$ such that $(H_{i_{j'}}, \Phi^k_{H_{i_{j'}}})$ is non-quasi-monochromatic on $(a,i_{j'})$. First, consider a $(1,(a,i_{j'}),\circ)$-reduction of $(H_{i_{j'}}, \Phi^k_{H_{i_{j'}}})$. We refer to Figure 7. Move $((a,i_{j'}),\circ) \rightarrow ((a,m),\bullet)$ and then $((a,m),\circ) \rightarrow ((a,i_j),\bullet)$ for some $i_j \leq r$. Then, apply a $(1,(b,i_j),\bullet)$-reduction of $(H_{i_j}, \Phi^k_{H_{i_j}})$ and move $((b,i_j),\bullet) \rightarrow ((b,m),\circ)$.

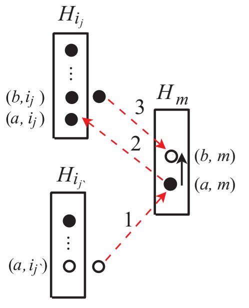

Figure 7: A move $\bullet \to \circ$.

$\blacksquare$

Our main result, Theorem 1.2, is a direct consequence of Claim 2.1.

3 Conclusion

The strong reducibility of Solitaire Clobber was studied for certain classes of graphs and the state of the art can be summarized in Table 1.

Graph classes	References
Hamming graphs, except hypercubes and $K_2 \square K_3$	Dorbec et al. [5]
All cliques of size $n \geq 3$	Dorbec et al. [5]
Powers of cycles $C_n^r, r \geq 3$	Pará et al. [8]
$G \square K_n, n \geq 4$, where G is $S1R$ with at least 4 vertices	Dorbec et al. [5]
$G \square H$, where both G and H are $S1R$ with at least 7 vertices	This paper

Table 1: Strongly 1-Reducible (S1R) graph classes.

In this paper we complete Table 1 by proving that if G and H are two connected S1R graphs with at least seven vertices, then the Solitaire Clobber game played on $G \square H$ is strongly 1-reducible. We verify that if $3 \leq |V(G)| \leq 7$ and $3 \leq |V(H)| \leq 7$, then Theorem 1.2 remains true; and these small cases can be checked by hand. There are graphs which admit a coloring for which they are not strongly 1-reducible: $K_2 \square K_2$ and

$K_2 \square K_3$, as shown by Dorbec et al. [5]. And, thanks to Theorem 1.1, if $|V(G)| = 2$ and $|V(H)| \geq 4$ then $G \square H$ is strongly 1-reducible.

References

[1] Albert, M. H., Grossman, J. P., Nowakowsky, R. J. and Wolfe, D., *An Introduction to Clobber*, Integers, Journal of Combinatorial Number Theory, **5(2)**, A1, pp. 1–12, 2005.

[2] Beaudou, L., Duchêne, E. and Gravier, S., *A Survey about Solitaire Clobber*, Games of No Chance 4 (to appear).

[3] Blondel, V.D., Kerchove, C. de, Hendrickx, J. M., and Jungers, R., *Solitaire Clobber as an Optimization Problem on Words*, Integers, Journal of Combinatorial Number Theory, **8(1)**, G04, pp. 1–12, 2008.

[4] Demaine, E. D., Demaine, M. L., and Fleischer, R., *Solitaire Clobber*, Theoretical Computer Science, **313**, pp. 325–338, 2004.

[5] Dorbec, P., Duchêne, E. and Gravier, S., *Solitaire Clobber played on Hamming graphs*, Integers, Journal of Combinatorial Number Theory, **8(1)**, G03, pp. 1–21, 2008.

[6] Duchêne, E., Gravier, S. and Moncel, J., *Solitaire Clobber on graphs*, RAIRO, Theoretical Informatics and Applications, **43**, pp. 463–482, 2009.

[7] Itaí, A., Papadimitriou, C. H. and Szwarcfiter, J. L., *Hamilton paths in grid graphs*, SIAM Journal of Computing, **11**, pp. 676–686, 1982.

[8] Pará, T., Gravier, S., and Dantas, S., *Strong reducibility of powers of paths and powers of cycles on Impartial Solitaire Clobber*, Eletronic Notes in Discrete Mathematics, **37**, pp. 177–182, 2011.

[9] Pará, T., Dantas, S. and Gravier, S., *Solitaire Clobber on Circulant Graphs*, Accepted for publication in Discrete Mathematics, 2013.

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