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(k,ℓ)-Sandwich Problems: why not ask for special kinds of bread?

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Abstract

In this work, we consider the Golumbic, Kaplan, and Shamir decision sandwich problem for a property Π: given two graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, the question is: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G satisfies Π? The graph G is called *sandwich graph*. Note that what matters here is just the "filling" of the sandwich. Our proposal is to try different kinds of "bread" for each chosen special sandwich filling. In other words, we focus on the complexity of sandwich problems when, beforehand, it is known that G^i satisfies a property Π^i, $i = 1, 2$. Let (Π^1, Π, Π^2)-SP denote the sandwich problem for property Π when G^i satisfies Π^i, called *sandwich problem with boundary conditions*. When G^i is not required to satisfy any special property, Π^i is denoted by $*$. A graph G is (k,ℓ) if there is a partition of $V(G)$ into k independent sets and ℓ cliques. It is known that $(*, (k,\ell), *)$-SP is NP-complete, for all $k+\ell$ greater than 2. In order to motivate this new work proposal, in this paper we describe polynomial-time algorithms for three sandwich problems with boundary conditions: (PERFECT, (k,ℓ), POLY-

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NOMIAL NUMBER OF MAXIMAL CLIQUES)-SP for all $k, \ell \in \mathbb{N}$, ($*$, $(2, 1)$, TRIANGLE-FREE)-SP, and ($*$, $(2, 1)$, BOUNDED DEGREE)-SP. The first problem includes the case (CHORDAL, (k, ℓ), CHORDAL)-SP.

1 Introduction

Golumbic, Kaplan and Shamir introduced in [15] the GRAPH SANDWICH PROBLEM FOR PROPERTY Π in its original form, as follows:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: Graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, such that $E^1 \subseteq E^2$.
Question: Is there $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G satisfies Π?

We call E^1 the set of *mandatory* edges, $E^2 \setminus E^1$ the set of *optional* edges, and $E(\overline{G^2})$ the set of *forbidden* edges. Hence, any sandwich graph $G = (V, E)$ for the pair G^1, G^2 must contain all mandatory edges and no forbidden edges. Graph sandwich problems have drawn much attention because they naturally generalize graph recognition problems and have many applications [8, 9, 10, 14, 19].

After studying many of them, we started questioning why not choose special kinds of "bread" for a particular special stuffed sandwich? After all, we can change its taste just changing its bread. So, we propose a generalized version of sandwich problems, that we call *sandwich problem with boundary conditions*, denoted by (Π^1, Π ,Π^2)-SP in which the input graphs G^1 and G^2 satisfy properties Π^1 and Π^2, respectively. We can formalize it as follows:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π WITH BOUNDARY CONDITIONS

Instance: Graphs $G^i = (V, E^i)$ satisfying Π^i, $i = 1, 2$, such that $E^1 \subseteq E^2$.
Question: Is there $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G satisfies Π?

If G^i is not required to satisfy any special property, we denote Π^i by $*$. The *recognition problem* for a class of graphs $\mathcal{C}$ is equivalent to a particular graph sandwich problem where $E^1 = E^2$. In this case, the goal is to decide wether $G^1 = G^2 = G$ satisfies Π, where Π is the property of belonging to $\mathcal{C}$. Golumbic et al. [15] remark that $(*, \Pi, *)$-SP is interesting when property Π is polynomially recognizable. In contrast, we observe that (Π^1, Π, Π^2)-SP is interesting both when $(*, \Pi, *)$-SP is polynomially solvable and when $(*, \Pi, *)$-SP is NP-complete:

- If $(*, \Pi, *)$-SP is polynomially solvable then there may exist algorithms with better complexity to solve (Π^1, Π, Π^2)-SP, using properties Π^1 or Π^2;

- If $(*, \Pi, *)$-SP is NP-complete, then there still remains the possibility that (Π^1, Π, Π^2)-SP admits a polynomial-time algorithm, since (Π^1, Π, Π^2)-SP is not more difficult than $(*, \Pi, *)$-SP.

In this paper we consider the following situation: $(*, \Pi, *)$-SP is NP-complete and (Π^1, Π, Π^2)-SP is polynomially solvable.

We focus on a particular property Π related to the (k, ℓ)-partition problem [11, 12]. A graph G is (k, ℓ) if $V(G)$ can be partitioned into k stable sets and ℓ cliques. In [1, 2, 4, 12], the problem of recognizing (k, ℓ)-graphs was shown to be NP-complete if $k \geq 3$ or $\ell \geq 3$ and polynomially solvable otherwise, while the problem $(*, (k, \ell), *)$-SP was shown to be NP-complete if $k + \ell \geq 3$ [7] and polynomially solvable otherwise.

A *perfect graph* G is a graph in which the chromatic number of every induced subgraph H is equal to the size of the largest clique of H. Bipartite, chordal, strongly chordal, comparability graphs and cographs are important classes of perfect graphs [16]. PERFECT GRAPH RECOGNITION is in P [5], and coloring a perfect graph G with $\chi(G)$ colors (the chromatic number of G) is a polynomial problem [3].

In Section 2, we study three different sandwich problems with boundary conditions: (PERFECT, (k, ℓ), POLYNOMIAL NUMBER OF MAXIMAL CLIQUES)-SP, $(*$, $(2, 1)$, TRIANGLE-FREE)-SP, and $(*$, $(2, 1)$, BOUNDED DEGREE)-SP. We will prove that these problems can be solved in polynomial time.

Several sandwich problems with boundary conditions fit on the framework (PERFECT, (k,ℓ), POLYNOMIAL NUMBER OF MAXIMAL CLIQUES)-SP , since there are many subclasses of perfect graphs and other important ones having a polynomial number of maximal cliques, such as chordal, triangle free, and bounded degree graphs. Therefore, as a byproduct, the problems (CHORDAL, (k,ℓ), CHORDAL)-SP, (COMPARABILITY, (k,ℓ), BOUNDED DEGREE)-SP, (COGRAPH, (k,ℓ), STRONGLY CHORDAL)-SP are examples of polynomially solvable problems.

2 Three Sandwich Problems with Boundary Conditions

Let $k, \ell \geq 0$ be fixed integers.

Definition 2.1. POLYNOMIAL NUMBER OF MAXIMAL CLIQUES, or simply PNMC, stands for any infinite family of graphs for which there exists a polynomial $q(n)$ such that the number of maximal cliques of any graph G in the family is bounded by $O(q(n))$, where $n = |V(G)|$. As an example, PNMC may stand for chordal graphs, by taking $q(n) = n - 1$.

Theorem 2.1. (PERFECT, (k,ℓ), PNMC)-SP is in P.

The proof of Theorem 2.1 is based on Algorithm 1.

Algorithm 1: Algorithm for solving (PERFECT, (k,ℓ), PNMC)-SP

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Let 𝒞 be the collection of maximal cliques of G_2 ;
for each subcollection {C_1, C_2, ··· , C_ℓ} of 𝒞 do
    let C' = V(C_1) ∪ V(C_2) ∪ ··· ∪ V(C_ℓ) ;
    if G^1\C' is k-colorable then
        return G = (V, E_1 ∪ E(C_1) ∪ ··· ∪ E(C_ℓ))
return there is no (k,ℓ)-graph G such that E^1 ⊆ E(G) ⊆ E^2
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Now we consider the problem ($*$, (2,1), TRIANGLE-FREE)-SP:

Theorem 2.2. (∗, (2,1), TRIANGLE-FREE)-SP is in P.

Algorithm 2: Algorithm for solving $(*, (2,1)$, TRIANGLE-FREE$)$.

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if G^1 ≠ G^2 then
    for each (u,v) ∈ E^2\E^1 do
        V' := V\{u,v} ;
        G' = G^1[V'] ;
        if G' is bipartite then
            return G = (V, E^1 ∪ {(u,v)})
    return there is no (2,1)-graph G such that E^1 ⊆ E(G) ⊆ E^2
if G^1 is (2,1) then
    return G = G^1 = G^2
return there is no (2,1)-graph G such that E^1 ⊆ E(G) ⊆ E^2
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Algorithm 2 solves (∗, (2,1), TRIANGLE-FREE) and runs in $O((n+m)m)$ time.

Finally, we state the following result:

Theorem 2.3. (∗ , (2,1), BOUNDED DEGREE)-SP *is in P.*

Algorithm 3 solves (∗ , (2,1), BOUNDED DEGREE)-SP by listing all maximal cliques of G^2 and testing, for each maximal clique, if the deletion of its vertices in G^1 yields a bipartite graph. It runs in $O(mn^{k+1})$ time, where $k = \Delta(G^2)$.

Algorithm 3: Algorithm for solving ($*$,(2,1), BOUNDED DEGREE)-SP

let $\mathcal{C} = \{C_1, \cdots, C_l\}$ be the collection of maximal cliques of G^2;
for *each* $C_i \in \mathcal{C}$ **do**
  if $G^1 \backslash V(C_i)$ *is bipartite* **then**
    return $G = (V, E^1 \cup E(C_i))$
return *there is no* $(2,1)$*-sandwich graph* $G = (V, E)$ *such that* $E^1 \subseteq E \subseteq E^2$

We can generalize this result to graphs G^2 having a polynomial number of maximal cliques, that can be listed in polynomial time (see for instance [20]).

Theorem 2.4. $(*, (2, 1)$, PNMC)-SP is in P.

3 Conclusions

We observe that $(k, \ell)-$CHORDAL is a polynomially recognizable property [13, 17, 18]. Recently the versions ($*$, CHORDAL$-(2, 1), *$)-SP and ($*$, STRONGLY CHORDAL-$(2, 1), *$)-SP have been proved to be NP-complete [6]. In an ongoing work, we are trying to choose properties Π^1 and Π^2 to analyze the complexity of (Π^1, CHORDAL-$(2, 1), \Pi^2$)-SP and (Π^1, STRONGLY CHORDAL-$(2, 1), \Pi^2$)-SP.

In Tables 1 and 2 we summarize the results of this paper and the corresponding results in the literature.

$(\Pi_1, (2,1), \Pi_2)$				
$\Pi_1 \backslash \Pi_2$	CHORDAL	TRIANGLE-FREE	$\Delta = k$	$*$
CHORDAL	$O(mn)$	$O(m^2)$	$O(mn^{k+1})$	?
TRIANGLE-FREE	$O(mn)$	$O(m^2)$	$O(mn^{k+1})$	?
$\Delta = k$	$O(mn)$	$O(m^2)$	$O(mn^{k+1})$	?
$*$	$O(mn)$	$O(m^2)$	$O(mn^{k+1})$	NP-c [7]

Table 1: Complexity results and open problems for $(\Pi_1, (2,1), \Pi_2)$, where properties Π_1, Π_2 are in {CHORDAL, TRIANGLE-FREE, BOUNDED DEGREE k , $*$}.

$(\Pi_1, (k,\ell), \Pi_2)$		
$\Pi_1 \backslash \Pi_2$	PNMC	$*$
PERFECT	P	?
$*$	?	NP-complete if $k + \ell \geq 3$ and P otherwise[7]

Table 2: Complexity results and open problems for $(\Pi_1, (k,\ell), \Pi_2)$, where properties Π_1, Π_2 are in {PERFECT, PNMC,$*$}.

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