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Interval Count of Generalizations of Threshold Graphs

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Abstract

The interval count is the smallest number of distint interval lengths over all models of a particular interval graph. Although the graphs having interval count one are well-known since the sixties, very few is known about the graphs having interval count $k \geq 2$, even for fixed k. Recently, a polynomial-time algorithm has been presented to compute the interval count of extended-bull-free graphs, letting open the efficient computation of interval count of graphs in which extended-bulls may exist. In this work, we investigate the interval count of subclasses of split graphs. Specifically, we introduce two proper subclasses of split graphs, each of them generalizing the class of threshold graphs, and show that interval count is bounded for graphs in those classes. In contrast, we show that the interval count of general split graphs can be unbounded.

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1 Motivation

Interval graphs are well-known both for their applications and variety of known structures [3, 5]. A graph G is an *interval graph* if there exists a family $\mathcal{R} = \{I_v \mid v \in V(G)\}$ of closed intervals of the real line such that, for all distinct $x, y \in V(G)$, $I_x \cap I_y \neq \emptyset$ precisely when $xy \in E(G)$. We call such a family $\mathcal{R}$ a *model* of G. The *interval count* of a given interval graph G is the smallest number of distinct interval lengths in $\mathcal{R}$ over all models $\mathcal{R}$ of G. The interval count problem, which is the complexity of computing such a number for a given input graph, is known since the seventies and several results for particular classes of graphs have been presented [3]. However, even the problem of deciding whether a graph admits a model using only two interval sizes is still open [3].

The interval count problem has recently been determined to be efficiently computable for extended-bull-free orders and graphs [2]. A natural continuation is to determine the interval count of graphs restricted to classes of graphs in which particular extended-bulls are found. For example, an interesting problem is to determine the complexity of computing the interval count of graphs which contain bulls and no other type of extended-bulls.

On the other hand, split graphs are largely known for their simple structure and, in spite of that, for being a challenge to some problems. That is, several NP-complete questions on graphs remain hard even when the input is restricted to split graphs [4]. Additionally, the only possible extended-bulls in split graphs are bulls. The aim of this paper is to investigate the interval count of split graphs.

2 Split and threshold graphs

Split graphs are generalizations of threshold graphs. The interval count of threshold graphs is known since the eighties whereas the interval count of split graphs is open. We present a structural property of models of split graphs in this section.

For $x \in V(G)$, the *open neighborhood* is defined as $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$. The *closed neighborhood* is given by $N_G[x] = \{x\} \cup N_G(x)$. For $x, y \in V(G)$, x, y are called *twins* if $N_G[x] = N_G[y]$ and *false twins* if $N_G(x) = N_G(y)$. An *extended-bull* of size $n \geq 1$ consists of a path P on $n + 3$ vertices and a vertex that is adjacent to every vertex of P but the extreme ones. Therefore, for $n = 1$ the extended-bull is a bull. A graph G is *split* with *split partition* (K, I) if $K \cup I = V(G)$ is a partition, K is a clique and I is an independent set of G. If $xy \in E(G)$ for all $x \in K$, $y \in I$, then G is *complete split*. It is easy to see that split graphs can contain bulls as induced subgraphs, but no extended-bulls for $n \geq 2$ since split graphs are P_5-free [2]. A graph G is *threshold* if it admits a split partition (K, I) such that there exists an ordering $v_1, \ldots, v_{|I|}$ of the vertices of I such that $N_G(v_i) \subseteq N_G(v_{i+1})$ for each $1 \leq i < |I|$ (or, equivalently, there exists an ordering $u_1, \ldots, u_{|K|}$ of the vertices of K such that $I \cap N_G(u_{i+1}) \subseteq I \cap N_G(u_i)$ for each $1 \leq i < |K|$). By definitions, the class of threshold graphs is clearly properly included in that of split graphs. The interval count of threshold graphs is known to be limited to 2 [6].

Since the interval count problem concerns only interval graphs, we consider graphs to be interval graphs unless stated otherwise. Therefore, when it is stated *the interval count of split graphs*, it is implicitly stated *the interval count of split* $\cap$ *interval graphs*. (In fact, there are split graphs that are not interval graphs, so this implicit assumption must be clear.)

In what follows, some properties regarding the split graphs or their models will be assumed. Regarding the interval count problem, we can assume that graphs do not have twins. This follows from the fact that one can remove a vertex of a pair of twins t_1 and t_2, say t_1, and, given a model realizing the interval count of the resulting graph, letting I_{t_1} to coincide with I_{t_2} clearly realizes the interval count of the original graph. Besides, we can assume that graphs do not have isolated vertices, since they can be removed and their corresponding intervals can be inserted at the end of a model which realizes the interval count of the resulting graph, such that the inserted intervals have the length of some other interval, say, the smallest one.

The following lemma addresses the uniqueness of a split partition:

Lemma 2.1. *Let G be a split graph without twins. If (K, I) is a split partition in which for any $x \in V(G)$, $N[x] \neq K$, then (K, I) is unique.*

Proof. Suppose (K_1, I_1) and (K_2, I_2) are two distinct split partitions. Without loss of generality, there is $u \in K_1$ such that $u \in I_2$. For all $a \in I_2 \setminus \{u\}$, $au \notin E(G)$, implying $a \notin K_1$. Therefore, $I_2 \setminus \{u\} \subseteq I_1$. Since $I_1 \setminus I_2 \subseteq K_2$ and in order to $N[u] \neq K_1$, $I_1 \setminus I_2$ must consist of a single vertex b such that $bu \in E(G)$. Consequently, $N[u] = K_1 \cup \{b\} = K_1 \setminus \{u\} \cup \{b\} \cup \{u\} = K_2 \cup \{u\} = N[b]$. Therefore, u and b are twins, a contradiction. ∎

An *order* $(X, \prec)$ is a transitive and irreflexive binary relation $\prec$ on X. Let $G = (K, I)$ be a split graph in which for any $x \in V(G)$, $N[x] \neq K$. Therefore, from Lemma 2.1, (K, I) is unique. Since G is interval and therefore a cocomparability graph, let $P = (V(G), \prec)$ be the order defined by a transitive orientation of $\overline{G}$ (such P is said to *agree* with G). Note that for all $u, v \in K$, $u \not\prec v$, whereas for all $u, v \in I$, either $u \prec v$ or $v \prec u$.

Let $V^+(P) = \{x \in K \mid x \prec y \text{ for some } y \in I\}$, and similarly $V^-(P) = \{x \in K \mid y \prec x \text{ for some } y \in I\}$. Note that $N_{\overline{G}}(V^+(P)) \cap N_{\overline{G}}(V^-(P)) = \emptyset$. We define $K_1(P) = V^+(P) \setminus V^-(P)$, $K_2(P) = V^-(P) \setminus V^+(P)$, and $K_3(P) = V^+(P) \cap V^-(P)$. We also define $I_1(P) = N_{\overline{G}}(V^-(P))$, $I_2(P) = N_{\overline{G}}(V^+(P))$, and $I_3(P) = (I \setminus I_1(P)) \setminus I_2(P)$. For the sake of simplicity, the argument P will be omitted when P is clear in the context.

From the above definitions, it can be readily verified that a possible model (called *canonical*) of a split graph has the general layout of that in Figure 1, in which the following subgraphs are clear from the canonical model:

1. $G[K_1 \cup I_1]$ and $G[K_2 \cup I_2]$ are complete split graphs;

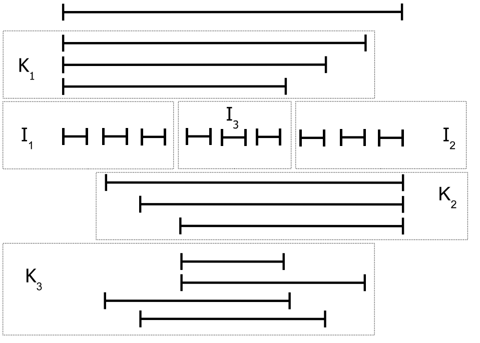

Figure 1: Canonical model of a split graph.

2. $G[K_i \cup I_3]$ are complete split graphs, for $1 \le i \le 3$;
3. $G[K_1 \cup I_2]$ and $G[K_2 \cup I_1]$ are threshold graphs;
4. $G[K_3 \cup I_1]$ and $G[K_3 \cup I_2]$ are threshold graphs.

We define SPLIT^2 and SPLIT^3, proper subclasses of split graphs, in the following way. We say that G is in SPLIT^3 if for all $x \in K_3$, either $N_G(x) \cap I \subseteq I_1 \cup I_3$, or $N_G(x) \cap I \subseteq I_2 \cup I_3$. Moreover, $G \in \text{SPLIT}^2$ if $K_3 = \emptyset$. Therefore, $\text{THRESHOLD} \subset \text{SPLIT}^2 \subset \text{SPLIT}^3 \subset \text{SPLIT}$.

Let G be a graph. The *clique ordering* of a model of G is the linear order of the cliques of G, reading them from left to right in the model [1]. The following theorem characterizes when the canonical model of a split graph is unique.

Theorem 2.1. Let $G = (K, I)$ be a split graph such that for any $x \in V(G)$, $N[x] \neq K$. Let P be an order that agrees with G. The clique ordering of models of G is unique up to reversal and reordering of false twins if and only if either $|I| < 3$ or $K_1(P) \neq \emptyset$ and $K_2(P) \neq \emptyset$.

3 Relationship to threshold dimension

The *threshold dimension* of a graph G is the minimum number of spanning threshold subgraphs of G whose union is G. Let 2-THRESHOLD be the class of graphs that have threshold dimension at most 2.

Lemma 3.1. SPLIT^3 $\subset$ 2-THRESHOLD

Proof. Let G be a split graph. In Figure 2, we schematically show a way to decompose G into two spanning threshold subgraphs G_1 and G_2 whose union is G that can be easily verified, implying the result. In this scheme, each graph is defined by its model. Moreover, intervals of K in G, G_1 and G_2 along a same horizontal line correspond to the same vertex. Intervals of I are identified by a label. ■

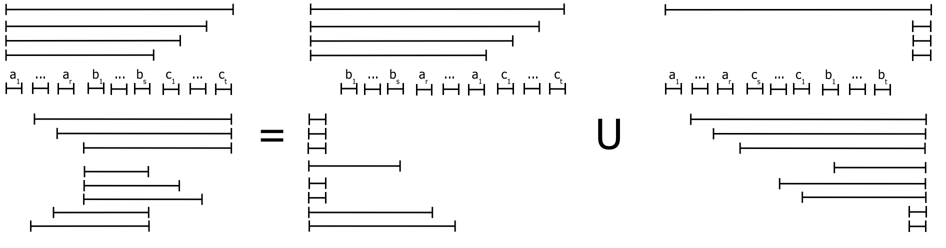

Figure 2: Schematic construction to show that SPLIT^3 $\subset$ 2-THRESHOLD.

Figure 3 presents a diagram relating these graph classes.

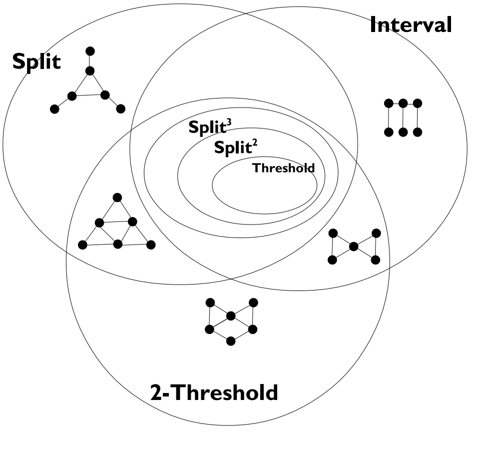

Figure 3: Inclusion diagram among the related classes.

4 Interval count of split graphs

In this section, we present the interval count of graphs in the classes of $\textsc{Split}^2$ and $\textsc{Split}^3$, besides commenting on the interval count of general split graphs.

A consequence of the unique clique ordering of a split graph is the following:

Corollary 4.1. The interval count of a split graph can be arbitrarily large.

Proof. From Theorem 2.1, the clique ordering of a split graph is unique when $K_1 \neq \emptyset$ and $K_2 \neq \emptyset$. In such a case, the only model is the canonical (up to reversal and reordering of false twins). Therefore, it is easy to build a graph having nested intervals in K_3, implying that the interval count increases arbitrarily for split graphs. ■

It is known that the interval count of threshold graphs are limited to 2. We extend this limitation to the more general classes of $\textsc{Split}^2$ and $\textsc{Split}^3$:

Theorem 4.1. If $G \in \textsc{Split}^2$, then $IC(G) \leq 2$.

Proof. Let $\mathcal{R} = \{I_x = [\ell(I_x), r(I_x)] \mid x \in V(G)\}$ be the canonical model of G, in which intervals of I are represented using a same length. Let $L = \max\{|I_x| \mid x \in K\}$. Note that for each $k \in K$, it is possible to decrease $\ell(I_k)$ (if $k \in K_1$) or increase $r(I_k)$ (if $k \in K_2$) until $|I_k| = L$ without changing comparabilities among intervals. Since $K_3 = \emptyset$, the resulting model has only 2 distinct lengths. ■

A more involved argument leads to the fact that the interval count of graphs in $\textsc{Split}^3$ is also bounded by a constant:

Theorem 4.2. If $G \in \textsc{Split}^3$, then $IC(G) \leq 3$.

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