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Searching for a NP-Complete Probe Graph Problem

R. B. Teixeira **C. M. H. de Figueiredo**
S. Dantas

Abstract

A *probe graph* for a graph class $\mathcal{C}$ is a graph $G = (V, E)$, such that it is possible to add some edges between vertices of an independent set $\mathcal{N}$ in order to obtain a graph G' belonging to $\mathcal{C}$ [7]. If the independent set $\mathcal{N} \subset V$ is given as input, we have a special case of *graph sandwich* problem [6], called *partitioned probe* problem. A *graph partition* of a graph $G = (V, E)$ is a partition of V into a number of parts. A *graph partition* problem consists in finding a graph partition where the parts satisfy some internal or external constraints. A *three nonempty part* problem is a graph partition problem, such that V must be partitioned in exactly three nonempty parts. All possible such nonempty part problems, reflecting the various combinations of internal and external constraints, are classified as polynomial or NP-complete in both recognition and sandwich versions [14]. We focus on *three nonempty part partitioned probe* problems, for which their sandwich version is NP-complete and their recognition version is polynomial. We show that most of those partitioned probe problems have a behavior strongly similar to their recognition version, despite

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of being a special case of a NP-complete sandwich problem. This result completely classifies into polynomial time or NP-complete all *three nonempty part partitioned probe* problems with the exception of the *clique cutset partitioned probe* problem.

1 Introduction

In the past, computer science researchers worked hard on hardware and software in order to make computers useful for people. Now, we note that the main need of the population turns out to be communication. The large scale computer's use on social networks leads the investment on science to problems like communities location. These problems can be modeled, with the use of graph theory, as graph partition problems which is the focus of this work [9].

We say that a graph $G = (V, E)$ is a *sandwich* graph for the pair $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ if $E_1 \subseteq E \subseteq E_2$. The GRAPH SANDWICH PROBLEM FOR PROPERTY Π is defined by Golumbic et al. as follows [6]:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π
Instance: Vertex set V, forced edge set E_1, optional edge set $E_2 \setminus E_1$.
Question: Is there a graph $G = (V, E)$ such that $E_1 \subseteq E$ and $E \subseteq E_2$, and G satisfies property Π?

Golumbic et al. [6] have considered sandwich problems with respect to several subclasses of perfect graphs, and proved that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS remains in P.

A three nonempty part partition of a graph is a partition of its vertex set into three nonempty subsets satisfying internal or external constraints. An internal constraint refers to constraints within the parts as to be a clique, or an independent set. An external constraint refers to constraints between different parts as to be completely adjacent or nonadjacent to other parts. The full complexity dichotomies into P or NP-complete of the three nonempty part for both recognition [8] and sandwich versions [14] have been determined.

Polynomial sandwich problems are rare, so the particular instance where the optional edges of $E_2 \setminus E_1$ are required to have both endpoints in a given independent set of the forced graph G_1 was considered as the PARTITIONED PROBE problem.

PARTITIONED PROBE PROBLEM FOR PROPERTY Π
Instance: Vertex set V, edge set E, partition of $V = (\mathcal{N}, \mathcal{P})$, where $\mathcal{N}$ is an independent set.
Question: Is there a graph $G' = (V, E')$ such that $E \subseteq E'$, all edges of $E' \setminus E$ have both endpoints in $\mathcal{N}$, and G' satisfies property Π?

Several partitioned probe problems have been studied, all of them so far classified as polynomial: cographs [12], P_4-parse [10], permutation graphs [11], threshold [1], chordal graphs [2], chain graphs [7], and trivially perfect graphs [1], leading to the Probe Graph Conjecture (PGC): "Partitioned probe graphs of $\mathcal{C}$ are polynomially recognizable whenever $\mathcal{C}$ is polynomially recognizable" [12]. More results and open problems on partitioned probe graphs can be found in [3].

Looking at sandwich problems as a generalization of recognition problems, the first and natural approach is to consider a property Π that can be recognized in polynomial time and to generalize the recognition algorithm. For partitioned probe problems, in addition to this approach, we consider sandwich problems for which property Π turns out to be NP-complete. Thus the studies on the three nonempty part partitioned probe problem were focused on those problems which are interesting in terms of their complexity, i.e., NP-complete sandwich problems and polynomial recognition problems. Such problems are twenty one: clique cutset [13], $(2, 1)$-partition [4] and other nineteen solved in [14].

We note that a graph G satisfies the complementary property $\overline{\Pi}$ if, and only if, $\overline{G}$ satisfies Π. While for GRAPH SANDWICH problem, property $\overline{\Pi}$

has trivially the same complexity as property Π, in PARTITIONED PROBE problem, it may not occur. In this paper, we establish the complexity dichotomy for all THREE NONEMPTY PART PARTITIONED PROBE problems, in which property Π is the membership to a class defined by three nonempty parts, with the exception of clique cutset and $(2, 1)$-partition problems, by proving that the remaining nineteen problems cited above and their complementary versions are in P.

2 Solution

Consider the nineteen problems depicted in Table 1, representing the three nonempty part sandwich problems classified as NP-complete by [14]. The first seven columns represent: the numbering of problems following [14]; and the internal and the external constraints imposed by each problem.

Let $U, U', U'' \in \{A, B, C\}$ with $U \neq U' \neq U''$. We note that a column entry corresponding to part U equal to 0 (resp., 1, $*$) requires that part U induces a stable set (resp., clique, arbitrary subgraph). An entry in a column UU' equal to 0 (resp., 1, $*$) requires 'no edges' (resp., 'all edges', 'no constraint') between a vertex placed in part U and a vertex placed in part U'.

Next, we prove that each corresponding partitioned probe version is classified as polynomial. In order to solve these problems, we present a single tool which uses similar techniques shown in [5, 15] for recognition or sandwich problems.

Let $V = (\mathcal{N}, \mathcal{P})$ be a partition of V. First, each vertex of V receives a list ABC. During the process, these vertices will have their lists reduced, according to the internal or external constraints imposed by each problem, so that a solution corresponds to a successful reduction of each list to a unitary list. Every time all list sizes are reduced to at most 2, we can solve the problem by using 2-SAT. The algorithm tries three possibilities for set $\mathcal{P}$ with respect to the target partition into parts A, B and C: first, all vertices of $\mathcal{P}$ belong to the same part; second, there is precisely one part without vertices of $\mathcal{P}$; and third there is no part without vertices of $\mathcal{P}$.

In the first possibility, the problem can be easily solved. In the second possibility, there exists only one part U without vertices of $\mathcal{P}$ then, each pair of vertices of $\mathcal{P}$ are placed in a distinct part, w.l.o.g., we place $x_{U'}, x_{U''} \in \mathcal{P}$ in parts U' and U'' respectively. So vertices $x_{U'}, x_{U''}$ have list U' and U'', respectively, each vertex of $\mathcal{P} \setminus \{x_U, x_{U''}\}$ receives list $U'U''$, and each vertex of $\mathcal{N}$ receives list ABC. This procedure can be applied to all problems with the exception of four (i.e., (14), (16), (42) and (43)) for which there exist vertices with list of size three. In these cases, this problem admits a solution if, and only if, there also exists a particular solution with the part $U = \{x_U\}$, with $x_U \in \mathcal{N}$, so we eliminate list U of all vertices but x_U.

In the third possibility, each triple of vertices of $\mathcal{P}$ are placed in a distinct part, w.l.o.g., we place $x_A, x_B, x_C \in \mathcal{P}$ in parts A, B, and C, respectively. So, vertices x_A, x_B and x_C have list A, B and C respectively, and each vertex of $V \setminus \{x_A, x_B, x_C\}$ receives list ABC. We refer to Table 1 for the nineteen problems and their respective lists. We recall that if a vertex v has list AC, this means that v cannot be placed in part B.

If during the process, the list of a vertex is reduced to the empty set, then the procedure stops with the answer NO, and the instance in question does not have such a corresponding three nonempty part partitioned probe partition.

We note that each one of the nineteen complementary problems for property $\overline{\Pi}$ has the same set of lists of the corresponding problem for Π. Hence, all problems can be solved by using the very same procedure.

3 Conclusion

We have shown a procedure for the nineteen THREE NONEMPTY PART PARTITIONED PROBE problems (and their complementary versions) for

Problem matrix							Vertex adjacent to							
Number in [14]	A	B	C	AB	AC	BC	$\emptyset$	x_C	x_B	x_B x_C	x_A	x_A x_C	x_A x_B	x_A x_B x_C
(7)	0	*	*	1	*	0	C	C	A	A	BC	C	B	
(12)	0	*	*	1	1	0				A	BC	C	B	
(14)	0	0	*	*	*	1	A	AB	AC	AC		B	C	C
(16)	0	1	*	*	1	*		A	B	AB	C	C	BC	BC
(20)	0	0	*	*	0	1	A	B	AC	C		B		
(23)	0	1	*	0	1	*		A	B	B	C	C	C	C
(28)	0	1	*	*	1	0		A	B	A	C	C	B	
(36)	0	1	*	0	1	0		A	B		C	C		
(39)	0	0	0	1	*	*	C		AC	A	BC	B	C	
(41)	0	0	1	1	*	*		C	A	AC	B	BC		C
(42)	0	0	1	*	0	*	AB	BC	A	C	B	B		
(43)	0	0	1	*	*	1	A	AB	A	AC		B		C
(45)	0	0	0	*	1	1		AB		A		B	C	
(48)	0	0	1	1	0	*		C	A	C	B	B		
(49)	0	0	1	1	*	1			A	AC		B		C
(55)	0	0	0	1	1	1				A		B	C	
(51)	0	0	1	*	0	1	A	B	A	C		B		
(59)	0	0	1	1	0	0		C	A		B			
(60)	0	0	1	1	0	1			A	C		B		
(28) (2,1) partition	0	0	1	*	*	0	AB	ABC	A	AC	B	BC		C
(3) $\overline{\text{Clique cutset}}$	0	*	*	*	*	1	BC	C	B		ABC	AC	AB	A

Table 1: Entries of the nineteen THREE NONEMPTY PART PARTITIONED PROBE problems and the list of each vertex according to the adjacencies to initial fixed vertices x_A, x_B and x_C.

which the corresponding GRAPH SANDWICH problems were classified as NP-complete [14]. For each $\mathcal{O}(n^3)$ feasible assignment of one vertex to each part, this process update the list of each vertex $v \in V(G)$ in $\mathcal{O}(n^2)$ in order to keep the internal and external constraints satisfied. The solution arises from a polynomial reduction to a 2-SAT instance that can be solved in linear time. Finally, we highlight in Table 2 the matrix entries of the constraints imposed by $(2,1)$ and clique cutset partition problems. We observe that $(2,1)$ PARTITIONED PROBE problem was recently shown to be polynomial [16], so this classifies into polynomial time or NP-complete all THREE NONEMPTY PART PARTITIONED PROBE problems with the exception of only one problem, the CLIQUE CUTSET PARTITIONED PROBE problem, which we set as candidate to be NP-complete.

Problem	A	B	C	AB	AC	BC	Complexity
Clique cutset	1	*	*	*	*	0	Open
$(2,1)$	0	0	1	*	*	*	**P**[16]

Table 2: The only open THREE NONEMPTY PART PARTITIONED PROBE problem.

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R. B. Teixeira
ICE,
Universidade Federal Rural do Rio de Janeiro
Brazil
rafaelbt@ufrrj.br

C. M. H. de Figueiredo
COPPE,
Universidade Federal do Rio de Janeiro
Brazil
celina@cos.ufrj.br

S. Dantas
Universidade Federal Fluminense
Brazil
sdantas@im.uff.br