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Total chromatic number of some families of graphs with maximum degree 3

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Abstract

The focus of this work is the total coloring of graphs with maximum degree 3. We 4-total-color five infinite families with maximum degree 3: a new infinite family of snarks constructed from Blanuša snarks; two new infinite families of cubic graphs constructed from 4-Möbius-ladder and 5-ladder, respectively; and two grid families, Hexagonal-grid and Pentagonal-grid.

1 Introduction

Let G be a simple graph. A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring. Clearly, $\chi_T \geq \Delta + 1$, where Δ is the maximum degree, and the Total Coloring Conjecture [1, 17] states that $\chi_T \leq \Delta + 2$. This conjecture has been proved for graphs with maximum degree 3 [12, 16], and so the total chromatic number of a cubic

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graph is either 4 or 5. Graphs with $\chi_T = \Delta + 1$ are said to be *Type 1*, and graphs with $\chi_T = \Delta + 2$ are said to be *Type 2*. The problem of deciding whether a graph is Type 1 has been shown NP-complete even for cubic bipartite graphs [14, 15]. There are few graph classes whose total chromatic number has been determined. Examples include cycle graphs [18], complete graphs [18], complete bipartite graphs [18], and grids [4].

We say that G contains a square if it has a chordless cycle on four vertices as an induced subgraph. A cubic graph is said *cyclically* 4-*edge-connected* if every edge-cutset of cardinality less than 4 consists of three edges incident to the same vertex. So a cyclically 4-edge-connected cubic graph may contain squares, but it does not contain triangles unless it is the complete graph on four vertices. The *girth* of a graph is the length of a shortest cycle contained in the graph.

Coloring is a challenging problem that models many real situations where the adjacencies represent conflicts. *Snarks* are cyclically 4-edge-connected cubic graphs with chromatic index 4, which had their origin in the search of counterexamples to the Four Color Conjecture. The importance of these graphs arises mainly from the fact that several conjectures would have snarks as minimal counterexamples, such as Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture [8]. Furthermore, eight published conjectures were recently refuted with the use of snarks [2]. The total coloring of this graph class and of other families of graphs with maximum degree 3 is the focus of this work.

In 2003, Cavicchioli et al. [6] reported that their extensive computer study of snarks showed that all snarks with girth greater than 4 and with less than 30 vertices are Type 1, and asked for the smallest order of a Type 2 snark with such girth. Brinkmann et al. [2] have recently extended the computer study up to order 36. In 2011, it was proved that all members of the two infinite families of Flower and Goldberg snarks are Type 1 [5]; recently, all members of the two additional infinite families of

Blanuša and Loupekhine snarks have been proved to be Type 1 [13]; all graphs of these four families are square-free.

Type 2 snarks have very recently been found, but so far every known Type 2 snark contains a square [3]. On the other hand, since 1988 and even considering bipartite graphs, infinite families of Type 2 cubic graphs were known [7], and all of them contain a square or a triangle. In fact, all Type 2 cubic graphs that we know (whatever their chromatic index or cyclic-edge-connectivity) have triangles or squares. So it could be that there exists no Type 2 cubic graph of girth greater than 4. The investigation of the relation between the presence of a square or a triangle in a cubic graph and its total chromatic number motivates this work. Due to space restrictions, proofs will be omitted.

2 Cubic graphs

In this section, we study three infinite families of cubic graphs and determine the total chromatic number of all members of both families. We consider families of cubic graphs with squares, trying to better understand the relation between the presence of a square in a cubic graph and its total chromatic number.

Snarks We define an infinite family of snarks with squares and show that all its members have total chromatic number 4. Given a cyclically 4-edge-connected cubic graph, a *brick* is a connected bridgeless subgraph with precisely four vertices of degree 2 and possibly vertices of degree 3. A *junction* of two bricks G_1 and G_2 is a cyclically 4-edge-connected cubic graph obtained by adding a matching of four edges between the four vertices of degree 2 of G_1 and the four vertices of degree 2 of G_2. If G_1 or G_2 has chromatic index 4, then a junction of these two bricks is a snark.

A *dot product* of two snarks G and H is any graph obtained as follows:

- delete two non adjacent edges e and f of G,
- delete two adjacent vertices x and y of H,

• make a junction of the so obtained bricks in such a way that the four new edges can be split into two *special pairs* of edges joining the extremities of e (respectively f) to vertices that are in H adjacent to the same extremity of xy (see Figure 1).

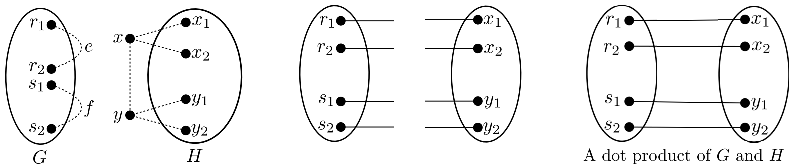

Figure 1: A dot product of graphs G and H with special pairs (r_1x_1, r_2x_2) and (s_1y_1, s_2y_2).

Isaacs has defined this construction and he proved that the resulting graph is a snark [10]. Two snarks may be obtained from the dot product of two Petersen graphs, they are called Blanuša snarks B_1 and B_2 [11].

We denote by $B_iS_k^j$ ($i = 1, 2$; $j, k \geq 0$) the graph obtained from B_i by "inserting" j squares into one special pair and k squares into the other special pair, as shown in Figure 2 for $j = 1, k = 0$ and $j = k = 1$.

It is easy to verify that deleting a special pair of the dot product of two snarks one obtains a brick of chromatic index 4. Similarly, it is not difficult to see that $B_iS_k^j$ is a snark for any values of i, j, k.

Theorem 2.1. *All graphs $B_iS_k^j$, $i = 1, 2$, $j, k \geq 0$, are Type 1.* ■

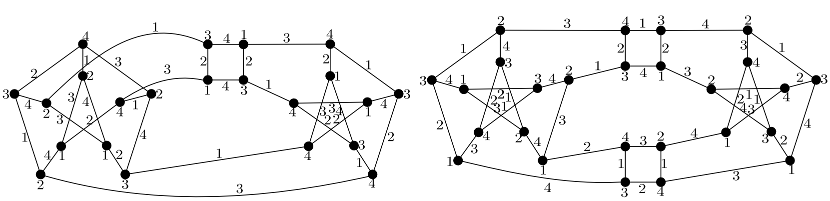

Figure 2: The two snarks $B_2S_0^1$ and $B_1S_1^1$ with 4-total-colorings.

Other interesting cubic families We define two infinite families of cubic graphs constructed from ladder families. Since the smallest members of these families are Type 2 [9], we investigate the other members and determine their total chromatic number to be 4. The 4-Möbius-ladder-extension family is constructed from the 4-Möbius-ladder graph by adding a sequence of squares as shown in Figure 3. Theorem 2.2 proves that all other members are Type 1 (see Figure 4).

Theorem 2.2. *Except for the smallest three members of 4-Möbius-ladder-extension family, all other members are Type 1.* ■

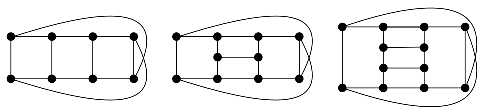

Figure 3: The three members of 4-Möbius-ladder-extension family that are Type 2 [9].

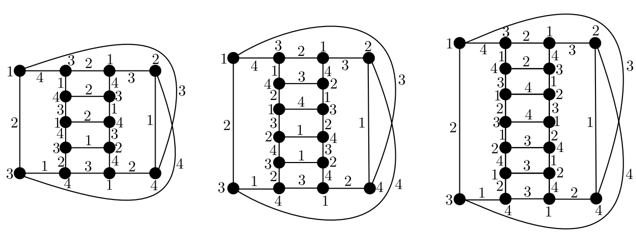

Figure 4: Three members of 4-Möbius-ladder-extension family with 4-total-colorings of Theorem 2.2.

Similarly, the 5-ladder-extension family is constructed from the 5-ladder graph by adding a sequence of squares as shown in Figure 5. Theorem 2.3 proves that all other members are Type 1 (see Figure 6).

Theorem 2.3. *Except for the smallest two members of 5-ladder-extension family, all other members are Type 1.* ■

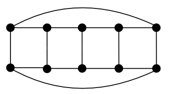

Figure 5: The two members of 5-ladder-extension family that are Type 2 [9].

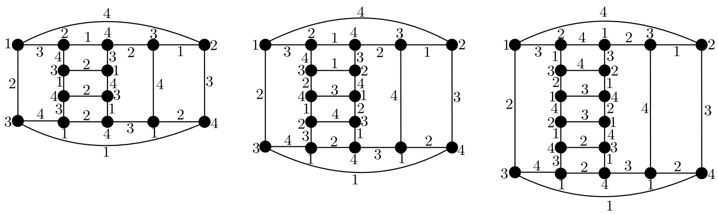

Figure 6: Three members of 5-ladder-extension family with 4-total-colorings of Theorem 2.3.

Concluding remarks Motivated by the search for Type 2 cubic graphs of girth greater than 4, and since not even a Type 2 graph with girth greater than 4 and maximum degree 3 is known, we started analyzing more general graphs as well. We proved that all members of two grid families of graphs with maximum degree 3 and girth greater than 4, named Hexagonal-grid and Pentagonal-grid families, have total chromatic number 4. The construction of each infinite family is very simple and based on adding one more cycle to obtain the following member. Some bipartite grids of girth 4 have already been total colored [4]. Figure 7 shows an example of each family.

In this work, we 4-total-colored five infinite families of graphs with maximum degree 3. The search for a square-free Type 2 snark proposed by Cavicchioli et al. [6], and, more generally, for a Type 2 cubic graph of girth at least 5 continues.

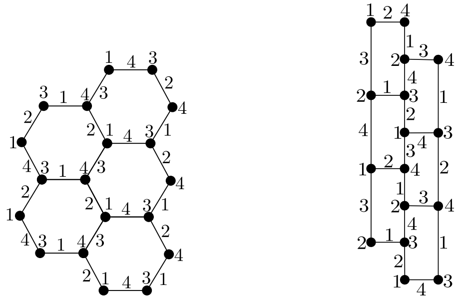

Figure 7: A 4-total-coloring of a member of Hexagonal-grid family and of Pentagonal-grid family, resp.

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