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Independence in Graphs A Medley of Popular Tunes

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Abstract

We survey some recent developments related to three classical lower bounds on the independence number of graphs; Caro's and Wei's bound for general graphs, Shearer's bound for triangle-free graphs, and Staton's bound for triangle-free graphs of maximum degree at most three. We discuss these bounds, their proofs, and some of our own research that evolved around them.

1 Prelude

Independent sets and the independence number form a well-established topic in graph theory. While the basic definitions are extremely simple, the complexity of the notion and the corresponding algorithmic issues lead to many challenging research problems.

For a finite, simple, and undirected graph G, an *independent set* of G is a set of pairwise non-adjacent vertices of G and the maximum cardinality of an independent set of G is the *independence number* $\alpha(G)$ of G, that is,

$$\alpha(G) = \max\{|I| : I \subseteq V(G) \text{ is independent}\}.$$

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In the present admittedly personal as well as all but comprehensive small survey we focus on the following three classical lower bounds on the independence number[1]:

- Caro's and Wei's bound for general graphs [3, 14]
- Shearer's bound for triangle-free graphs [11]
- Staton's bound for triangle-free graphs of maximum degree at most three [12]

Our aim is to discuss these bounds, their proofs, and some of our own recent research that evolved around them. We limit the citations to an absolute minimum and refer to the cited papers for further references.

2 First Tune: Caro-Wei

Here is the complete statement of Caro's and Wei's bound for general graphs.

Theorem 2.1 (Caro 1979 [3] and Wei 1981 [14])**.**
If G is a graph, then

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{d_G(u)+1} \tag{1}$$

with equality if and only if every component of G is a clique.

The intuition behind this bound is obvious; the larger the degree $d_G(u)$ of some vertex u in a graph G is, the lower should be u's contribution to G's independence number. The bound itself is just the sum of all contributions of the vertices of G.

We present three well-known simple ways to prove this bound; two greedy algorithms and a probabilistic argument.

[1]The survey essentially corresponds to my invited lecture at the 5th Latin American Workshop on Cliques in Graphs held on November 5, 2012 in Buenos Aires, Argentina.

Here is the first greedy algorithm.

Input: A graph G.

Output: An independent set I.

$I \leftarrow \emptyset$;

while $V(G) \neq \emptyset$ **do**

  Select $u \in V(G)$ of minimum degree;

  $I \leftarrow I \cup \{u\}$;

  $G \leftarrow G - N_G[u]$;

end

return I;

Algorithm 1: Best-In-Greedy

Best-In-Greedy iteratively selects vertices to be added to the independent set I. Whenever some vertex u is added to I, the vertex u together with all its neighbors is removed from G. In order to construct a large independent set in this way, it is intuitively plausible to select u among the minimum degree vertices.

To turn Best-In-Greedy into a proof of Theorem 2.1, we have to control the loss in the weight function $\sum\limits_{v \in V(G)} \frac{1}{d_G(v)+1}$ caused by removing the closed neighborhood $N_G[u]$ of u from G.

Lemma 2.1. *If G is a graph, u is a vertex of minimum degree in G, and $G' = G - N_G[u]$, then*

$$\sum_{v \in V(G)} \frac{1}{d_G(v)+1} - \sum_{v \in V(G')} \frac{1}{d_{G'}(v)+1} \leq 1,$$

with equality if and only if $N_G[u]$ induces a complete component of G.

Proof: Consider the following inequality chain bounding the loss in the weight function.

$$\begin{aligned}\sum_{v\in V(G)}\frac{1}{d_G(v)+1}-\sum_{v\in V(G')}\frac{1}{d_{G'}(v)+1} &\overset{(2)}{\leq} \sum_{v\in N_G[u]}\frac{1}{d_G(v)+1}\\ &\overset{(3)}{\leq} \sum_{v\in N_G[u]}\frac{1}{d_G(u)+1}\\ &\overset{(4)}{=} 1\end{aligned}$$

Inequality (2) holds because going from G to G', we lose the contribution of all vertices in $N_G[u]$. Note that the contribution to the weight function of the remaining vertices in $V(G)\setminus N_G[u]$ might increase because their degrees might go down. This means that (2) is satisfied with equality exactly if there is no edge between $N_G[u]$ and $V(G)\setminus N_G[u]$.

The inequality (3) follows immediately from the choice of u as a vertex of minimum degree.

The equality (4) follows since there are $d_G(u)+1$ terms in the sum that all have value $1/(d_G(u)+1)$.

If the loss in the weight function is exactly 1, then (1) and (2) hold with equality. This implies that there is no edge between $N_G[u]$ and $V(G)\setminus N_G[u]$, and all vertices in $N_G(u)$ have the same degree as u, that is, $N_G[u]$ induces a complete component of G. □

Since every iteration of Best-In-Greedy adds one vertex to the independent set I and the weight loss per iteration is at most one, a simple inductive argument implies Theorem 2.1. Furthermore, the statement about the extremal graphs also follows.

Here is the second greedy algorithm.

Input: A graph G.
Output: An independent set I.
while $V(G)$ *is not independent* **do**
    Select $u \in V(G)$ of maximum degree;
    $G \leftarrow G - \{u\}$;
end
$I \leftarrow V(G)$;
return I;

Algorithm 2: WORST-OUT-GREEDY

WORST-OUT-GREEDY produces an independent set not by iteratively selecting its elements but by iteratively selecting the elements of a vertex cover, that is, WORST-OUT-GREEDY removes vertices and all incident edges until the remaining graph is edge-less. The choice of the vertex to be deleted among the maximum degree vertices guarantees that in each individual iteration, the maximum possible number of edges is removed from G.

Similar arguments as in the proof of Lemma 2.1 allow to control the loss in the weight function caused by removing a vertex u of maximum degree from G. In fact, if we denote the resulting graph by G' again, then

$$\sum_{v \in V(G')} \frac{1}{d_{G'}(v)+1} \geq \sum_{v \in V(G)} \frac{1}{d_G(v)+1},$$

that is, there is no loss at all. Since WORST-OUT-GREEDY stops when all edges have been removed from G, and since then for every remaining vertex u, we have $1/(d_G(u)+1) = 1/(0+1) = 1$, the set of remaining vertices is the desired independent set whose order is at least the initial weight.

The third proof of Theorem 2.1 relies on a probabilistic argument.

If $\pi : v_1, \ldots, v_n$ is a random linear order of the vertices of G, then the set

$$I_\pi = \{v_i : \forall j \in [n] : v_i v_j \in E(G) \Rightarrow i < j\}$$

is independent. A vertex u of G belongs to I_π exactly if it comes first within the linear order among all $d_G(u)+1$ vertices of $N_G[u]$. Since this happens exactly with probability $\frac{1}{d_G(v_i)+1}$, the first-moment-method implies

$$\begin{aligned}\alpha(G) &\geq \mathbf{E}[|I_\pi|]\\ &= \sum_{i\in[n]} \mathbf{P}[v_i \in I_\pi]\\ &= \sum_{i\in[n]} \mathbf{P}[\forall j \in [n] : v_i v_j \in E(G) \Rightarrow i < j]\\ &= \sum_{i\in[n]} \frac{1}{d_G(v_i)+1}.\end{aligned}$$

After discussing the bound and its proofs we proceed to our related results.

Apart from the complete graph, all extremal graphs for (1) are disconnected. Therefore, assuming connectivity should lead to a better lower bound on the independence number. After a first contribution of Harant and Schiermeyer [4], Harant and Rautenbach [6] obtained the following best-possible result.

Theorem 2.2 (Harant and Rautenbach 2011 [6])**.**
If G is a connected graph, then there exist a positive integer $k \in \mathbb{N}$ and a function $h : V(G) \to \mathbb{N}_0$ with non-negative integer values such that $h \leq d_G$,

$$\alpha(G) \geq k \geq \sum_{u\in V(G)} \frac{1}{d_G(u)+1-h(u)}, \text{ and } \sum_{u\in V(G)} h(u) \geq 2(k-1).$$

The idea behind Theorem 2.2 is to increase the degree-dependent contribution of each vertex to the lower bound by reducing the denominator $d_G(u)+1$ of the corresponding fraction by some $h(u)$ and to show that the total reduction $\sum\limits_{u\in V(G)} h(u)$ is not too small.

The proof relies on a variant of Best-In-Greedy, which uses additional criteria for the selection of the vertex u within the **while**-loop of Best-In-Greedy. Let G_0 denote the initial input graph and let G denote the current graph at the beginning of some **while**-loop. Among all vertices of minimum degree of G the vertex u is chosen such that — with decreasing priority —

- $\sum\limits_{v\in N_G[u]}(d_{G_0}(v) - d_G(v))$ is as large as possible,

  which allows to exploit the fact that by removing vertices during previous iterations, the degrees of some of the remaining vertices were reduced;

- $\sum\limits_{v\in N_G[u]}(d_G(v) - \delta(G))$ is as large as possible,

  which allows to exploit degree variations within the neighborhoods of minimum degree vertices leading to improvements of inequality (3) in the proof of Lemma 2.1; and

- $\delta(G - N_G[u])$ is as small as possible,

  which improves the situation for the subsequent iterations.

Note that all these additional criteria make sense even if the input graph is not connected. For connected graphs though, their analysis leads to Theorem 2.2. In fact, the value of k within that result corresponds to the number of executions of the **while**-loop of the modified Best-In-Greedy. By setting $\gamma(\cdot) = d_{G_0}(\cdot) - d_G(\cdot)$ and $\beta(\cdot) = d_G(\cdot) - \delta(G)$, and suitably defining aggregated values Γ_i and B_i for each of the k executions, we have

$$k = \sum_{u\in V(G)} \frac{1}{d_G(u) + 1 - (\gamma(u) + \beta(u))}$$

and

$$\sum_{u\in V(G)} (\gamma(u) + \beta(u)) = \sum_{i=1}^{k} (\Gamma_i + B_i).$$

The proof of Theorem 2.2 is completed by showing that

- $\Gamma_i + B_i$ is 0 at most once in the very first iteration and that

- $\Gamma_i + B_i = 1$ for some i implies that $\Gamma_{i+1} + B_{i+1} \geq 3$, that is, $\Gamma_i + B_i$ is about 2 on average.

Applying Jensen's inequality to the somewhat unexplicit bound in Theorem 2.2 yields

$$\frac{\alpha(G)}{n(G)} \geq \frac{2}{\left(d(G)+1+\frac{2}{n(G)}\right)+\sqrt{\left(d(G)+1+\frac{2}{n(G)}\right)^2-8}}$$

for a connected graph G of order $n(G)$ and average degree $d(G)$.

A nice way to extract the essence of this bound is to consider asymptotic independence ratios. If $\mathcal{P}$ is an infinite class of graphs and $d \in \mathbb{R}_{\geq 0}$, then

$$\alpha_{\mathcal{P}}(d) = \lim_{n\to\infty} \inf \left\{ \frac{\alpha(G)}{n(G)} : G \in \mathcal{P},\ d(G) \leq d,\ n(G) \geq n \right\}$$

expresses the best-possible asymptotic lower bound on the independence ratio $\frac{\alpha(G)}{n(G)}$ for graphs G in $\mathcal{P}$ whose average degree is at most d. Considering $n(G) \geq n$ and $n \to \infty$ ensures that the value of $\alpha_{\mathcal{P}}(d)$ is not determined by some exceptional graphs of small order. Theorems 2.1 and 2.2 imply

$$\begin{aligned} \alpha_{\mathcal{G}}(d) &\geq \frac{1}{d+1} \text{ and} \\ \alpha_{\mathcal{G}_{\text{conn}}}(d) &\geq \left(\frac{2}{1+\sqrt{1-\frac{8}{(d+1)^2}}}\right)\frac{1}{d+1} \end{aligned}$$

where $\mathcal{G}$ denotes the class of all graphs and $\mathcal{G}_{\text{conn}}$ denotes the class of all connected graphs. Both these estimates are best possible for infinitely many discrete values of d. Between these values of d, the true values of the asymptotic independence ratios are obtained by linear interpolation [6]. See Figure 1 for an illustration.

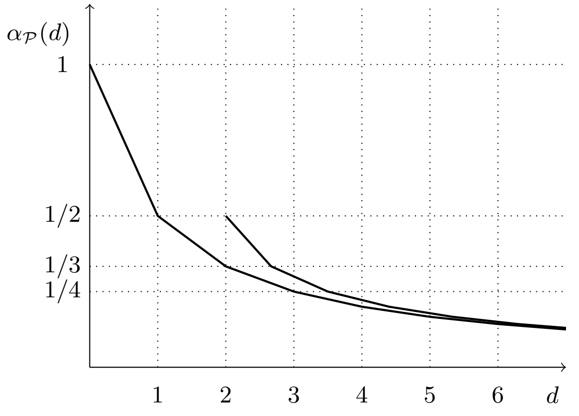

Figure 1: The asymptotic independence ratios of all graphs and all connected graphs.

Let us reconsider our strategy for improving Caro' and Wei's bound from a more general point of view by asking the following question:

> *Given a graph G, for which functions $p_G : V(G) \to \mathbb{N}_0$ can we prove*
>
> $$\alpha(G) \quad \geq \quad \sum_{u \in V(G)} \frac{1}{p_G(u)+1}, \tag{5}$$
>
> *while the values $p_G(u)$ are typically smaller than the degrees $d_G(u)$?*

In [1] we identify conditions on the function p_G such that suitable modifications of Best-In-Greedy and Worst-Out-Greedy allow the same kind of analysis as above replacing the degree function d_G with a smaller function p_G. The next result gives some examples where this approach works; the first two examples are based on versions of Best-In-Greedy while the third example is based on a version of Worst-Out-Greedy.

Theorem 2.3 (Borowiecki et al. 2012 [1]).
If G is a graph and $p_G : V(G) \to \mathbb{N}_0$ is such that

- either $p_G(u) = |\{v \in N_G(u) : d_G(v) \geq d_G(u)\}|$ for every vertex u of G,
- or $p_G(u) \geq |\{v \in N_G(u) : p_G(v) \geq p_G(u)\}|$ for every vertex u of G,
- or $p_G(u)$ is the maximum integer k such that there are k distinct neighbors $v_1, v_2, \ldots, v_k$ of u in G with $d_G(v_i) \geq i$ for every $1 \leq i \leq k$ and every vertex u of G,

then (5) holds.

Another interesting choice for p_G is related to greedy colorings and the Grundy number. For a graph G, a coloring $f : V(G) \to \mathbb{N}_0$ is a *greedy coloring* if there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that

$$f(v_i) = \min\left(\mathbb{N}_0 \setminus \{f(v_j) : j \in [i-1] \text{ with } v_j \in N_G(v_i)\}\right)$$

for $1 \leq i \leq n$, that is, the vertices of G are colored in the given order and each vertex receives the smallest possible available non-negative integer as a color. The *Grundy number* $\Gamma(G)$ and the *local Grundy numbers* $\Gamma_G(u)$ are defined as

$$\begin{aligned}\Gamma_G(u) &= \max\{f(u) : f \text{ is a greedy coloring}\} \text{ and} \\ \Gamma(G) &= \max\{\Gamma_G(u) : u \in V(G)\}.\end{aligned}$$

Analyzing a suitable version of Worst-Out-Greedy we are able to prove the following result.

Theorem 2.4 (Borowiecki et al. 2012 [1])**.**
If G is such that $\{v \in V(G) : \mathrm{dist}_G(u,v) \leq d_G(u)\}$ induces a tree in G for every vertex u of G, then

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{\Gamma_G(u)+1}.$$

As an interesting open problem in this context we pose the following.

Conjecture 2.1 (Borowiecki et al. 2012 [1])**.**
For every graph G,

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{\Gamma_G(u)+1}.$$

As our results illustrate, variants of the two greedy algorithms lead to improvements of Theorem 2.1. Also the probabilistic argument given above has further potential. In [1] we illustrate that it can be used to obtain an elegant and very short proof of Thiele's lower bound on the independence number of hypergraphs [13]. Thiele's original argument relied on a greedy algorithm and his analysis was quite involved.

3 Second Tune: Shearer

The general form of Shearer's bound is similar to Caro's and Wei's bound; the lower bound receives a degree-dependent contribution from every vertex.

Here is the complete statement.

Theorem 3.1 (Shearer 1991 [11])**.**
If G is a triangle-free graph, then

$$\alpha(G) \quad \geq \quad \sum_{u \in V(G)} f(d_G(u)) \tag{6}$$

where $f(0) = 1$ and $f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$ for $d \in \mathbb{N}$.

Note that $f(d) = \Omega\left(\frac{\log d}{d}\right)$, that is, the degree-dependent contribution of each vertex u to (6) is by a $\Omega(\log d_G(u))$ factor larger than in (1).

The approach taken by Shearer to prove his result is based on a version of Best-In-Greedy. For a function $f : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$, he considers the loss in the weight function $\sum\limits_{u \in V(G)} f(d_G(u))$ caused by removing the closed neighborhood $N_G[u]$ of some vertex u. Since G is triangle-free, all edges incident with neighbors of u that are not incident with u, connect $N_G[u]$

with $V(G) \setminus N_G[u]$. In fact, the removal of $N_G[u]$ reduces the degree of each vertex w in $V(G) \setminus N_G[u]$ by exactly $|N_G(u) \cap N_G(w)|$, which allows the $\Omega(\log d)$ improvement over Caro's and Wei's bound. For the choice of f given in Theorem 3.1, Shearer shows that the sum of

$$1 - \sum_{v \in V(G)} f(d_G(v)) + \sum_{w \in V(G) \setminus N_G[u]} f(d_G(w) - |N_G(w) \cap N_G(u)|)$$

over all vertices u of G is non-negative, which implies the existence of some vertex u such that removing $N_G[u]$ results in a loss of at most 1.

In [2] we consider a possible common generalization of (1) and (6) of the form

$$\alpha(G) \geq \sum_{u \in V(G)} g(d_G(u), t_G(u)), \tag{7}$$

where $t_G(u)$ denotes the number of distinct triangles of G that contain u, that is, $t_G(u)$ is the number of edges of G that completely lie in $N_G(u)$.

If $t_G \equiv 0$, then G is triangle-free and (7) should reduce to (6), that is, $g(d,0) = f(d)$. Similarly, if there is no bound on $t_G(u)$, then (7) should imply (1), that is, $\lim\limits_{t \to \infty} g(d,t) = \frac{1}{d+1}$.

The following reasonable choice for g is proposed in [2].

$$g(d,t) \quad = \quad \begin{cases} \frac{1}{d+1} & , t \geq \binom{d}{2}, \\ \frac{1+(d^2-d-2t)g(d-1,t)}{d^2+1-2t} & , t < \binom{d}{2}. \end{cases}$$

Note that for $t \geq \binom{d}{2}$, we allow enough edges in the neighborhood of some vertex to make the neighborhood complete, that is, we cannot expect an improvement over (1). Once $t < \binom{d}{2}$, the neighborhood cannot induce a complete graph and a similar recursion as in Shearer's result gives the values of g.

The following is our main result in [2].

Theorem 3.2 (Boßecker and Rautenbach 2010 [2]).
Let $T \in \mathbb{N}_0$. If G is a graph such that every vertex of G belongs to at most T triangles, then

$$\alpha(G) \geq \sum_{u \in V(G)} g(d_G(u), T).$$

While Theorem 3.2 actually is a common generalization of Theorems 2.1 and 3.1, we pose the following stronger conjecture.

Conjecture 3.1 (Boßecker and Rautenbach 2010 [2]).
If G is a graph, then

$$\alpha(G) \geq \sum_{u \in V(G)} g(d_G(u), t_G(u)).$$

4 Third Tune: Staton

Staton determined the independence ratio of cubic triangle-free graphs.

Theorem 4.1 (Staton 1979 [12]).
If G is a triangle-free graph of maximum degree at most three, then

$$\alpha(G) \quad \geq \quad \frac{5n(G)}{14}. \tag{8}$$

There are only the two cubic graphs in Figure 2 for which (8) is achieved with equality.

Figure 2: Extremal cubic graphs for Theorem 4.1.

In view of these graphs, it was quite a surprise when in 2001 Heckman and Thomas [8] found an extremely simple and elegant proof for Theorem 4.1. They consider a block of a graph as *difficult* if it is isomorphic to one of the two graphs G_2 and G_3 in Figure 3.

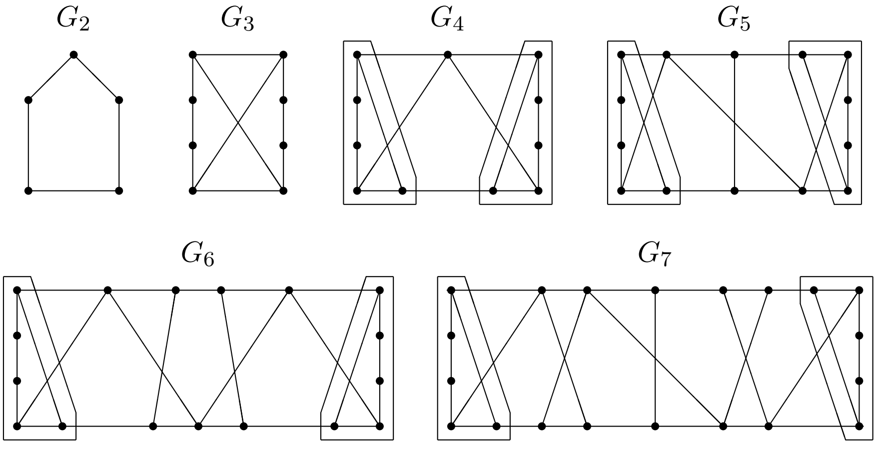

Figure 3: Some difficult blocks.

Furthermore, they consider a connected graph as *bad* if each of its blocks is either difficult or is a bridge between two difficult blocks. For a graph G, $\lambda(G)$ counts the number of bad components. Building on these notions they proved the following statement.

Theorem 4.2 (Heckman and Thomas 2001 [8]).
If G is a triangle-free graph of maximum degree at most three, then

$$\alpha(G) \geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right). \tag{9}$$

Since for a triangle-free cubic graph G we have

$$\begin{aligned}\alpha(G) &\geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right)\\ &= \frac{1}{7}\left(4n(G) - \frac{3}{2}n(G) - 0\right)\\ &= \frac{5}{14}n(G),\end{aligned}$$

Theorem 4.1 follows easily from Theorem 4.2.

We extended Theorem 4.2 in two ways.

In [5] we proved that adding to G_2 and G_3 the two difficult blocks shown in Figure 4 and denoting by $tr(G)$ the maximum number of vertex-disjoint triangles in G, then

$$\alpha(G) \geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G) - tr(G)\right)$$

for every K_4-free graph of maximum degree at most three. Note that $tr(G)$ can be determined efficiently for graphs of maximum degree at most three.

Figure 4: Two difficult blocks with triangles.

In [9] we showed that by extending the list of difficult blocks from the two elements G_2 and G_3 to an infinite sequence of well-described graphs $(G_i)_{i\geq 2}$, the bound (9) from Theorem 4.2 holds without any assumption on the maximum degree. The first seven graphs in the sequences $(G_i)_{i\geq 2}$ are shown in Figure 3 and each element of that sequence is obtained from the previous one by a simple extension operation.

As a consequence of this last result we can determine the asymptotic independence ratio of connected triangle-free graphs for average degrees up to $\frac{10}{3}$. See Figure 5 for an illustration.

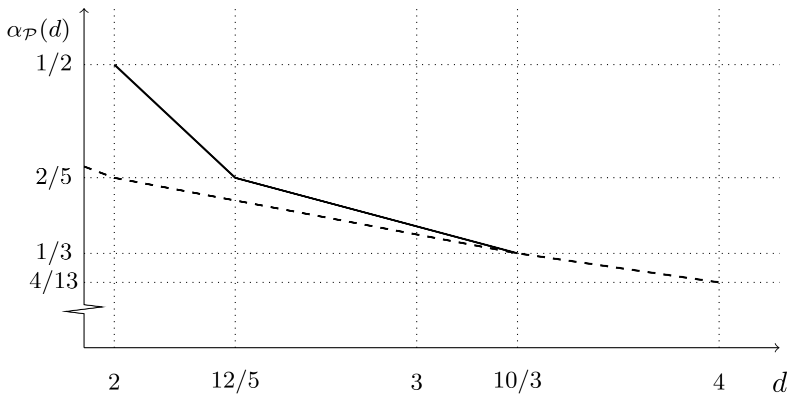

Figure 5: The dashed line and the line above it give the asymptotic independence ratio of triangle-free graphs and of connected triangle-free graphs, respectively.

5 Postlude

We close with one further result that fits well into the context and nicely relates the independence number and the matching number, denoted here by $\alpha'(G)$, of a graph G. Confirming a conjecture posed by Anders Sune Pedersen, we prove the following.

Theorem 5.1 (Henning, Löwenstein, and Rautenbach 2012 [7]).
If G is a triangle-free graph of maximum degree at most three, then

$$\frac{3}{2}\alpha(G) + \alpha'(G) \quad \geq \quad n(G). \tag{10}$$

The intuitive message of Theorem 5.1 is clear. Graphs with small maximum degree necessarily have few edges. In order to avoid a large independent set in such a graph, the few edges have to be well-spread in some sense. But this will force the existence of a large matching. In other words, either $\alpha(G)$ or $\alpha'(G)$ cannot be too small.

The proof of Theorem 5.1 follows the Heckman and Thomas' approach that we already exploited successfully for our results in the last section. A nice consequence is a best-possible χ-binding function for a certain class of graphs.

Corollary 5.1 (Henning, Löwenstein, and Rautenbach 2012 [7]).
If G is a $K_1 \cup K_4$-free graph with $\alpha(G) \leq 2$, then $\chi(G) \leq \frac{3}{2}\omega(G)$.

Proof: Since the complement $\overline{G}$ of G is triangle-free and of maximum degree at most three, Theorem 5.1 implies $\frac{3}{2}\alpha\left(\overline{G}\right) + \alpha'\left(\overline{G}\right) \geq n(G)$. Since $\alpha(G) \leq 2$, we obtain $\chi(G) = n(G) - \alpha'\left(\overline{G}\right) \leq \frac{3}{2}\alpha\left(\overline{G}\right) = \frac{3}{2}\omega(G)$. □

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References

[1] P. Borowiecki, F. Göring, J. Harant, and D. Rautenbach, The Potential of Greed for Independence, *J. Graph Theory* **71** (2012), 245-259.

[2] A. Boßecker and D. Rautenbach, Interpolating between Bounds on the Independence Number, *Discrete Math.* **310** (2010), 2398-2403.

[3] Y. Caro, New Results on the Independence Number, Technical Report, Tel-Aviv University, 1979.

[4] J. Harant and I. Schiermeyer, On the independence number of a graph in terms of order and size, *Discrete Math.* **232** (2001), 131-138.

[5] J. Harant, M.A. Henning, D. Rautenbach, and I. Schiermeyer, Independence number in graphs of maximum degree three, *Discrete Math.* **308** (2008), 5829-5833.

[6] J. Harant and D. Rautenbach, Independence in Connected Graphs, *Discrete Appl. Math.* **159** (2011), 79-86.

[7] M.A. Henning, C. Löwenstein, and D. Rautenbach, Independent Sets and Matchings in Subcubic Graphs, *Discrete Math.* **312** (2012), 1900-1910.

[8] C.C. Heckman and R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, *Discrete Math.* **233** (2001), 233-237.

[9] C. Löwenstein, A.S. Pedersen, D. Rautenbach, and F. Regen, Independence, Odd Girth, and Average Degree, *J. Graph Theory* **67** (2011), 96-111.

[10] V. Lozin and D. de Werra, Foreword: Special issue on stability in graphs and related topics, *Discrete Appl. Math.* **132** (2003), 1-2.

[11] J.B. Shearer, A note on the independence number of triangle-free graphs. II, *J. Comb. Theory, Ser. B* **53** (1991), 300-307.

[12] W. Staton, Some Ramsey-type numbers and the independence ratio, *Trans. Amer. Math. Soc.* **256** (1979), 353-370.

[13] T. Thiele, A lower bound on the independence number of arbitrary hypergraphs, *J. Graph Theory* **30** (1999), 213-221.

[14] V.K. Wei, A Lower Bound on the Stability Number of a Simple Graph, Technical memorandum, TM 81 - 11217 - 9, Bell laboratories, 1981.

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