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On the Hochschild-Kostant-Rosenberg theorem for differentiable manifolds

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Abstract

In these notes we will provide a set of techniques which can help one to understand the proof of the Hochschild-Kostant-Rosenberg theorem for differentiable manifolds. Precise definitions of multi-differential operators and polyderivations on an algebra are given, allowing to work on these concepts, when the algebra is an algebra of functions on a differentiable manifold, in a coordinate free description. Also, we will construct a cup product on polyderivations which corresponds on (Hochschild) cohomology to the wedge product on multivector fields. At the end, a proof of the above mentioned theorem will be given.

1 Introduction

In [5] the authors stablish some results relating the Hochschild homology groups of an algebra with the exterior power of universal differentials on that algebra in the context of regular affine algebras. From this, the authors also deduce a dual version, which relates the Hochschild cohomology modules of an algebra with the exterior power of derivations on that algebra. This is what we call today the Hochschild-Kostant-Rosenberg theorem. Since then many authors furnished similar results with other hypothesis on the algebra. For example, in [2] the author gives the cohomological version of the theorem in the case when the algebra is

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the algebra of (complex) functions on a compact smooth manifold. The motivation for the present work was [6], in which the author furnished the cohomological version of the theorem in the case when the algebra is the algebra of functions on a smooth manifold, as a first step for the proof of his *formality theorem*. When one is dealing with topological algebras (as algebras of functions on topological spaces, for example), a direct proof of the cohomological version of the theorem is desirable, because the dualisation process involves continuity and completion considerations, turning the arguments difficult to carry on.

The main purpose of these notes is to provide a set of techniques which can help one to understand the proof of the Hochschild-Kostant-Rosenberg theorem for differentiable manifolds. On the way to do that, we will give precise definitions of multidifferential operators and polyderivations on an algebra. When the algebra is an algebra of functions on a differentiable manifold, this allows us to work on these concepts in a coordinate free description. Also, we will construct a cup product on polyderivations which corresponds on (Hochschild) cohomology to the wedge product on multivector fields. For the theorem itself, we will follow the main ideas presented in [1].

To fully understand these notes, some background on algebra and differentiable manifolds is desirable. In section 2 some basic concepts are presented and some notations are fixes. So, the reader who has some knowledge on differential geometry, derivations and differential operators on associative algebras may go directly to section 3. In section 3 we will define the concept of multiderivations on an associative, commutative, unital algebra and state some results on algebras of smooth functions on a manifold. In section 4 we will define the concept of iterated derivations and relate to derivations of higher orders on the algebra of smooth functions on a manifold. In section 5 we will define the concept of polyderivations of an associative, commutative, unital algebra and we will use it to provide a coordinate free version of polyderivations on a manifold. Section 5 ends with a proof of the Hochschild-Kostant-Rosenberg theorem for differentiable manifolds which uses the tools introduced in the previous sections.

Throughout these notes, $Hom_{\mathbf{C}}(A,B)$ will denote the morphisms from A to B in the category $\mathbf{C}$ and $A \approx_{\mathbf{C}} B$ will mean that A is isomorphic to B in the category $\mathbf{C}$. As an example, $Hom_{\mathbf{Vec}^{\mathbb{K}}}(A,B)$ is the set of $\mathbb{K}$-linear transformations between the $\mathbb{K}$-vector spaces A and B.

2 Algebras, Derivations and the Tangent Bundle

In this section we will provide some of the standard concepts needed to understand the techniques developed in the final sections of these notes. It was made for setting some notations and some "ways of thinking". From now on, k will denote a commutative unital ring and $\mathbb{K}$ will denote a field.

Definition 2.1 (Derivation on an algebra)**.** *Let (A, μ, e) be an associative k-algebra with unit e (briefly A, when the product μ is clear from the context). A derivation D on A is a k-linear map $D : A \to A$ such that*

$$D(\mu) = \mu(D \otimes id) + \mu(id \otimes D)$$

where $id\colon A \to A$ is the identity on A.

The condition above is known as "Leibniz rule".

Remark 2.2. *The unit e on A provides an immersion $e : k \to A$, so elements in k can be viewed as elements in A through such immersion. If D is a derivation on A, by the Leibniz rule we have for all $a \in k$*

$$D(a) = aD(e) = aD(\mu(e \otimes e)) = a(\mu(D \otimes e) + \mu(e \otimes D)) = D(a) + D(a)$$

therefore $D(a) = 0$.

Notation 2.3. *The k-module of all derivations on A will be denoted by $Der(A)$.*

Definition 2.4 (Inner derivations)**.** *Let (A, μ, e) be an associative k-algebra with unit e. A k-linear map $f : A \to A$ is an inner derivation on A if and only if there exists $a \in A$ such that*

$$f = \mu(a \otimes id) - \mu(id \otimes a),$$

where $a \otimes id$ denotes $a \otimes id : A \to A \otimes A$ such that $(a \otimes id)(b) = a \otimes b$.

Remark 2.5. *By denoting $IDer(A)$ the k-module of inner derivations on A, we have $IDer(A) \subset Der(A)$.*

Now we make precise the notion of high order differential operator and high order derivation on a commutative associative k-algebra. The definition, naturally, is recursive.

Definition 2.6 (Higher order differential operator)**.** *Let* (A, μ, e) *be a commutative associative* k*-algebra with unit* e*. A* k*-linear map* $D : A \to A$ *is a differential operator of order* $\leq r$ *on* A*, with* $r \in \mathbb{N} \setminus \{0\}$ *if and only if for all* $a \in A$ *the map*

$$\tilde{D} = D(\mu(a \otimes id)) - \mu(a \otimes D)$$

is a differential operator of order $\leq r - 1$*. A map* $D : A \to A$ *is a differential operator of order 0 if and only if there exists* $a \in A$ *such that* $D = \mu(a \otimes id)$*.*

Definition 2.7 (Higher order derivation)**.** *Let* (A, μ, e) *be a commutative associative* k*-algebra with unit* e*. A derivation of order* $\leq r$ *on* A *is a differential operator of order* $\leq r$ D *such that* $D(a) = 0$*,* $\forall a \in k$*.*

Theorem 2.8. *Let* (A, μ, e) *be a commutative associative* k*-algebra with unit* e*. Then derivations on* A *are exactly the derivations of order* ≤ 1 *on* A *and a differential operator of order* ≤ 1 D *can be written uniquely as* $D = \partial + D(e)$ *with* $\partial \in Der(A)$*.*

Proof. Here it is convenient to use de juxtaposition to denote the product μ. If $D \in Der(A)$ then D is a derivation of order ≤ 1 because given $a \in A$

$$\tilde{D}(b) = D(ab) - aD(b) = D(a)b$$

and if $a \in k$ we have $D(a) = 0$.

Conversely, if $D : A \to A$ is a derivation of order ≤ 1 then the map $\partial = D - D(e)$, *i.e.*, $\partial(a) = D(a) - aD(e)$, $\forall a \in A$ satisfies

$$\partial(a) = D(a) - aD(e) = f_a$$

were f_a is the element of A given by

$$D(ab) - aD(b) = f_a b$$

(just put $b = e$) thus

$$D(ab) - aD(b) = \partial(a)b$$

hence

$$\begin{aligned} \partial(ab) &= D(ab) - abD(e) = D(ab) - aD(b) + aD(b) - abD(e) = \\ &= \partial(a)b + aD(b) - abD(e) = \partial(a)b + a(D(b) - bD(e)) = \\ &= \partial(a)b + a\partial(b) \end{aligned}$$

which means $\partial \in Der(A)$. □

When the algebra (A, μ) is graded we can define a derivation which "respects" such structure.

Definition 2.9 (Graded derivation)**.** *Let (A, μ) be a graded k-algebra. A k-linear map $D : A \to A$ is a graded derivation on A of degree p if and only if for all homogeneous elements $a \in A_i$ and for all $b \in A$ it satisfies*

$$D(\mu(a \otimes b)) = \mu(D(a) \otimes b) + (-1)^{pi} \mu(a \otimes D(b))$$

Now it is convenient to fix some notions (and notations).

Definition 2.10 (Superalgebra)**.** *We say that a graded k-algebra (A, μ) is a superalgebra (or supercommutative) if and only if μ satisfies for all $a \in A_i$, $b \in A_j$*

$$\mu(a \otimes b) = (-1)^{ij} \mu(b \otimes a)$$

Definition 2.11 (Lie superalgebra)**.** *A Lie superalgebra is a pair $(L, [\ ,\ ])$ where L is a $\mathbb{Z}$-graded $\mathbb{K}$-vector space and $[\ ,\ ] : L \times L \to L$ is a bilinear map such that*

i) $[L_i, L_j] \subset L_{i+j}$, $\forall i, j \in \mathbb{Z}$;

ii) $[a, b] = -(-1)^{ij}[b, a]$, $\forall a \in L_i$, $\forall b \in L_j$, *(graded antisymmetry);*

iii) $[a, [b, c]] = [[a, b], c] + (-1)^{ij}[b, [a, c]]$, $\forall a \in L_i$, $\forall b \in L_j$, $\forall c \in L$, *(graded Jacobi).*

Definition 2.12 (Derivation on a Lie superalgebra)**.** *Let $(L, [\ ,\ ])$ be a Lie superalgebra. A $\mathbb{K}$-linear map $D : L \to L$ is a derivation of degree p on L if and only if it satisfies for all $a \in L_i$ and for all $b \in L$*

$$D([a, b]) = [D(a), b] + (-1)^{ip}[a, D(b)]$$

Remark 2.13. *It is easy to see that if $(L, [\ ,\ ])$ is a Lie superalgebra, then for any $a \in L_i$, $ad(a) : L \to L$ given by $ad(a)(b) = [a, b]$ is a degree i derivation on L. This is exactly what the Jacobi identity is about.*

In differential geometry, the concept of tangent vector on a manifold at a point is often given by using equivalence classes of curves or simply stating the property of being a derivation at a point. In these notes we will use an equivalent (for finite dimensional differentiable manifolds) and well known definition for tangent vectors which is slightly different from the usual definition, but reveals some interesting aspects. The construction given here follows [10].

Let $\mathbf{M}$ be a differentiable manifold. For each $p \in \mathbf{M}$, define the $\mathbb{R}$-vector space V_p tangent at p as the following. Define the relation $\sim$ on $C^\infty(\mathbf{M})$ by $f \sim g$ if and only if there exists an open neighbourhood U of p such that $f|_U = g|_U$. This is an equivalence relation (the reader is invited to prove this, it is not hard) and the equivalence classes induced are called germs of functions at p. The algebraic operations on $C^\infty(\mathbf{M})$ can be used to induce a $\mathbb{R}$-algebra structure on $\mathcal{F}_p = C^\infty(\mathbf{M})/\sim$. Let I_p be the ideal of $\mathcal{F}_p$ of germs of functions that vanish at p. As I_p is an ideal of $\mathcal{F}_p$ and I_p^2 is an ideal of I_p, we have that I_p/I_p^2 is a $\mathbb{R}$-vector space. Then we define $V_p = (I_p/I_p^2)^*$, *i.e.*, V_p is the vector space dual to I_p/I_p^2. We will prove that V_p is finite dimensional. From now on, we will denote by (U, φ, m) a local chart on a differentiable manifold $\mathbf{M}$ where U is an open subset of $\mathbf{M}$, $\varphi : U \to U_0$ is a diffeomorphism from U to an open subset U_0 of $\mathbb{R}^m$ or simply (U, φ) if the dimension of $\mathbf{M}$ is clear.

Proposition 2.14. *Let (U, φ, m) be a local chart of $\mathbf{M}$ around p. Denoting the i-th canonical projection on $\mathbb{R}^m$ by $t^i : \mathbb{R}^m \to \mathbb{R}$ and the i-th coordinate function on U by $x^i = t^i \circ \varphi$ then the set of equivalence classes of x^i for $i = 1, \ldots, m$ in I_p/I_p^2 constitute a basis for such space.*

Proof. Given $\mathbf{f} \in I_p/I_p^2$, let $f \in C^\infty(\mathbf{M})$ be a representing element. Note that $f(p) = 0$. Without loss of generality we can suppose $\varphi(U)$ convex[1] and $\varphi(p) = 0$. The coordinate expression of f is given by $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$, which by the Taylor formula gives, for $y = \varphi(q), q \in U$

$$\begin{aligned}(f \circ \varphi^{-1})(y) &= \sum_{i=1}^{m} \left.\frac{\partial (f \circ \varphi^{-1})}{\partial t^i}\right|_0 t^i(y) + \\ &+ \sum_{i,j=1}^{m} t^i(y) t^j(y) \int_0^1 (1-s) \left.\frac{\partial^2 (f \circ \varphi^{-1})}{\partial t^i \partial t^j}\right|_{sy} ds\end{aligned}$$

$$\begin{aligned}(f \circ \varphi^{-1})(\varphi(q)) &= \sum_{i=1}^{m} \left.\frac{\partial (f \circ \varphi^{-1})}{\partial t^i}\right|_0 t^i(\varphi(q)) + \\ &+ \sum_{i,j=1}^{m} t^i(\varphi(q)) t^j(\varphi(q)) \int_0^1 (1-s) \left.\frac{\partial^2 (f \circ \varphi^{-1})}{\partial t^i \partial t^j}\right|_{sy} ds\end{aligned}$$

[1] It is necessary because we want to use Taylor's formula with integral remainder.

$$
\begin{aligned}
f(q) &= \sum_{i=1}^{m} \frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\bigg|_{\varphi(p)} x^i(q) + \\
&+ \sum_{i,j=1}^{m} x^i(q)x^j(q)\int_0^1 (1-s)\frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i\partial t^j}\bigg|_{sy} ds
\end{aligned}
$$

Because $f \in C^\infty(\mathbf{M})$ and $x^i(p) = 0$, the term

$$
\sum_{i,j=1}^{m} x^i x^j \int_0^1 (1-s)\frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i\partial t^j}\bigg|_{sy} ds
$$

represents the zero class in I_p/I_p^2. From this we infer that we can write

$$
\mathbf{f} = \sum_{i=1}^{m} \frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\bigg|_{\varphi(p)} \mathbf{x}^i
$$

where $\mathbf{x}^i$ is the class of x^i in I_p/I_p^2. Hence the set $\{\mathbf{x}^i\}, i = 1, \dots m$ spans I_p/I_p^2. To show the linear independence, notice that

$$
\sum_{i=1}^{m} a_i\mathbf{x}^i = 0 \Rightarrow \sum_{i=1}^{m} a_i[x^i] \in I_p^2
$$

where $[x^i]$ is a representative of $\mathbf{x}^i$ in I_p. In coordinates, we have:

$$
\left(\sum_{i=1}^{m} a_i x^i\right)\circ\varphi^{-1} = \sum_{i=1}^{m} a_i(x^i\circ\varphi^{-1}) = \sum_{i=1}^{m} a_i t^i
$$

which shows that $\sum_{i=1}^m a_i[t^i] \in I_{\varphi(p)}^2$, because the map $\varphi^{-1} : \varphi(U) \to U$ induces an algebra homomorphism $(\varphi^{-1})^* : \mathcal{F}_p \to \mathcal{F}_{\varphi(p)}$ given by $(\varphi^{-1})^*([f]) = [f\circ\varphi^{-1}]$. Thus, the first order terms vanish, which means that for each $j = 1, \dots, m$

$$
\frac{\partial}{\partial t^j}\left(\sum_{i=1}^{m} a_i t^i\right)\bigg|_0 = 0
$$

and therefore $a_i = 0$, for all $i = 1, \dots, m$. □

From this we conclude that V_p is finite dimensional and $dim(V_p) = m$. An element $\xi_p \in V_p$ is called a tangent vector on $\mathbf{M}$ at p.

For each $p \in \mathbf{M}$ we can associate to each tangent vector $\xi_i \in V_p$ a linear map $v_p : \mathcal{F}_p \to \mathbb{R}$, given by

$$v_p(f) = \begin{cases} 0 & \text{, if } \exists\, c \in f \mid c(x) = c \;\forall x \in \mathbf{M} \\ \xi_p([f]) & \text{, if } f \in I_p \end{cases}$$

where $[f]$ denotes the class corresponding to the germ f in I_p/I_p^2. Since all germs can be written as $f = \tilde{f} + f(p)$, where $\tilde{f} \in I_p$ and $f(p)$ is the germ of the constant function whose value is $f(p)$, v_p satisfies the following property

$$\begin{aligned} v_p(fg) &= v_p((\tilde{f} + f(p))(\tilde{g} + g(p))) = \\ &= v_p(\tilde{f}\tilde{g} + f(p)\tilde{g} + g(p)\tilde{f} + f(p)g(p)) = \\ &= v_p(\tilde{f}\tilde{g}) + v_p(f(p)\tilde{g}) + v_p(g(p)\tilde{f}) + v_p(f(p)g(p)) = \\ &= \xi_p(\tilde{f}\tilde{g}) + f(p)\xi_p(\tilde{g}) + g(p)\xi_p(\tilde{f}) + 0 = \\ &= f(p)\xi_p(\tilde{g}) + g(p)\xi_p(\tilde{f}) = \\ &= f(p)v_p(\tilde{g}) + g(p)v_p(\tilde{f}) = \\ &= g(p)v_p(f) + f(p)v_p(g) \end{aligned}$$

When a linear map $w : \mathcal{F}_p \to \mathbb{R}$ obeys

$$w(fg) = g(p)w(f) + f(p)w(g)$$

we call w a *derivation on $\mathcal{F}_p$ at the point p.*

Conversely, if w is a derivation on $\mathcal{F}_p$ at p, we can associate to w a unique tangent vector η_p such that $\eta_p([f]) = w(f)$ for all $f \in \mathcal{F}_p$. To see this, note that if c is the germ of a constant function

$$w(c) = w(c \cdot 1) = cw(1) = cw(1 \cdot 1) = cw(1) + cw(1) = 2w(c)$$

and therefore $w(c) = 0$. By writing f as $f = \tilde{f} + f(p)$, we have:

$$w(f) = w(\tilde{f} + f(p)) = w(\tilde{f}) + w(f(p)) = w(\tilde{f})$$

therefore the value of w is determined by its value at I_p. If $f \in I_p^2$, there exists $g, h \in I_p$ such that $f = gh$. Hence

$$w(f) = w(gh) = h(p)w(g) + g(p)w(h) = 0$$

thus, w vanishes on I_p^2. However, $f = \tilde{f} + f(p)$ gives

$$w(f) = w(f - f(p)) = w(\tilde{f})$$

which shows that if $\tilde{f} \equiv \tilde{g} \ mod \ I_p^2$, then $w(f) = w(g)$ and w induces a unique linear map η_p taking elements in I_p/I_p^2 to real values. In other words $\eta_p \in V_p$.

So we stablish a one-one correspondence between derivations on $\mathcal{F}_p$ at p and elements in $(I_p/I_p^2)^*$. It is not hard to see that the set of such derivations at a point, with the usual operations of addition and product by scalars, turns out to be an $\mathbb{R}$-vector space and the association that sends elements in $(I_p/I_p^2)^*$ to derivations on $\mathcal{F}_p$ at p above mentioned is a $\mathbb{R}$-vector space isomorphism. Thus, we can speak of elements in V_p acting on a germ $f \in \mathcal{F}_p$, once we implicitly understood the association above constructed. Further, define the action of a tangent vector $v_p \in V_p$ on a function $f \in C^\infty(\mathbf{M})$ by

$$v_p(f) = v_p(\mathbf{f})$$

where $\mathbf{f}$ is the class of f in $\mathcal{F}_p$. So, $v_p(g) = v_p(f)$ when $g \in \mathbf{f}$. Linearity and Leibniz rule follow straightforwardly from the definition.

From those facts, if $\mathbf{M}$ is a m-dimensional differentiable manifold, we can construct the differentiable vector bundle $(\mathcal{E}, \mathbf{M}, \pi, GL(\mathbb{R}^m))$ with total space $\mathcal{E} = \cup_{p \in \mathbf{M}} V_p$, base space $\mathbf{M}$, projection π given by $\pi(v_p) = p$ and typical fibre $\mathbb{R}^m$, with differentiable structure obtained from the structure on $\mathbf{M}$. Indeed, let (U, φ) be a local chart on the m-dimensional differentiable manifold $\mathbf{M}$ around $p \in \mathbf{M}$. By using the same notations of proposition 2.14, given $f \in C^\infty(\mathbf{M})$, take elements $\frac{\partial}{\partial x^i}|_p \in V_p$ for $i = 1, \ldots, m$, such that

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial t^i}\right|_{\varphi(p)}$$

Now, given $\eta \in \pi^{-1}(U)$, for all $f \in C^\infty(\mathbf{M})$

$$\eta(f) = \eta\left(\sum_{i=1}^m \left.\frac{\partial(\tilde{f} \circ \varphi^{-1})}{\partial t^i}\right|_{\varphi(\pi(\eta))} \mathbf{x}^i\right) = \sum_{i=1}^m \left.\frac{\partial f}{\partial x^i}\right|_{\pi(\eta)} \eta(x^i)$$

allowing to write

$$\eta = \sum_{i=1}^m \eta(x^i) \left.\frac{\partial}{\partial x^i}\right|_{\pi(\eta)} \tag{2.1}$$

and we call this formula the *coordinate expression* of η with respect to the local chart (U, φ) and we call the values $\eta(x^i)$ *coordinates of* η. Hence we can define a map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^m$ given by

$$\tilde{\varphi}(\eta) = (\eta(x^1), \dots, \eta(x^m))$$

Finally, define the map $\phi : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ given by

$$\phi(\eta) = ((\varphi \circ \pi)(\eta), \tilde{\varphi}(\eta)) \tag{2.2}$$

Let $\mathfrak{A}(\mathbf{M})$ be the atlas of $\mathbf{M}$. To each local chart $(U_\alpha, \varphi_\alpha)$ we associate the map $\phi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2m}$, constructed as above. Declare a set $V \subset \mathcal{E}$ open on $\mathcal{E}$ if and only if there exists an open set V_0 on $\mathbb{R}^m$ and an index α such that $V = \phi_\alpha^{-1}(V_0)$. The collection of those sets is a base for the topology on $\mathcal{E}$ which makes $\mathcal{E}$ a topological manifold. Besides, if $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathfrak{A}(\mathbf{M})$, then the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\pi^{-1}(U_\alpha)) \to \phi_\beta(\pi^{-1}(U_\beta))$ is C^∞. This shows that the collection $(\pi^{-1}(U_\alpha), \phi_\alpha)$ defines a differentiable atlas on $\mathcal{E}$.

With this structures, we see that $(\mathcal{E}, \mathbf{M}, \pi, GL(\mathbb{R}^m))$ is a differentiable vector bundle with typical fibre $\mathbb{R}^m$, total space $\mathcal{E}$, which is a $2m$-dimensional differentiable manifold, base space $\mathbf{M}$, projection $\pi : \mathcal{E} \to \mathbf{M}$, which is a differentiable surjective submersion, trivializations given by the maps $\phi_\alpha = (\varphi_\alpha \circ \pi, \tilde{\varphi}_\alpha)$, structure group $GL(\mathbb{R}^m)$ in which compatibility conditions of trivializations are satisfied, since map such as $\phi_\beta \circ \phi_\alpha^{-1}$ are diffeomorphisms. To fix notations, we will write $T\mathbf{M} = \mathcal{E}$ and call $T\mathbf{M}$ the *tangent bundle* of $\mathbf{M}$, since its elements can be regarded as tangent vectors at points in $\mathbf{M}$. From now on we will denote the vector space tangent at p by $T_p\mathbf{M} = V_p$.

We will use this

Definition 2.15 (Vector fields)**.** *Let* $\mathbf{M}$ *be a* m*-dimensional differentiable manifold. The* C^∞ *sections of* π *of* $(T\mathbf{M}, \mathbf{M}, \pi, GL(\mathbb{R}^m))$ *are called vector fields on* $\mathbf{M}$*. We denote the* $\mathbb{R}$*-vector space of vector fields with the usual operations of addition and product by scalars pointwise by* $\mathfrak{X}(\mathbf{M})$*. Also,* $\mathfrak{X}$ *has a* $C^\infty(\mathbf{M})$*-module structure given by pointwise product by functions.*

This leads to the following

Theorem 2.16. *Let* $\mathbf{M}$ *be a* m*-dimensional differentiable manifold and* $C^\infty(\mathbf{M})$ *the* $\mathbb{R}$*-algebra of* C^∞ *functions on* $\mathbf{M}$*. Then*

$$\mathfrak{X}(\mathbf{M}) \approx_{\mathbf{Vec}^{\mathbb{R}}} Der(C^\infty(\mathbf{M}))$$

The reader is invited to prove the above theorem using the tools (and definitions) given here as an exercise.

Following these steps, we can now construct a differentiable vector bundle over a differentiable manifold $\mathbf{M}$, whose differentiable sections correspond to derivations of order $\leq r$ on the algebra of functions $C^\infty(\mathbf{M})$ (definition 2.7).

First, we will need the notion of higher order derivation at a point. Let $p \in \mathbf{M}$ and $\mathcal{F}_p$ be the $\mathbb{R}$-algebra of germs of functions at p. If I_p denotes the ideal of germs of functions vanishing at p we have (as above) a natural $\mathbb{R}$-vector space structure on I_p/I_p^r, since I_p^r (here $r \in \mathbb{Z},\ r \geq 1$) is an ideal of I_p. By denoting $J_p^r = (I_p/I_p^{r+1})^*$, we can repeat all what we have done on V_p and define derivations of order $\leq r$ at p.

Definition 2.17. *Let $\mathcal{F}_p$ be the germs of functions at a point p in an m-dimensional differentiable manifold $\mathbf{M}$. We say that an $\mathbb{R}$-linear map $D_p : \mathcal{F}_p \to \mathbb{R}$ is a differential operator of order $\leq r$, $r \geq 1$, at p if and only if for all $g \in \mathcal{F}_p$, the map $d_g : \mathcal{F}_p \to \mathbb{R}$ given by*

$$d_g(f) = D_p(gf) - g(p)D_p(f)$$

is a differential operator of order $\leq r-1$ at p, and called a differential operator of order 0 at p if it is a product by a germ of functions at p. D_p is called a derivation of order $\leq r$ at p if, in addition, it is identically zero on germs of constant functions.

The previous definition should be compared with Definition 2.7.

Proposition 2.18. *J_p^r is finite dimensional.*

Proof. Let (U, φ) be a local chart around $p \in \mathbf{M}$. Without loss of generality, we can suppose $\varphi(U)$ convex and $\varphi(p) = 0$. As before, denote the coordinates on U by $x^i = t^i \circ \varphi$, where $t^i : \mathbb{R}^m \to \mathbb{R}$ is the i-th projection on $\mathbb{R}^m$. The claim follows by showing that the set of classes of the functions x^i, $i = 1, \ldots, m$, $x^i x^j, 1 \leq i \leq j \leq n, \ \ldots, x^{i_1} \cdot \ldots \cdot x^{i_r}$, $i_1 \leq \ldots \leq i_r$ is a basis for I_p/I_p^{r+1}. Let $f \in I_p/I_p^{r+1}$. Let $f \in C^\infty(\mathbf{M})$ be a representing element of this class. By the Taylor formula, the coordinate expression of f on U is written by

$$\begin{aligned} f &= \sum_{i=1}^m \left.\frac{\partial(f \circ \varphi^{-1})}{\partial t^i}\right|_0 x^i + \frac{1}{2}\sum_{i,j=1}^m \left.\frac{\partial^2(f \circ \varphi^{-1})}{\partial t^i \partial t^j}\right|_0 x^i x^j + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^m x^{i_1}\ldots x^{i_{r+1}} \int_0^1 (1-s)^r \left.\frac{\partial^{r+1}(f \circ \varphi^{-1})}{\partial t^{i_1}\ldots \partial t^{i_{r+1}}}\right|_{sy} ds \end{aligned}$$

By taking the quotient, we note that the last term on the right hand side vanishes on I_p/I_p^{r+1}. Hence, the class of f is written

$$\begin{aligned} f &= \sum_{i=1}^{m} \frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\bigg|_0 x^i + \frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i\partial t^j}\bigg|_0 x^i x^j + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_r=1}^{m}\frac{\partial^r(f\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_r}}\bigg|_0 x^{i_1}\ldots x^{i_r} \end{aligned}$$

which shows that x^i, $i=1,\ldots,m$, $x^ix^j, 1\le i\le j\le n$, $\ldots, x^{i_1}\cdot\ldots\cdot x^{i_r}$, $i_1\le\ldots\le i_r$ spans I_p/I_p^{r+1}, since $f\circ\varphi^{-1}$ is differentiable.

To show the linear independence, note that

$$\begin{aligned} &\sum_{i=1}^{m} a_i x^i + \sum_{1\le i\le j\le m} a_{ij}x^ix^j + \ldots + \sum_{i_1\le\ldots\le i_r} a_{i_1\ldots i_r}x^{i_1}\ldots x^{i_r} = 0 \Rightarrow \\ &\Rightarrow \sum_{i=1}^{m} a_i x^i + \sum_{1\le i\le j\le m} a_{ij}x^ix^j + \ldots + \sum_{i_1\le\ldots\le i_r} a_{i_1\ldots i_r}x^{i_1}\ldots x^{i_r} \in I_p^{r+1} \Rightarrow \\ &\Rightarrow \sum_{i=1}^{m} a_i t^i + \sum_{1\le i\le j\le m} a_{ij}t^it^j + \ldots + \sum_{i_1\le\ldots\le i_r} a_{i_1\ldots i_r}t^{i_1}\ldots t^{i_r} \in I_{\varphi(p)}^{r+1} \end{aligned}$$

Thus, the terms of order $\le r$ are zero, which leads to

$$\begin{aligned} &\frac{\partial^r}{\partial t^{j_1}\ldots\partial t^{j_r}}\left(\sum_{i_1\le\ldots\le i_r} a_{i_1\ldots i_r}t^{i_1}\ldots t^{i_r}\right)\bigg|_0 = 0 \\ &\vdots \\ &\frac{\partial^2}{\partial t^k\partial t^l}\left(\sum_{1\le i\le j\le m} a_{ij}t^it^j\right)\bigg|_0 = 0 \\ &\frac{\partial}{\partial t^j}\left(\sum_{i=1}^{m} a_i t^i\right)\bigg|_0 = 0 \end{aligned}$$

hence

$$a_{j_1\ldots j_r} = 0, \ldots,\ \ a_{kl} = 0,\ \ a_j = 0$$

for all possible combinations of indices. So, I_p/I_p^{r+1} is finite dimensional, therefore J_p^r is finite dimensional also. □

Let $\xi_p \in J^r_p$. Associate to ξ_p a linear map $D_p : \mathcal{F}_p \to \mathbb{R}$ given by

$$D_p(f) = \begin{cases} 0 & \text{, if } \exists\, c \in f \mid c(x) = c \;\forall x \in \mathbf{M} \\ \xi_p([f]) & \text{, if } f \in I_p \end{cases}$$

where $[f]$ denotes the class of f in I_p/I_p^{r+1}. By writing germs $f \in \mathcal{F}_p$ as $f = \tilde{f} + f(p)$, with $\tilde{f} \in I_p$, given $g \in \mathcal{F}_p$ we have

$$\begin{aligned} &\Delta_g(f) = D_p(gf) - g(p)D_p(f) = \\ &= D_p(\tilde{g}\tilde{f}) + g(p)D_p(\tilde{f}) + f(p)D_p(\tilde{g}) + 2f(p)g(p)D_p(1) - g(p)D_p(\tilde{f}) = \\ &= D_p(\tilde{g}\tilde{f}) + f(p)D_p(\tilde{g}) = \xi_p(\tilde{g}\tilde{f}) + f(p)\xi_p(\tilde{g}). \end{aligned} \tag{2.3}$$

Note that, given $f_1 \in I_p$, for all $f_0 \in I_p$, we have

$$\delta^{r-1}_{f_1}(f_0) = \xi_p(f_1 f_0) - f_1(p)\xi_p(f_0) = \xi_p(f_1 f_0)$$

and successively we can see that, given $f_1, \ldots, f_i$

$$\delta^{r-i}_{f_i}(f_0) = \xi_p(f_i f_{i-1} \ldots f_0)$$

for all $f_0 \in I_p$. Now

$$\delta^{0}_{f_r}(f_0) = \xi_p(f_r \ldots f_0) = 0$$

which shows that $\delta^1_{f_{r-1}}$ is a differential operator of order ≤ 1 at the point p, viewed as restricted to I_p. Restricting to I_p, by construction, $\delta^{r-i}_{f_i}$ being a differential operator of order $\leq r-i$ at p implies that $\delta^{r-i+1}_{f_{i-1}}$ is a differential operator of order $\leq$ to $r-i+1$ at p. Hence, we have ξ_p differential operator of order $\leq$ to r at p and also, if $\delta : I_p \to \mathbb{R}$ is an operator such that for all $f \in I_p$, $\delta(f) = \xi_p(f_1 \ldots f_k f)$, with $f_1, \ldots, f_k \in I_p$, then δ is a differential operator of order $\leq r-k$ at p. Putting on equation 2.3, we see that Δ_g is a differential operator of order $\leq r-1$ at p, leading to the conclusion that D_p is a differential operator of order $\leq$ r at the point p.

Conversely, let $\omega : \mathcal{F}_p \to \mathbb{R}$ be a derivation of order $\leq r$ at the point p. Then $f \in \mathcal{F}_p$ gives

$$\omega(f) = \omega(\tilde{f} + f(p)) = \omega(\tilde{f})$$

which shows that the value of ω depends on its evaluation at I_p only. We also have that if $f \in I_p^{r+1}$ then there exists $f_1, \ldots, f_{r+1} \in I_p$ such that $f = f_1 \ldots f_{r+1}$ and therefore

$$\begin{aligned}
&\omega(f) = \omega(f_1 \ldots f_{r+1}) = \delta^{r-1}_{f_{r+1}}(f_1 \ldots f_r) + f_{r+1}(p)\omega(f_1 \ldots f_r) = \\
&= \delta^{r-1}_{f_{r+1}}(f_1 \ldots f_r) = \delta^{r-2}_{f_r}(f_1 \ldots f_{r-1}) = \ldots = \delta^0_{f_2}(f_1) = \\
&= f_1(p)g = 0
\end{aligned}$$

for some $g \in \mathcal{F}_p$, where $\delta^{r-i}_{f_{r-i+2}}$ is a differential operator of order $\leq r - i$, for $i = 1, \ldots, r$. So, if $f \equiv g \ mod\ I_p^{r+1}$ in I_p then $\omega(f) = \omega(g)$ and ω can be viewed as an element in $(I_p/I_p^{r+1})^*$. It follows that there exists an one to one correspondence between derivations of order $\leq r$ at the point p and elements in J_p^r.

We define the action of a derivation of order $\leq r$ at the point p, D_p, on a function $f \in C^\infty(\mathbf{M})$ as given by

$$D_p(f) = \xi_p([f])$$

where ξ_p is the element related to D_p by the above correspondence and $[f]$ is the equivalence class of the function f in I_p/I_p^{r+1}.

We can now prove the following theorem.

Theorem 2.19. *Let* $\mathbf{M}$ *be an* m*-dimensional differentiable manifold. There exists a differentiable vector bundle* $J^r(\mathbf{M})$*, whose base space is* $\mathbf{M}$ *and whose space of differentiable sections* $\Gamma(J^r(\mathbf{M}))$ *is isomorphic, as* $\mathbb{R}$*-vector space, to the space of derivations of order* $\leq r$ *on* $C^\infty(\mathbf{M})$*.*

Proof. Let us take the differentiable vector bundle $J^r(\mathbf{M}) = \bigcup_{p \in \mathbf{M}} J_p^r$, with $J_p^r = (I_p/I_p^{r+1})^*$ whose coordinate functions $\phi : \pi^{-1}(U) \to \mathbb{R}^K$, with π the projection of $J^r(\mathbf{M})$ on $\mathbf{M}$ and U an open subset of $\mathbf{M}$, are of the form $\phi(\omega_p) = (x^i(\pi(\omega_p)), \omega_p(x^i), \omega_p(x^i x^j), \ldots, \omega_p(x^{i_1} \ldots x^{i_r}))$, where the indices are increasing, 1 to m, for all $\omega_p \in J_p^r$. Note that $K = \sum_{k=1}^r \binom{m+k-1}{k}$.

Let $\omega : \mathbf{M} \to J^r(\mathbf{M})$ a differentiable section of $J^r(\mathbf{M})$. To ω we associate the map $D : C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$ given by

$$D(f)(p) = \omega(p)(f) \quad \forall f \in C^\infty(\mathbf{M})$$

The map D above defined is a derivation of order $\leq r$ on $C^\infty(\mathbf{M})$. To see this, first note that if c is a constant function, then

$$D(c)(p) = \omega(p)(c) = 0 \quad \forall p \in \mathbf{M}$$

hence D vanishes on constants. Second, given $g \in C^\infty(\mathbf{M})$, the operator Δ_g given by

$$\Delta_g(f) = D(gf) - gD(f) \quad \forall f \in C^\infty(\mathbf{M})$$

is such that

$$\begin{aligned}\Delta_g(f)(p) &= D(gf)(p) - g(p)D(f)(p) = \omega(p)(gf) - g(p)\omega(p)(f) = \\ &= \delta_g(p)(f)\end{aligned}$$

which is a differentiable section (by construction) of $J^{r-1}(\mathbf{M})$. However, $\Gamma(J^1(\mathbf{M})) = \mathfrak{X}(\mathbf{M})$ and theorem 2.16 shows that $\Gamma(J^1(\mathbf{M})) \approx_{\mathbf{Vec}^{\mathbb{R}}} Der(C^\infty(\mathbf{M}))$. Therefore, by induction, D is a derivation of order $\leq r$ on $C^\infty(\mathbf{M})$.

The assignment $\omega \mapsto D$ is clearly linear. Let's show it is injective. Suppose D associated to ω is identically zero. We have

$$\begin{aligned} D(f) = 0 \quad &, \quad \forall f \in C^\infty(\mathbf{M}) \\ D(f)(p) = 0 \quad &, \quad \forall f \in C^\infty(\mathbf{M}),\ \forall p \in \mathbf{M} \\ \omega(p)(f) = 0 \quad &, \quad \forall p \in \mathbf{M},\ \forall f \in C^\infty(\mathbf{M}) \\ \omega(p) = 0 \quad &, \quad \forall p \in \mathbf{M} \\ \omega = 0 \quad & \end{aligned}$$

Let's show it is surjective. Let $D : C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$ be a derivation of order $\leq r$ on $C^\infty(\mathbf{M})$. For each $p \in \mathbf{M}$, define $\omega_p : \mathcal{F}_p \to \mathbb{R}$ by

$$\omega_p(f) = D(f)(p) \quad \forall f \in C^\infty(\mathbf{M})$$

where f on the left hand side is for the equivalence class in $\mathcal{F}_p$ of the function represented by the symbol f on the right hand side. ω_p is well defined because if $f, g \in C^\infty(\mathbf{M})$ are such that $f \equiv g$ in $\mathcal{F}_p$, we can take a local chart (U, φ) around p such that $U \subset W$, where W is an open subset of $\mathbf{M}$ in which f and g coincide, $\varphi(p) = 0$ and $\varphi(U)$ is convex. Now, on U, f and g are written

$$\begin{aligned} f &= f(p) + \sum_{i=1}^{m} \left.\frac{\partial(f \circ \varphi^{-1})}{\partial t^i}\right|_0 x^i + \frac{1}{2}\sum_{i,j=1}^{m} \left.\frac{\partial^2(f \circ \varphi^{-1})}{\partial t^i \partial t^j}\right|_0 x^i x^j + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} x^{i_1}\ldots x^{i_{r+1}} \int_0^1 (1-s)^r \left.\frac{\partial^{r+1}(f \circ \varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\right|_{sy} ds \end{aligned}$$

and

$$\begin{aligned} g &= g(p) + \sum_{i=1}^{m} \left.\frac{\partial(g\circ\varphi^{-1})}{\partial t^i}\right|_0 x^i + \frac{1}{2}\sum_{i,j=1}^{m} \left.\frac{\partial^2(g\circ\varphi^{-1})}{\partial t^i \partial t^j}\right|_0 x^i x^j + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} x^{i_1}\ldots x^{i_{r+1}} \int_0^1 (1-s)^r \left.\frac{\partial^{r+1}(g\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\right|_{sy} ds \end{aligned}$$

where $t^i : \mathbb{R}^m \to \mathbb{R}$ is the i-th canonical projection on $\mathbb{R}^m$. Hence, on U,

$$\begin{aligned} D(f) &= \sum_{i=1}^{m} \left.\frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\right|_0 D(x^i) + \frac{1}{2}\sum_{i,j=1}^{m} \left.\frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i \partial t^j}\right|_0 D(x^i x^j) + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} D(x^{i_1}\ldots x^{i_{r+1}}) \int_0^1 (1-s)^r \left.\frac{\partial^{r+1}(f\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\right|_{sy} ds \end{aligned}$$

and

$$\begin{aligned} D(g) &= \sum_{i=1}^{m} \left.\frac{\partial(g\circ\varphi^{-1})}{\partial t^i}\right|_0 D(x^i) + \frac{1}{2}\sum_{i,j=1}^{m} \left.\frac{\partial^2(g\circ\varphi^{-1})}{\partial t^i \partial t^j}\right|_0 D(x^i x^j) + \ldots + \\ &+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} D(x^{i_1}\ldots x^{i_{r+1}}) \int_0^1 (1-s)^r \left.\frac{\partial^{r+1}(g\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\right|_{sy} ds \end{aligned}$$

But since D is a derivation of order $\leq r$, we have

$$D(x^{i_1}\ldots x^{i_{r+1}})(p) = 0$$

for all relevant combinations of indices. As f and g are in the same germ of functions at p, all partial derivatives up to order r of its coordinate expressions coincide, leading to

$$\begin{aligned}
&\omega_p(f) = D(f)(p) = \\
&= \sum_{i=1}^{m} \frac{\partial(f\circ\varphi^{-1})}{\partial t^i}\bigg|_0 D(x^i)(p) + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2(f\circ\varphi^{-1})}{\partial t^i\partial t^j}\bigg|_0 D(x^ix^j)(p) + \ldots + \\
&+ \frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} D(x^{i_1}\ldots x^{i_{r+1}})(p)\int_0^1 (1-s)^r \frac{\partial^{r+1}(f\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\bigg|_{sy} ds = \\
&+ \sum_{i=1}^{m} \frac{\partial(g\circ\varphi^{-1})}{\partial t^i}\bigg|_0 D(x^i)(p) + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2(g\circ\varphi^{-1})}{\partial t^i\partial t^j}\bigg|_0 D(x^ix^j)(p) + \ldots + \\
+ \quad &\frac{1}{r!}\sum_{i_1,\ldots,i_{r+1}=1}^{m} D(x^{i_1}\ldots x^{i_{r+1}})(p)\int_0^1 (1-s)^r \frac{\partial^{r+1}(g\circ\varphi^{-1})}{\partial t^{i_1}\ldots\partial t^{i_{r+1}}}\bigg|_{sy} ds = \\
&= D(g)(p) = \omega_p(g)
\end{aligned}$$

As D is a derivation of order $\leq r$, it follows by induction on r that ω_p is a derivation of order r at p. Hence, $\omega_p \in J_p^r$, for all $p \in \mathbf{M}$. Let $\omega : \mathbf{M} \to J^r(\mathbf{M})$ be the map given by $\omega(p) = \omega_p$. For all $f \in C^\infty(\mathbf{M})$, we have

$$\omega(p)(f) = \omega_p(f) = D(f)(p)$$

showing that $p \mapsto \omega(p)(f)$ is differentiable, because $D(f)$ is, and so ω is a differentiable section of $J^r(\mathbf{M})$.

Hence, $\Gamma(J^r(\mathbf{M}))$ is isomorphic, as $\mathbb{R}$-vector space, to the space of derivations of order $\leq r$ on $C^\infty(\mathbf{M})$. □

It follows from the last theorem that if $\mathbf{M}$ is a m-dimensional manifold, a derivation of order $\leq r$, $D : C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$ is written locally as

$$D(f) = \sum_{k=1}^{r} \sum_{1\leq i_1\leq\ldots\leq i_k\leq m} a_{i_1\ldots i_k}(x_1,\ldots,x_m)\frac{\partial^k f}{\partial x^{i_1}\ldots\partial x^{i_k}}$$

with $a_{i_1\ldots i_k}$ differentiable [7].

3 Multidifferential Operators

Let (A, μ, e) be an associative commutative unital $\mathbb{K}$-algebra. Denote

$$C^n(A,A) = Hom_{\mathbf{Vec}^{\mathbb{K}}}(A^{\otimes n}, A), \quad \forall n \in \mathbb{Z}, n \geq 0$$

$$C^\bullet(A,A)=\bigoplus_{n\geq 0} C^n(A,A)$$

Definition 3.1 (Partial composition). *Let* $f\in C^{m+1}(A,A)$ *and* $g\in C^{n+1}(A,A)$. *For* $1\leq i\leq m+1$, *the* i*-th partial composition of* f *and* g *is the linear map* $\circ_i : C^{m+1}(A,A)\otimes C^{n+1}(A,A)\to C^{m+n+1}(A,A)$ *given by*

$$f\circ_i g = f(id_A^{\otimes(i-1)}\otimes g\otimes id_A^{\otimes(m-i+1)})$$

where id_A *denotes the identity of* A.

Definition 3.2 (Total composition). *The total composition (or dot product)* $\circ : C^\bullet(A,A)\otimes C^\bullet(A,A)\to C^\bullet(A,A)$ *is the linear map which, for each* $f\in C^{m+1}(A,A)$ *and* $g\in C^{n+1}(A,A)$, *associates* $f\circ g\in C^{m+n+1}(A,A)$ *given by*

$$f\circ g=\sum_{i=1}^{m+1}(-1)^{n(i+1)}f\circ_i g$$

Definition 3.3 (Cup product). *The cup product is the degree 0* $\mathbb{K}$*-linear map* $\smile: C^\bullet(A,A)\otimes C^\bullet(A,A)\to C^\bullet(A,A)$, *which to each* $f\in C^{m+1}(A,A)$ *and* $g\in C^{n+1}(A,A)$, *associates*

$$f\smile g=(-1)^{(m+1)(n+1)}\mu\circ(f\otimes g)$$

i.e., *if* $a_0,\dots,a_m,a_{m+1},\dots,a_{m+n+1}\in A$, *we have*

$$\begin{aligned}&(f\smile g)(a_0\otimes\dots\otimes a_m\otimes a_{m+1}\otimes\dots\otimes a_{m+n+1})=\\&=\mu(f(a_0\otimes\dots\otimes a_m)\otimes g(a_{m+1}\otimes\dots\otimes a_{m+n+1}))\end{aligned}$$

Proposition 3.4. $(C^\bullet(A,A),\smile)$ *is an associative graded* $\mathbb{K}$*-algebra.*

Proof. By construction, $C^\bullet(A,A)$ is a graded $\mathbb{K}$-vector space. By $\mathbb{K}$-linearity and 0 degree of cup product, we only have to prove the associativity. Note that $\mu\in C^2(A,A)$, which leads to

$$\mu^2=\mu\circ\mu=\mu(\mu\otimes id_A - id_A\otimes\mu)=0$$

So, if $f\in C^{m+1}(A,A)$, $g\in C^{n+1}(A,A)$, $h\in C^{l+1}(A,A)$, and denoting $\sigma=(m+1)(n+1)+(m+1)(l+1)+(n+1)(l+1)$ we have

$$(f\smile g)\smile h - f\smile(g\smile h)=(-1)^\sigma\mu^2(f\otimes g\otimes h)=0 \qquad\square$$

Definition 3.5 (Hochschild cohomology)**.** *The Hochschild cohomology of an associative $\mathbb{K}$-algebra A with coefficients in A is the cohomology of the complex*

$$0 \longrightarrow A \xrightarrow{\delta_H} C^1(A,A) \xrightarrow{\delta_H} \dots \xrightarrow{\delta_H} C^n(A,A) \xrightarrow{\delta_H} \dots$$

where the coboundary operator δ_H, called the Hochschild differential, is given by

$$\begin{aligned}&(\delta_H f)(a_0 \otimes \ldots \otimes a_n) := \mu(a_0 \otimes f(a_1 \otimes \ldots \otimes a_n)) + \\ &+\sum_{i=0}^{n-1}(-1)^{i+1} f(a_0 \otimes \ldots \otimes \mu(a_i \otimes a_{i+1}) \otimes \ldots \otimes a_n) + \\ &+(-1)^{n+1}\mu(f(a_0 \otimes \ldots \otimes a_{n-1}) \otimes a_n)\end{aligned}$$

for any $f \in C^n(A,A)$, for all $a_i \in A,\ i = 0, \ldots, n$.

Proposition 3.6. *The Hochschild differential δ_H is a degree 1 derivation for $(C^\bullet(A,A), \smile)$.*

Proof. For simplicity, we will denote the product μ of the algebra A by juxtaposition. By linearity, it is enough to consider the evaluation of δ_H on products of homogeneous terms. Let $f \in C^{m+1}(A,A)$ and

$g \in C^{n+1}(A,A)$. For any $a_i \in A,\ i = 0,\dots,m+n+2$, we have

$$\begin{aligned}
& \delta_H(f \smile g)(a_0 \otimes \ldots \otimes a_{m+n+2}) = a_0(f \smile g)(a_1 \otimes \ldots \otimes a_{m+n+2}) + \\
+ & \sum_{i=0}^{m+n+1} (-1)^{i+1}(f \smile g)(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & (-1)^{m+n+3}(f \smile g)(a_0 \otimes \ldots \otimes a_{m+n+1})a_{m+n+2} = \\
= & a_0 f(a_1 \otimes \ldots \otimes a_{m+1}) g(a_{m+2} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & \sum_{i=0}^{m} (-1)^{i+1} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+1}) g(a_{m+2} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & \sum_{i=m+1}^{m+n+1} (-1)^{i+1} f(a_0 \otimes \ldots \otimes a_m) g(a_{m+1} \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & (-1)^{m+n+3} f(a_0 \otimes \ldots \otimes a_m) g(a_{m+1} \otimes \ldots \otimes a_{m+n+1}) a_{m+n+2} = \\
= & (a_0 f(a_1 \otimes \ldots \otimes a_m) + \sum_{i=0}^{m} (-1)^{i+1} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+1}) + \\
+ & (-1)^{m+2} f(a_0 \otimes \ldots \otimes a_m) a_{m+1}) g(a_{m+2} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & (-1)^{m+1} f(a_0 \otimes \ldots \otimes a_m)(a_{m+1} g(a_{m+2} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & \sum_{i=0}^{n} (-1)^{i+1} g(a_{m+1} \otimes \ldots \otimes a_{m+i+1} a_{m+i+2} \otimes \ldots \otimes a_{m+n+2}) + \\
+ & (-1)^{n+2} g(a_{m+1} \otimes \ldots \otimes a_{m+n+1}) a_{m+n+2}) = \\
= & ((\delta_H f) \smile g)(a_0 \otimes \ldots \otimes a_{m+n+2}) + (-1)^{m+1}(f \smile \delta_H g)(a_0 \otimes \ldots \otimes a_{m+n+2})
\end{aligned}$$

□

Definition 3.7 (Gerstenhaber bracket)**.** *The Gerstenhaber bracket is the degree -1 $\mathbb{K}$-linear map $[\ ,\ ] : C^\bullet(A,A) \otimes C^\bullet(A,A) \to C^\bullet(A,A)$ which, for each $f \in C^{m+1}(A,A)$ and $g \in C^{n+1}(A,A)$*

$$[f,g] = f \circ g - (-1)^{mn} g \circ f$$

Proposition 3.8. *$(C^\bullet(A,A), [\ ,\ ])$ is a Lie superalgebra with respect to the reduced (by one) degree.*

The reader can find a proof of this fact in [3].

Proposition 3.9. *Let (A,μ) be an associative $\mathbb{K}$-algebra. For any $f \in C^{m+1}(A,A)$*

$$\delta_H(f) = (-1)^m[\mu, f]$$

where $[\ ,\ ]$ *is the Gerstenhaber bracket.*

Proof. Let $f \in C^{m+1}(A, A)$. Since $\mu \in C^2(A, A)$, it follows that

$$\begin{aligned}[\mu, f] &= \mu \circ f - (-1)^m f \circ \mu = \mu(f \otimes id_A) + (-1)^m \mu(id_A \otimes f) + \\ &+ (-1)^m \sum_{i=0}^{m} (-1)^{i+1} f(id_A^{\otimes i} \otimes \mu \otimes id_A^{\otimes(m-i)}) = \\ &= (-1)^m \delta_H(f) \quad \square\end{aligned}$$

Proposition 3.10. *Let* A *be a* $\mathbb{K}$*-algebra and* $\nu \in C^2(A, A)$ *be a product in* A*. Then* ν *is associative if and only if* $\nu \circ \nu = 0$*.*

Proof. By definition, the total composition gives

$$\nu \circ \nu = \nu(\nu \otimes id_A) - \nu(id_A \otimes \nu)$$

which is the expression of associativity of ν. $\square$

Corollary 3.11. *Let* A *be a* $\mathbb{K}$*-algebra, where* $\mathbb{K}$ *has characteristic different from 2. Given a product* $\nu \in C^2(A, A)$*,* ν *is associative if and only if* $[\nu, \nu] = 0$*.*

Proof. By definition, the Gerstenhaber bracket gives

$$[\nu, \nu] = \nu \circ \nu + \nu \circ \nu = 2\, \nu \circ \nu$$

Hence, by the last proposition ν is associative if and only if $[\nu, \nu] = 0$. $\square$

Proposition 3.12. *Let* A *be a* $\mathbb{K}$*-algebra, where* $\mathbb{K}$ *has characteristic different from 2. Fix a product* $\nu \in C^2(A, A)$ *and let* δ_H *be the Hochschild operator defined as in definition 3.5. Then* δ_H *is a differential if and only if* $[\nu, \nu] = 0$*.*

Proof. Given $f \in C^{m+1}(A, A)$, we have

$$\begin{aligned}\delta_H^2(f) &= \delta_H(\delta_H(f)) = \\ &= \delta_H((-1)^m[\nu, f]) = (-1)^{2m+2}[\nu, [\nu, f]] = [[\nu, \nu], f] - [\nu, [\nu, f]]\end{aligned}$$

Therefore,

$$\delta_H^2(f) = \frac{1}{2}[[\nu, \nu], f]$$

The result holds true by corollary 3.11. $\square$

Remark 3.13. *If* (A, μ) *is an associative* $\mathbb{K}$*-algebra and* $id_A : A \to A$ *denotes the identity map on* A*, then*

$$\delta_H(id_A) = \mu(id_A \otimes id_A) + \mu(id_A \otimes id_A) - id_A(\mu) = \mu$$

Definition 3.14 (Multiderivations)**.** *The space of the multiderivations on the associative unital* $\mathbb{K}$*-algebra* (A, μ, e)*, denoted by* $MDer(A)$*, is the subalgebra of* $(C^\bullet(A, A), \smile)$ *generated by* $Der(A)$*.*

Note that $MDer(A)$ is a graded algebra with

$$MDer(A) = \bigoplus_{n \geq 1} MDer^n(A)$$

where $MDer^n(A) = MDer(A) \cap C^n(A, A)$.

Theorem 3.15 (The $MDer(A)$ subcomplex)**.** *Every multiderivation is a Hochschild cocycle.*

Proof. Let's proceed by induction to prove that δ_H is identically zero on $MDer(A)$. Let $X \in Der(A)$. For all $a, b \in A$

$$\delta_H X(a \otimes b) = \mu(a \otimes X(b)) - X(\mu(a \otimes b)) + \mu(X(a) \otimes b) = 0$$

Now, suppose the result holds true for elements in $MDer^{n-1}(A)$ and consider
$D \in MDer^n(A)$. The space $MDer(A)$ is generated by $Der(A)$, so D can be written as linear combinations of elements of the form $X \smile \tilde{D}$, where $X \in Der(A)$ and $\tilde{D} \in MDer^{n-1}(A)$. By linearity, it's enough to consider the evaluation of δ_H at such elements. It follows from the fact that δ_H is a degree 1 derivation on $(C^\bullet(A, A), \smile)$ that

$$\delta_H(X \smile \tilde{D}) = (\delta_H X) \smile \tilde{D} - X \smile \delta_H \tilde{D} = 0$$

Hence, $MDer(A)$ is a subcomplex of $(C^\bullet(A, A), \smile)$ and δ_H is identically zero on $MDer(A)$. □

The next theorem relates contravariant tensor fields on a differentiable manifold $\mathbf{M}$ with multiderivations on the algebra $C^\infty(\mathbf{M})$. Before stating the results it is worthy to relate such fields with multiderivations at a point, an analogous to the concept of derivation at a point (see section 2). Let's make precise the notion of multiderivation at a point.

Let $\mathbf{M}$ be a m-dimensional differentiable manifold. The tangent space at every point of $\mathbf{M}$, being finite dimensional, allows to identify $(T_p\mathbf{M})^{\otimes n}$ and $(I_p/I_p^2)^{*\otimes n}$, where I_p denotes the ideal of germs of functions which vanish at p.

Consider the differentiable tensor bundle $(T\mathbf{M})^{\otimes n}$. Let $p \in \mathbf{M}$ and $\tau_p \in (T\mathbf{M})^{\otimes n}$ such that $\tau_p \in V_p^{\otimes n} = (I_p/I_p^2)^{*\otimes n}$. By denoting $\mathcal{F}_p$ the $\mathbb{R}$-vector space of germs of functions at the point p, define the linear map $\vartheta_p : \mathcal{F}_p^{\otimes n} \to \mathbb{R}$ given by

$$\vartheta_p(f_1 \otimes \ldots \otimes f_n) = \begin{cases} 0 & \text{, if } \exists\, c \in f_i \mid c(x) = c \;\forall x \in \mathbf{M}, \\ & \text{for any } i,\; i = 1, \ldots, n \\ \tau_p([f_1] \otimes \ldots \otimes [f_n]) & \text{, if } f_i \in I_p,\; \forall i = 1, \ldots, n \end{cases}$$

where $[f_i]$ denotes the equivalence class of the germ f_i in I_p/I_p^2. Since every germ f can be written as $f = \tilde{f} + f(p)$, where $\tilde{f} \in I_p$ and $f(p)$ is the germ of the constant function whose value is $f(p)$, ϑ_p satisfies the following:

$$\begin{aligned} & \vartheta_p(f_1 \otimes \ldots \otimes f_i g_i \otimes \ldots \otimes f_n) = \\ = \;& g_i(p)\vartheta_p(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n) + f_i(p)\vartheta_p(f_1 \otimes \ldots \otimes g_i \otimes \ldots \otimes f_n) \end{aligned}$$

for every i. A linear map $\omega : \mathcal{F}_p^{\otimes n} \to \mathbb{R}$ such that

$$\begin{aligned} & \omega(f_1 \otimes \ldots \otimes f_i g_i \otimes \ldots \otimes f_n) = \\ = \;& g_i(p)\omega(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n) + f_i(p)\omega(f_1 \otimes \ldots \otimes g_i \otimes \ldots \otimes f_n) \end{aligned}$$

for every $i = 1, \ldots, n$ is called a *multiderivation of degree n, at the point p.*

On the other hand, if $\omega : \mathcal{F}_p^{\otimes n} \to \mathbb{R}$ is a multiderivation at p, one can relate to this a unique element $\eta_p \in V_p^{\otimes n}$, such that $\eta_p([f_1] \otimes \ldots \otimes [f_n]) = \omega(f_1 \otimes \ldots \otimes f_n)$, for all $f_1 \otimes \ldots \otimes f_n \in \mathcal{F}_p^{\otimes n}$. To see this, first note that if c represents a constant function, then

$$\begin{aligned} & \omega(f_1 \otimes \ldots \otimes c \otimes \ldots \otimes f_n) = c\,\omega(f_1 \otimes \ldots \otimes 1 \cdot 1 \otimes \ldots \otimes f_n) = \\ = \;& c(\omega(f_1 \otimes \ldots \otimes 1 \otimes \ldots \otimes f_n) + \omega(f_1 \otimes \ldots \otimes 1 \otimes \ldots \otimes f_n)) = \\ = \;& 2\,\omega(f_1 \otimes \ldots \otimes c \otimes \ldots \otimes f_n) \end{aligned}$$

Hence, $\omega(f_1 \otimes \ldots \otimes c \otimes \ldots \otimes f_n) = 0$. It follows that if $f_1 \otimes \ldots \otimes f_n \in \mathcal{F}_p^{\otimes n}$, writing every f_i as $f_i = \tilde{f}_i + f_i(p)$, with $\tilde{f}_i \in I_p$, by linearity of ω, we have

$$\omega(f_1 \otimes \ldots \otimes f_n) = \omega(\tilde{f}_1 \otimes \ldots \otimes \tilde{f}_n)$$

which shows that the value of ω is determined only by its value in $I_p^{\otimes n}$.

Now, suppose $f_i \in I_p^2$ for some i. Then there exists $g_i, h_i \in I_p$ such that $f_i = g_i h_i$, resulting in

$$\begin{aligned} & \omega(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n) = \omega(f_1 \otimes \ldots \otimes g_i h_i \otimes \ldots \otimes f_n) = \\ = \ & g_i(p)\omega(f_1 \otimes \ldots \otimes h_i \otimes \ldots \otimes f_n) + h_i(p)\omega(f_1 \otimes \ldots \otimes g_i \otimes \ldots \otimes f_n) = 0 \end{aligned}$$

So, if $\tilde{f}_i \equiv \tilde{g}_i \ mod \ I_p^2$, $\forall i = 1, \ldots, n$, then $\omega(f_1 \otimes \ldots \otimes f_n) = \omega(g_1 \otimes \ldots \otimes g_n)$ and ω induces an unique linear map $\eta_p \in (I_p/I_p^2)^{*\otimes n}$. Therefore there exists a bijection between $V_p^{\otimes n}$ and multiderivations of degree n at p. Hence, an element $\eta_p \in V_p^{\otimes n}$ can be regarded as a multiderivation of degree n at p.

Define the action of $\vartheta_p \in V_p^{\otimes n}$ on elements in $(C^\infty(\mathbf{M}))^{\otimes n}$ by $\vartheta_p(F_1 \otimes \ldots \otimes F_n) = \vartheta_p(f_1 \otimes \ldots \otimes f_n)$ on decomposable elements and extended it by linearity, where f_i denotes a representing element of $F_i \in C^\infty(\mathbf{M})$ in $\mathfrak{F}_p$, $i = 1, \ldots, n$.

We can now prove the following

Theorem 3.16. *Let* $\mathbf{M}$ *be an* m*-dimensional differentiable manifold. Then*

$$\Gamma((T\mathbf{M})^{\otimes n}) \approx_{\mathbf{Vec}^{\mathbb{R}}} MDer^n(C^\infty(\mathbf{M})), \quad \forall n \geq 1.$$

Proof. Given $\tau \in \Gamma((T\mathbf{M})^{\otimes n})$, define a linear map $\bar{\tau} : C^\infty(\mathbf{M})^{\otimes n} \to C^\infty(\mathbf{M})$, given by

$$\bar{\tau}(f_1 \otimes \ldots \otimes f_n)(p) = \tau_p(f_1 \otimes \ldots \otimes f_n), \ \forall p \in \mathbf{M}$$

Note that $\tau_p \in (I_p/I_p^2)^{*\otimes n}$ means that τ_p can be written as linear combination of elements of the form $v_p^1 \otimes \ldots \otimes v_p^n$ with $v_p^i \in (I_p/I_p^2)^*$, $\forall i = 1, \ldots, n$, and $(v_p^1 \otimes \ldots \otimes v_p^n)(f_1 \otimes \ldots \otimes f_n) = v_p^1(f_1) \ldots v_p^n(f_n)$. Since τ is a differentiable section, we have $\bar{\tau} \in MDer^n(C^\infty(\mathbf{M}))$.

The assignment $\tau \mapsto \bar{\tau}$ is clearly linear. We claim it is a bijection.

To show that it is injective, it is enough to see that

$$\begin{aligned} \bar{\tau}(f_1 \otimes \ldots \otimes f_n) = 0 \quad &, \quad \forall f_i \in C^\infty(\mathbf{M}), \ i = 1, \ldots, n \Rightarrow \\ \Rightarrow \bar{\tau}(f_1 \otimes \ldots \otimes f_n)(p) = 0 \quad &, \quad \forall p \in \mathbf{M}, \ \forall f_i \in C^\infty(\mathbf{M}), \ i = 1, \ldots, n \Rightarrow \\ \Rightarrow \tau_p(f_1 \otimes \ldots \otimes f_n) = 0 \quad &, \quad \forall p \in \mathbf{M}, \ \forall f_i \in C^\infty(\mathbf{M}), \ i = 1, \ldots, n \Rightarrow \\ \Rightarrow \tau_p = 0 \quad &, \quad \forall p \in \mathbf{M} \Rightarrow \\ \Rightarrow \tau = 0 \quad & \end{aligned}$$

where in the one but last step we used the following fact. Taking a local chart (U, φ) at the point p, with $\varphi(p) = 0$ and taking $x^i = t^i \circ \varphi$, with $t^i : \mathbb{R}^m \to \mathbb{R}$ the canonical projection on the i-th component (see section 2), we can write

$$\tau_p(f_1 \otimes \ldots \otimes f_n) = \sum_{(i_1,\ldots,i_n)} a^{i_1\ldots i_n} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_n}}$$

where $(i_1, \ldots, i_n)$ under the summation symbol means that the sum must be evaluated for each i_j, with $j = 1, \ldots, n$, $i_j = 1, \ldots, m$. Evaluating τ_p on elements of the form $(x^{i_1} \otimes \ldots \otimes x^{i_n})$ successively, we have $a^{i_1\cdots i_n} = 0$ for any combination of indices i_j.

To show that it is surjective, consider $D \in MDer^n(C^\infty(\mathbf{M}))$. For each $p \in \mathbf{M}$, define the linear map $\tau_p : \mathcal{F}_p^{\otimes n} \to \mathbb{R}$ given by

$$\tau_p(f_1 \otimes \ldots \otimes f_n) = D(f_1 \otimes \ldots \otimes f_n)(p), \ \forall f_1 \otimes \ldots \otimes f_n \in \mathcal{F}_p$$

where f_i on the left hand side denotes the germ of the function f_i written on the right. For now on in this proof we will denote germs and functions by the same symbol to avoid cumbersome notation. It is clear when a symbol denotes a function or a germ, from the operator acting on the symbol. Let's show that τ_p is well defined. Let $f_i, g_i \in C^\infty(\mathbf{M})$ with $i = 1, \ldots, n$, such that $f_i \equiv g_i$ on $\mathcal{F}_p$, for each $i = 1, \ldots, n$. Let W_i be open neighbourhoods of $p \in \mathbf{M}$ such that $f_i|_{W_i} = g_i|_{W_i}$, $i = 1, \ldots, n$. Let (V, φ) be a local chart such that $\varphi(p) = 0$. Taking $U = V \cap W_1 \cap \ldots \cap W_n$ we have (U, φ) still a local chart around p. If necessary, we can shrink U to make $\varphi(U)$ open and convex on $\mathbb{R}^m$. As f_i and g_i coincides on U for each i, it also coincides $\tilde{f}_i = f_i - f_i(p)$ and $\tilde{g}_i = g_i - g_i(p)$ on U for each i. Hence,

$$\left.\frac{\partial(\tilde{f}_i \circ \varphi^{-1})}{\partial t^j}\right|_0 = \left.\frac{\partial(\tilde{g}_i \circ \varphi^{-1})}{\partial t^j}\right|_0, \ \forall j = 1, \ldots, m, \ \forall i = 1, \ldots, n.$$

By $D \in MDer^n(C^\infty(\mathbf{M}))$, we have

$$D(f_1 \otimes \ldots \otimes f_n) = D((\tilde{f}_1 + f_1(p)) \otimes \ldots \otimes (\tilde{f}_n + f_n(p))) = D(\tilde{f}_1 \otimes \ldots \otimes \tilde{f}_n)$$

which results in

$$\tau_p(f_1 \otimes \ldots \otimes f_n) = D(f_1 \otimes \ldots \otimes f_n)(p) = D(\tilde{f}_1 \otimes \ldots \otimes \tilde{f}_n)(p) =$$
$$= \sum_{j_1,\ldots,j_n=1}^{m} \frac{\partial(\tilde{f}_i \circ \varphi^{-1})}{\partial t^{j_1}}\Big|_0 \cdots \frac{\partial(\tilde{f}_i \circ \varphi^{-1})}{\partial t^{j_n}}\Big|_0 D(x^{j_1} \otimes \ldots \otimes x^{j_n})(p) +$$
$$+ \sum_{\substack{j_1,\ldots,j_n=1\\ l_1,\ldots,l_n=1}}^{m} \int_0^1 (1-s) \frac{\partial^2(\tilde{f}_1 \circ \varphi^{-1})}{\partial t^{j_1}\partial t^{l_1}}\Big|_{sy} ds \ldots \int_0^1 (1-s) \frac{\partial^2(\tilde{f}_n \circ \varphi^{-1})}{\partial t^{j_n}\partial t^{l_n}}\Big|_{sy} ds \cdot$$
$$\cdot D(x^{j_1}x^{l_1} \otimes \ldots \otimes x^{j_n}x^{l_n})(p) =$$

$$= \sum_{j_1,\ldots,j_n=1}^{m} \frac{\partial(\tilde{g}_i \circ \varphi^{-1})}{\partial t^{j_1}}\Big|_0 \cdots \frac{\partial(\tilde{g}_i \circ \varphi^{-1})}{\partial t^{j_n}}\Big|_0 D(x^{j_1} \otimes \ldots \otimes x^{j_n})(p) +$$
$$+ \sum_{\substack{j_1,\ldots,j_n=1\\ l_1,\ldots,l_n=1}}^{m} \int_0^1 (1-s) \frac{\partial^2(\tilde{g}_1 \circ \varphi^{-1})}{\partial t^{j_1}\partial t^{l_1}}\Big|_{sy} ds \ldots \int_0^1 (1-s) \frac{\partial^2(\tilde{g}_n \circ \varphi^{-1})}{\partial t^{j_n}\partial t^{l_n}}\Big|_{sy} ds \cdot$$
$$\cdot D(x^{j_1}x^{l_1} \otimes \ldots \otimes x^{j_n}x^{l_n})(p) = D(\tilde{g}_1 \otimes \ldots \otimes \tilde{g}_n)(p) =$$
$$= D(g_1 \otimes \ldots \otimes g_n)(p) = \tau_p(g_1 \otimes \ldots \otimes g_n)$$

because $D(x^{j_1}x^{l_1} \otimes \ldots \otimes x^{j_n}x^{l_n})(p) = 0$ for any combination of indices (j_k, l_k). Thus, τ_p is well defined as a linear map on $\mathcal{F}_p^{\otimes n}$ taking real values, for all $p \in \mathbf{M}$. Besides that, we have

$$\tau_p(f_1 \otimes \ldots \otimes f_i g_i \otimes \ldots \otimes f_n) = D(f_1 \otimes \ldots \otimes f_i g_i \otimes \ldots \otimes f_n)(p) =$$
$$= g_i(p) D(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n)(p) + f_i(p) D(f_1 \otimes \ldots \otimes g_i \otimes \ldots \otimes f_n)(p) =$$
$$= g_i(p)\tau_p(f_1 \otimes \ldots \otimes f_i \otimes \ldots \otimes f_n) + f_i(p)\tau_p(f_1 \otimes \ldots \otimes g_i \otimes \ldots \otimes f_n)$$

on each entry. Therefore τ_p is a multiderivation of degree n at the point p, for each $p \in \mathbf{M}$. Finally, let's construct the map $\tau : \mathbf{M} \to (T\mathbf{M})^{\otimes n}$ given by $\tau(p) = \tau_p$. The map τ is a differentiable section, because for each $p \in \mathbf{M}$

$$\tau(p)(f_1 \otimes \ldots \otimes f_n) = \tau_p(f_1 \otimes \ldots \otimes f_n) = D(f_1 \otimes \ldots \otimes f_n)(p)$$

and $D(f_1 \otimes \ldots \otimes f_n) \in C^\infty(\mathbf{M})$ for any linear combination of elements $f_1 \otimes \ldots \otimes f_n \in C^\infty(\mathbf{M})$.

Thus, the stated assignment is an isomorphism of $\mathbb{R}$-vector spaces between $\Gamma((T\mathbf{M})^{\otimes n})$ and $MDer^n(C^\infty(\mathbf{M}))$. □

The last theorem reveals that if $\mathbf{M}$ is an m-dimensional differentiable manifold, an element in $D \in MDer^n(C^\infty(\mathbf{M}))$ can be written in local coordinates as

$$D = \sum_{j_1,\ldots,j_n=1}^{m} D(x^{j_1} \otimes \ldots \otimes x^{j_n}) \frac{\partial}{\partial x^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_n}}$$

4 Iterated Derivations

Definition 4.1 (Iterated Derivation)**.** *Let A be a commutative associative unital $\mathbb{K}$-algebra. The space of iterated derivations, denoted by $SDer(A)$, is the subalgebra of $(C^1(A,A),\circ)$ generated by $Der(A)$. We denote by $SDer^n(A)$ the set of elements $D \in SDer(A)$ which can be written as linear combinations of elements of the form $X_1 \circ \ldots \circ X_r$, with $X_i \in Der(A)$, $\forall i = 1,\ldots,r$, $r \leq n$.*

Remark 4.2. *Note that $SDer(A)$ can not be written as direct sum of the spaces $SDer^n(A)$. However, if $r \leq n$ then we have $SDer^r(A) \subset SDer^n(A)$. Hence, we have a filtration on the algebra $SDer(A)$.*

Theorem 4.3. *If $D \in SDer^n(A)$, then D is a derivation of order $\leq n$.*[2]

Proof. Denote the product on A by juxtaposition. We proceed by induction on n. Surely, if $X \in Der(A)$, then X is a derivation of order ≤ 1. Suppose $D \in SDer^n(A)$ and the result is valid for $n-1$. By linearity, it is enough to consider D as $D = \tilde{D} \circ X_n$, where $\tilde{D} \in SDer^{n-1}(A)$ and $X_n \in Der(A)$. By the induction hypothesis and the fact that $SDer^{r-1}(A) \subset SDer^r(A)$ for all $r \geq 1$, it is enough to consider the high order terms of $\tilde{D}$, *i.e.* terms like $X_1 \circ \ldots \circ X_{n-1}$. To show that $X_1 \circ \ldots \circ X_{n-1}$ is a differential operator of order $\leq n$, given $a \in A$, we must show that the operator Δ_a, given by

$$\Delta_a(b) = (X_1 \circ \ldots \circ X_n)(ab) - a(X_1 \circ \ldots \circ X_n)(b)$$

[2] A similar notion for Lie algebras can be found in [9].

for all $b \in A$, is a differential operator of order $\leq n-1$. We have

$$\begin{aligned}
&\Delta_a(b) = (X_1 \circ \ldots \circ X_n)(ab) - a(X_1 \circ \ldots \circ X_n)(b) = \\
&= (X_1 \circ \ldots \circ X_n)(a) \cdot b + \sum_{i=1}^{n} (X_1 \circ \ldots \circ \hat{X}_i \circ \ldots \circ X_n)(a) X_i(b) + \\
&+ \sum_{1 \leq i < j \leq n} (X_1 \circ \ldots \circ \hat{X}_i \circ \ldots \circ \hat{X}_j \circ \ldots \circ X_n)(a)(X_i \circ X_j)(b) + \ldots + \\
&+ \sum_{I_k} X_{\hat{I}_k}(a) X_{I_k}(b) + \ldots + \sum_{i=1}^{n} X_i(a)(X_1 \circ \ldots \circ \hat{X}_i \circ \ldots \circ X_n)(b)
\end{aligned}$$

where I_k represents a set of indices, subset of $\{1, \ldots, n\}$, with exactly k elements $\{i_1, \ldots, i_k\}$, such that $i_1 < \ldots < i_k$, $X_{\hat{I}_k}$ represents the composition $X_1 \circ \ldots \circ \hat{X}_{i_j} \circ \ldots \circ X_n$ in which all elements $X_{i_1}, \ldots, X_{i_k}$ in this order are absent, and X_{I_k} represents the composition $X_{i_1} \circ \ldots \circ X_{i_k}$. Hence, we have the operator Δ_a acting on b with composites having at most $n-1$ factors. Therefore $\Delta_a \in SDer^{n-1}(A)$, which is by the induction hypothesis a differential operator of order $\leq n-1$, for all $a \in A$. Thus D is a differential operator of order $\leq n$. By considering operators as $\tilde{D} \circ X$, with $X \in Der(A)$, it is clear that $\tilde{D}(X(\alpha)) = 0$, for all $\alpha \in \mathbb{K}$ (properly identified as element of A). Hence, D is a derivation of order $\leq n$. □

Theorem 4.4. *Let $\mathbf{M}$ be an m-dimensional differentiable manifold. If D is a derivation of order $\leq r$ on $C^\infty(\mathbf{M})$, then $D \in SDer^r(C^\infty(\mathbf{M}))$.*

Proof. We proceed by induction. If D is a derivation of order ≤ 1, then $D \in Der(C^\infty(\mathbf{M}))$ therefore $D \in SDer^1(C^\infty(\mathbf{M}))$. Suppose the result holds for $r-1$. Let D be a derivation of order $\leq r$ on $C^\infty(\mathbf{M})$. Then, by theorem 2.19, D can be related to an element $D \in \Gamma(J^r(\mathbf{M}))$. For each $p \in \mathbf{M}$, define the linear map $\Phi_{r,p} : I_p/I_p^{r+1} \to I_p/I_p^r$ which associates the equivalence class of a germ of a function f in I_p/I_p^{r+1} to its class in I_p/I_p^r. This is well defined, because $I_p^{r+1} \subset I_p^r$ and it is a projection because, by Taylor's formula, if f has a representative in I_p/I_p^r, then it has a representing in I_p/I_p^{r+1} such that $[f]_r = \Phi_{r,p}([f]_{r+1})$. Note that if $f \in I_p^r \; mod \; I_p^{r+1}$, then $\Phi_{r,p}(f) = 0$, and on the other hand, if $\Phi_{r,p}(f) = 0$, then $f \in I_p^r \; mod \; I_p^{r+1}$. Thus, $Ker(\Phi_{r,p}) \approx_{\mathbf{Vec}^{\mathbb{R}}} I_p^r/I_p^{r+1}$. We have, naturally, $I_p/I_p^{r+1} \approx_{\mathbf{Vec}^{\mathbb{R}}} I_p/I_p^r \oplus I_p^r/I_p^{r+1}$.

The dual map to $\Phi_{r,p}$ is $\Phi^*_{r,p} : J^{r-1}_p \to J^r_p$, given by

$$(\Phi^*_{r,p}(u))(f) = u(\Phi_{r,p}(f))$$

remembering that $J^r_p = (I_p/I^{r+1}_p)^*$. $\Phi^*_{r,p}$ is injective. This follows from the fact of being dual to a surjective linear map between vector spaces, because if $u \in J^{r-1}_p$ is such that $\Phi^*_{r,p}(u) = 0$, then $(\Phi^*_{r,p}(u))(f) = 0$ for all $f \in I_p/I^{r+1}_p$ and then, $u(\Phi_{r,p}(f)) = 0$ for all $f \in I_p/I^{r+1}_p$. As $\Phi_{r,p}$ is surjective, given $g \in I_p/I^r_p$, there exists $f \in I_p/I^{r+1}_p$ such that $g = \Phi_{r,p}(f)$. Hence, $u(g) = 0$ for all $g \in I_p/I^r_p$ therefore $u = 0$.

The map $\Phi^*_r : J^{r-1}(\mathbf{M}) \to J^r(\mathbf{M})$ such that $\Phi^*_r(\xi) = \Phi^*_{r,\pi(\xi)}(\xi)$ is a morphism of differentiable vector bundles. We have Φ^*_r fibre preserving and linear on fibres by construction. Furthermore, if $\xi \in J^{r-1}(\mathbf{M})$, locally, ξ is written as

$$\xi = \sum_{k=1}^{r-1} \sum_{1 \le i_1 \le \ldots \le i_k \le m} \xi(x^{i_1} \ldots x^{i_k}) \frac{\partial^k}{\partial x^{i_1} \ldots \partial x^{i_k}}$$

But $\Phi^*_r(\xi)$ is written locally as

$$\xi = \sum_{k=1}^{r-1} \sum_{1 \le i_1 \le \ldots \le i_k \le m} \xi(y^{i_1} \ldots y^{i_k}) \frac{\partial^k}{\partial y^{i_1} \ldots \partial y^{i_k}}$$

because terms of order r do not belong to the range of Φ^*_r. By the fact that local charts on $J^{r-1}(\mathbf{M})$ and $J^r(\mathbf{M})$ are fibred charts, there exists a diffeomorphism sending the coordinate expression of ξ in terms of y^i and derivatives, to the coordinate expression of ξ in terms of x^i and derivatives. By the match of those expressions it follows that Φ^*_r is differentiable.

Given $p \in \mathbf{M}$, by the induction hypothesis and the inclusion above, it is enough to consider derivations of order $\le r$ such that in a neighbourhood of p have only terms of order r. Let η be a derivation and let $(U, x^1, \ldots, x^m)$ be a local chart around p for which this occurs. η being a derivation of order $\le r$ leads to

$$\eta(x^{i_1} \ldots x^{i_r}) = \Delta_{i_r}(x^{i_1} \ldots x^{i_{r-1}}) + x^{i_r} \eta(x^{i_1} \ldots x^{i_{r-1}})$$

with Δ_{i_r} a differential operator of order $\le r - 1$. By the choice of the local chart, we have $\eta(x^{i_1} \ldots x^{i_{r-1}}) = 0$, because η has only terms of order r. Therefore

$$\eta(x^{i_1} \ldots x^{i_r}) = \Delta_{i_r}(x^{i_1} \ldots x^{i_{r-1}})$$

and from this it follows that Δ_{i_r} is a derivation of order $\leq r-1$. η can be written in terms of Δ_{i_r} in this way

$$\begin{aligned}
\eta &= \sum_{k=1}^{m} \sum_{i_1 \leq \ldots \leq i_{r-1}} \frac{\eta(x^{i_1} \ldots x^{i_{r-1}} x^k)}{r!} \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_{r-1}} \partial x^k} = \\
&= \sum_{k=1}^{m} \sum_{i_1 \leq \ldots \leq i_{r-1}} \frac{\Delta_k(x^{i_1} \ldots x^{i_{r-1}})}{r!} \frac{\partial^{r-1}}{\partial x^{i_1} \ldots \partial x^{i_{r-1}}} \frac{\partial}{\partial x^k} = \\
&= \sum_{k=1}^{m} \frac{\Delta_k}{r!} \frac{\partial}{\partial x^k} \qquad (4.1)
\end{aligned}$$

As Δ_k is a derivation of order $\leq r-1$, the induction hypothesis allows to write

$$\Delta_k = v_k \circ u_k$$

where v_k is a vector field and u_k is a derivation of order $\leq r-2$, both defined on U.

By the equation 4.1, we have

$$\eta = \sum_{k=1}^{m} \frac{\Delta_k}{r!} \frac{\partial}{\partial x^k} = \sum_{k=1}^{m} \frac{(v_k \circ u_k)}{r!} \frac{\partial}{\partial x^k} = \sum_{k=1}^{m} v_k \left(\frac{u_k}{r!} \frac{\partial}{\partial x^k} \right)$$

as the term $\frac{u_k}{r!} \frac{\partial}{\partial x^k}$ is a composition of derivations, it is itself a derivation of order $\leq r-1$ defined on U. Therefore

$$\eta = \sum_{k=1}^{m} v_k \circ w_k$$

with $v_k \in \Gamma(J^1(U))$ and $w_k \in \Gamma(J^{r-1}(U))$, for each $k = 1, \ldots, m$. For the sake of simplicity, we denote this by $\eta = v \circ u$.

Let $\{U_\alpha\}$ be a locally finite open covering of $\mathbf{M}$ and $\{\rho_\alpha\}$ a partition of unity subordinated to such covering. For each index α, we can find v_α and u_α as above, such that

$$\eta = v_\alpha \circ u_\alpha$$

Let's construct the fields $\zeta \in \mathfrak{X}(\mathbf{M})$, $\xi, \theta \in \Gamma(J^{r-1}(\mathbf{M}))$ by

$$\zeta = \sum_{\lambda} \rho_\lambda v_\lambda, \quad \xi = \sum_{\nu} \rho_\nu u_\nu, \quad \theta = \sum_{\beta} \gamma_\beta u_\beta$$

where $\gamma_\beta = \sum_\alpha \rho_\alpha v_\alpha(\rho_\beta)$. Note that θ is well defined, because if ρ_β has support on U_β, so are its derivatives and then, given $p \in \mathbf{M}$, $\gamma_\beta(p)$ does not vanish only for a finite number of indices β. Furthermore, given $f \in C^\infty(\mathbf{M})$

$$\rho_\lambda v_\lambda(\rho_\nu u_\nu(f)) = \rho_\lambda \rho_\nu v_\lambda(u_\nu(f)) + \rho_\lambda v_\lambda(\rho_\nu) u_\nu(f)$$

leading to

$$\rho_\lambda \rho_\nu \eta(f) = \rho_\lambda v_\lambda(\rho_\nu u_\nu(f)) - \rho_\lambda v_\lambda(\rho_\nu) u_\nu(f)$$

because if $U_\lambda \cap U_\nu = \varnothing$, then either ρ_λ or ρ_ν vanish, and then $\rho_\lambda \rho_\nu v_\lambda \circ u_\nu = \rho_\lambda \rho_\nu \eta$, and if $U_\lambda \cap U_\nu \neq \varnothing$, then $u_{\nu p} = u_{\lambda p}$ at each $p \in U_\lambda \cap U_\nu$, leading to $\rho_\lambda \rho_\nu v_\lambda \circ u_\nu = \rho_\lambda \rho_\nu \eta$.

Hence,

$$\begin{aligned} \eta(f) &= \sum_{\lambda,\nu} \rho_\lambda \rho_\nu \eta(f) = \sum_{\lambda,\nu} \rho_\lambda v_\lambda(\rho_\nu u_\nu(f)) - \sum_{\lambda,\nu} \rho_\lambda v_\lambda(\rho_\nu) u_\nu(f) = \\ &= \sum_\lambda \rho_\lambda v_\lambda \left(\sum_\nu \rho_\nu u_\nu(f) \right) - \sum_\nu \gamma_\nu u_\nu(f) = \\ &= (\zeta \circ \xi)(f) - \theta(f) \end{aligned}$$

By the induction hypothesis, $\xi, \theta \in \Gamma(J^{r-1}(\mathbf{M}))$ can be related to elements in $SDer^{r-1}(C^\infty(\mathbf{M}))$, leading to $\eta \in SDer^r(C^\infty(\mathbf{M}))$. As an arbitrary element $D \in \Gamma(J^r(\mathbf{M}))$ is a linear combination of elements in $\Gamma(J^{r-1}(\mathbf{M}))$ and elements of order r, it follows that

$$D \in SDer^r(C^\infty(\mathbf{M})) \qquad \square$$

5 Polydifferential Operators

The Hochschild-Kostant-Rosenberg theorem is usually stated as an isomorphism of graded algebras between Hochschild homology and universal differential forms (given by the Kähler differentials) of a smooth algebra. A proof of this version can be found in [8]. However, we want a dual version of this fact, by relating Hochschild cohomology of an algebra and its multilinear transformations. The process of taking duals often involves some restriction to a nice subspace. For infinite dimensional cases, the dual of a vector space is too big and an analogous copy of the original space that retains or preserves the desired properties lies in a specific kind of subspace. In the case of the Hochschild-Kostant-Rosenberg theorem we must restrict the Hochschild cohomology to the subcomplex of polyderivations.

Definition 5.1 (Polyderivations on an algebra)**.** *Let A be a commutative associative unital $\mathbb{K}$-algebra. The space of polyderivations on the algebra A, denoted by $D_{poly}(A)$, is the subalgebra of $(C^\bullet(A,A),\smile)$ generated by $SDer(A)$. We denote $D^n_{poly}(A) = D_{poly}(A) \cap C^n(A,A)$. Also, we denote by $D^{n,r}_{poly}(A)$ the space of polyderivations of degree n and order $\leq r$ i.e. elements in $C^n(A,A)$ which are polyderivations generated by $SDer^r(A)$.*

Theorem 5.2. *$(D_{poly}(A), \delta_H)$ is a filtered subcomplex of $(C^\bullet(A,A), \delta_H)$.*

Proof. For the sake of simplicity, we denote the product on A by juxtaposition. Take an element $D \in D^{n,r}_{poly}(A)$. Then D is a linear combination of elements of the form $D_1 \smile \ldots \smile D_n$, with $D_i \in SDer^r(A)$, for all $i = 1,\ldots,n$. However, if $D_i \in SDer^r(A)$, then it is a linear combination of elements of the form $X^i_1 \circ \ldots \circ X^i_j$, $j \leq r$, with $X^i_j \in Der(A)$, for all $i = 1,\ldots,n$, for all $j \leq r$. Then, if $a, b \in A$

$$\begin{aligned}
&\delta_H(X^i_1 \circ \ldots \circ X^i_j)(a \otimes b) = \\
&= a(X^i_1 \circ \ldots \circ X^i_j)(b) - (X^i_1 \circ \ldots \circ X^i_j)(ab) + (X^i_1 \circ \ldots \circ X^i_j)(a)b = \\
&= -\sum_{k=1}^{j-1} \sum_{I_k} (X^i_{\hat{I}_k})(a)(X^i_{I_k})(b) \qquad (5.1)
\end{aligned}$$

where I_k denotes a set of indices, subset of $\{1,\ldots,j\}$, with exactly k elements $l_1,\ldots,l_k$ such that $l_1 < \ldots < l_k$, for $k \leq j$, $X^i_{\hat{I}_k}$ denotes the composite $X^i_1 \circ \ldots \circ \hat{X}^i_{l_s} \circ \ldots \circ X^i_j$, in which all elements $X^i_{l_s}$, $l_s \in I_k$, in that order, are absent and $X^i_{I_k}$ denotes the composite $X^i_{l_1} \circ \ldots \circ X^i_{l_k}$ in that order.

Hence, $\delta_H(X^i_1 \circ \ldots \circ X^i_j) \in D^{2,j-1}_{poly}(A)$. As δ_H is a degree 1 derivation on $(C^\bullet(A,A),\smile)$, it follows that

$$\delta_H(D_1 \smile \ldots \smile D_n) = \sum_{i=1}^{n} (-1)^{i+1} D_1 \smile \ldots \smile \delta_H(D_i) \smile \ldots \smile D_n \qquad (5.2)$$

By linearity, $D \in D^{n,r}_{poly}(A)$, results in $\delta_H(D) \in D^{n+1,r}_{poly}(A)$. This shows that

$(D_{poly}(A), \delta_H)$ is subcomplex of $(C^\bullet(A,A), \delta_H)$, filtered by order of derivations. $\square$

Definition 5.3 (Alternator on $D^{n,r}_{poly}(A)$)**.** *If A is a commutative associative unital $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field with characteristic 0, we define for $n \geq 1$ the linear map $Alt : D^{n,r}_{poly}(A) \to D^{n,r}_{poly}(A)$ given, on decomposable elements, by*

$$Alt(D_1 \smile \ldots \smile D_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) D_{\sigma(1)} \smile \ldots \smile D_{\sigma(n)}$$

where σ denotes a permutation in S_n, the set of all permutations on n elements, and $\varepsilon(\sigma)$ denotes the signal of this permutation.

Proposition 5.4. *Let $D \in D^{n,r}_{poly}(C^\infty(\mathbf{M}))$ such that D is closed for the Hochschild differential. Then there exists a cochain $E \in D^{n-1,r+1}_{poly}(C^\infty(\mathbf{M}))$ and an alternating element $\eta \in MDer^n(C^\infty(\mathbf{M}))$ such that*

$$D = \delta_H(E) + \eta \tag{5.3}$$

The proof of the proposition is given in appendix A.

Definition 5.5 (Polydifferential operator)**.** *Let A be a commutative associative unital $\mathbb{K}$-algebra. An element in the space $\mathcal{D}(A) = A \oplus D_{poly}(A)$ is called a polydifferential operator on A.*

Note that $(\mathcal{D}(A), \delta_H)$ is subcomplex of the Hochschild complex $(C^\bullet(A,A), \delta_H)$.

Theorem 5.6 (The Hochschild-Kostant-Rosenberg theorem for differentiable manifolds[3])**.** *Let $\mathbf{M}$ be a m-dimensional differentiable manifold. There is a quasi-isomorphism between the complexes $(\mathcal{D}(C^\infty(\mathbf{M})), \delta_H)$ and $(\Omega_\bullet(\mathbf{M}), d)$, where $d : \Omega_\bullet(\mathbf{M}) \to \Omega_\bullet(\mathbf{M})$ is the zero differential on polyvector fields $\Omega_\bullet(\mathbf{M}) = \Gamma(\Lambda T\mathbf{M})$.*

Proof. Let $Alt(MDer^n(C^\infty(\mathbf{M})))$ be the range of the alternator on $MDer^n(C^\infty(\mathbf{M})) = D^{n,1}_{poly}(C^\infty(\mathbf{M}))$. Define the linear map $\psi : \Omega_n(\mathbf{M}) \to Alt(MDer^n(C^\infty(\mathbf{M})))$ given, on decomposable elements, by

$$\psi(X_1 \wedge \ldots \wedge X_n) = Alt(X_1 \smile \ldots \smile X_n)$$

for $n \geq 1$. Note that ψ is fibre preserving. Let's show that ψ is injective. Let $\eta \in \Omega_n(\mathbf{M})$ be such that $\psi(\eta) = 0$. At each $p \in \mathbf{M}$, η_p is written

[3] This proof follows the technique in [1]

as linear combination of elements in a base for $\Lambda_p(T_p\mathbf{M})$, of the form $X_{i_1p} \wedge \ldots \wedge X_{i_np}$. However,

$$\psi(X_{i_1p} \wedge \ldots \wedge X_{i_np}) = Alt(X_{i_1p} \smile \ldots \smile X_{i_np}) = Alt(X_{i_1p} \otimes \ldots \otimes X_{i_np}) = \\ = X_{i_1p} \wedge \ldots \wedge X_{i_np}$$

because at each point the cup product $\smile$ coincides with tensor product, once each X_{ip} can be viewed as a linear functional. Thus, $\psi(\eta) = 0$ results $\eta_p = 0$ for all p, and then $\eta = 0$. Let's show that ψ is surjective. Let $N \in Alt(MDer^n(C^\infty(\mathbf{M})))$. By linearity, it is enough to consider N of the form $\frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma)X_{\sigma(1)} \smile \ldots \smile X_{\sigma(n)}$. Now, take $\eta \in \Omega_n(\mathbf{M})$ as $X_1 \wedge \ldots \wedge X_n$. It follows that

$$\psi(X_1\wedge\ldots\wedge X_n) = Alt(X_1 \smile \ldots \smile X_n) = \frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma)X_{\sigma(1)} \smile \ldots \smile X_{\sigma(n)}$$

By linearity, $\psi(\eta) = N$. Hence, we have an one-one correspondence between alternating elements in $MDer^n(C^\infty(\mathbf{M}))$ and n-vector fields. For now on, we shall no longer distinguish between such elements. We call J_n the family of maps taking cochains $D \in D^{n,r}_{poly}(C^\infty(\mathbf{M}))$ and sending them to $J_n(D) = Alt(D)$, for $n \geq 1$ and J_0 as identity on $C^\infty(\mathbf{M})$. As $C^\infty(\mathbf{M})$ is commutative, δ_H vanishes on $C^\infty(\mathbf{M})$. Thus, $J_1 \circ \delta_H = d \circ J_0$. Let D be a n-coboundary, $n > 1$. Then there exists an $(n-1)$-cochain E such that $D = \delta_H(E)$. The formulae 5.1 and 5.2 show that $\delta_H(E)$ is a linear combination of terms which are symmetric on two entries, hence we must have that $Alt(\delta_H(E)) = 0$. It follows that $J_n \circ \delta_H = d \circ J_{n-1}$, because d is identically zero. Hence, each J_n induces a morphism on cohomology $J_n^* : H^n(\mathcal{D}(C^\infty(\mathbf{M}))) \to H^n(\Omega_n(\mathbf{M}))$.

By the fact that d is the zero differential on $(\Omega_n(\mathbf{M}), d)$ we have $H^n(\Omega_n(\mathbf{M}))$ isomorphic as $\mathbb{R}$-vector space to $\Omega_n(\mathbf{M})$, for all $n \geq 0$.

It is clear that J_0^* is an isomorphism. Let D be a n-cocycle, $n \geq 1$. From proposition 5.4 we have $D = \delta_H(E) + \eta$, where E is a $(n-1)$-cochain and $\eta \in \Omega_n(\mathbf{M})$. It follows that if $\theta \in H^n(D_{poly}(C^\infty(\mathbf{M})))$, whose representing element in $D_{poly}(C^\infty(\mathbf{M}))$ is D, then D can be written as $D = \delta_H(E) + \eta$ and thus

$$J_n^*(\theta) = [J_n(D)] = [J_n(\delta_H(E) + \eta)] = [\eta] = \eta$$

J_n^* is injective. Indeed, if θ is such that $J_n^*(\theta) = 0$, then $[J_n(D)] = 0$ hence $J_n(\delta_H(E) + \eta) = J_n(\eta) = 0$, resulting $\eta = 0$ because $J_n(\eta) = \eta$. Thus,

$D = \delta_H(E)$ and then θ is the zero class. Now, J_n^* is surjective. To show this, note that $\Omega_n(\mathbf{M})$ is isomorphic to $Alt(MDer^n(C^\infty(\mathbf{M})))$, which is contained in $MDer^n(C^\infty(\mathbf{M}))$, which is contained in $D_{poly}^{n,r}(C^\infty(\mathbf{M}))$, for all $r \geq 1$. Hence, given $\eta \in \Omega_n(\mathbf{M})$ we can regard η as being an element of $D_{poly}^{n,1}(C^\infty(\mathbf{M}))$. However, by theorem 3.15, η is a n-cocycle. Thus, $J_n(\eta) = \eta$. Also, since η is alternating and by formulae 5.1 and 5.2, η can not be a coboundary, therefore the class of η in $H^n(D_{poly}(C^\infty(\mathbf{M})))$ can not be the zero class. It follows that J_n^* is an isomorphism on cohomology for all n and hence $(\mathcal{D}(C^\infty(\mathbf{M})), \delta_H)$ and $(\Omega_\bullet(\mathbf{M}), d)$ are quasi-isomorphic. □

A Appendix

The aim of this section is to give a proof of proposition 5.4. This proposition allows us to write a Hochschild cocycle of polyderivations of the algebra $C^\infty(\mathbf{M})$, where $\mathbf{M}$ is a differentiable manifold, as a sum of a coboundary and an alternating multiderivation. In other words, every cocycle is cohomologous to an alternating multiderivation. The proof[4] will be preceded by two auxiliary propositions. From now on, A is the $\mathbb{R}$-algebra $C^\infty(\mathbf{M})$ and the product in A is denoted by juxtaposition. $MDer(A)$ is the space of multiderivations on A, $D_{poly}(A)$ is the space of polyderivations on A and δ_H is the Hochschild differential.

Lemma A.1. *Let* $C \in MDer^n(A)$. *Then there exist* $E \in D_{poly}^{n-1,2}(A)$ *and* $\omega \in Alt(MDer^n(A))$ *such that*

$$C = \delta_H(E) + \omega$$

Proof. The result is trivial for $n = 1$. Let $C \in MDer^n(A)$, with $n \geq 2$. Then C is a linear combination of elements of the form

$$X_1 \smile \ldots \smile X_n$$

with $X_i \in Der(A)$ for each $i = 1, \ldots, n$. For each $i = 1, \ldots, n-1$, let $\Phi_i : MDer^n(A) \to D_{poly}^{n-1,2}(A)$, given on decomposable elements by

$$\Phi_i(X_1 \smile \ldots \smile X_n) = (-1)^i (X_1 \smile \ldots \smile (X_i \circ X_{i+1}) \smile \ldots \smile X_n)$$

[4]This proof is done in [4]

and extended by linearity. From the properties of δ_H it follows that

$$\delta_H(\Phi_i(X_1 \smile \ldots \smile X_n)) = -(X_1 \smile \ldots \smile \delta_H(X_i \circ X_{i+1}) \smile \ldots \smile X_n) =$$
$$= X_1 \smile \ldots \smile X_i \smile X_{i+1} \smile \ldots \smile X_n + X_1 \smile \ldots \smile X_{i+1} \smile X_i \smile \ldots \smile X_n$$

By denoting τ_i the exchange of elements in consecutive positions i and $i+1$, we can write

$$\delta_H(\Phi_i(X_1 \smile \ldots \smile X_n)) = X_1 \smile \ldots \smile X_n + \tau_i X_1 \smile \ldots \smile X_n$$

which, by linearity, leads to

$$\delta_H(\Phi_i C) = C + \tau_i C$$

Hence, for consecutive transpositions we have

$$\begin{aligned} \delta_H(\Phi_i(\tau_j C) - \Phi_j C) &= \delta_H(\Phi_i(\tau_j C)) - \delta_H(\Phi_j C) = \\ &= \tau_j C + \tau_i \cdot \tau_j C - C - \tau_j C = \\ &= \tau_i \cdot \tau_j C - C \end{aligned}$$

Let σ be a permutation on $\{1, \ldots, n\}$ *i.e.* $\sigma \in S_n$. Then σ can be written as a finite number of consecutive transpositions. Write $\sigma = \tau_{i_1} \cdots \tau_{i_k}$. Define the map Φ_σ, for this permutation, by

$$\Phi_\sigma(C) = \Phi_{i_1 \ldots i_k}(C) = \Phi_{i_1}(\tau_{i_2} \cdots \tau_{i_k} C) - \Phi_{i_2 \ldots i_k}(C) =$$
$$= \sum_{l=1}^{k} (-1)^{l+1} \Phi_{i_l}(\tau_{i_{l+1}} \cdots \tau_{i_k} C)$$

And now we have

$$\delta_H(\Phi_\sigma(C)) = \sum_{l=1}^{k} (-1)^{l+1} (\tau_{i_{l+1}} \cdots \tau_{i_k} C + \tau_{i_l} \cdots \tau_{i_k} C) = \sigma \cdot C + (-1)^{k+1} C$$

Denoting the signal of σ by $\varepsilon(\sigma)$ we have

$$\varepsilon(\sigma)\delta_H(\Phi_\sigma(C)) = \varepsilon(\sigma)\sigma \cdot C - C$$

Taking

$$\Phi(C) = -\frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma)\Phi_\sigma(C)$$

we have

$$\begin{aligned}
\delta_H(\Phi(C)) &= -\frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\delta_H(\Phi_\sigma(C)) = \\
&= -\frac{1}{n!}\sum_{\sigma\in S_n}(\varepsilon(\sigma)\sigma\cdot C - C) = \\
&= C - \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\sigma\cdot C
\end{aligned}$$

which shows that

$$C = \delta_H(\Phi(C)) + \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\sigma\cdot C \qquad \square$$

Lemma A.2. *Let* $\mathbf{M} = \mathbb{R}^m$ *and* $C \in D^{n,r}_{poly}(A)$ *be such that* $\delta_H C = 0$. *Then there exist* $E \in D^{n-1,r+1}_{poly}(A)$ *and* $\omega \in Alt(MDer^n(A))$ *such that*

$$C = \delta_H(E) + \omega$$

Proof. We proceed by induction on the degree of the derivation. If C has degree 1, then $\delta_H(C) = 0$ means

$$C(ab) = C(a)b + aC(b)$$

in other words, $C \in Der(A)$, and we can take $E = 0$ and $\omega = C$. Suppose the result holds for $n-1$. Let $C \in D^{n,r}_{poly}(A)$, with $n > 1$, be a coboundary. Then C can be written as

$$C = \sum_{|I_1|=r}\frac{\partial^r}{\partial x^{I_1}} \smile D_{I_1} + \tilde{C}$$

where $I_1 = (i_1, \ldots, i_r)$ is a multi-index denoting which partial derivatives are involved on the differential operator, $|I_1|$ denotes de order of the multi-index (how many elements are in it) and $\tilde{C}$ is the part of C with order less than r on the first entry. There is no loss by supposing $|I_1| = r$.

Since $\delta_H C = 0$, we have

$$\sum_{|I_1|=r}\frac{\partial^r}{\partial x^{I_1}} \smile \delta_H D_{I_1} + C' = 0$$

where C' has the terms of δ_H whose orders of derivations on the first entry are less than r. Hence, C is an n-cocycle in which coefficients of the greatest order on the first entry are $(n-1)$-cocycles. By the induction hypothesis, we can write $D_{I_1} = \delta_H(E_{I_1}) + F_{I_1}$, with $E_{I_1} \in D_{poly}^{n-2,r+1}(A)$ and $F_{I_1} \in Alt(MDer^{n-1}(A))$. Now, make

$$G = \sum_{|I_1|=r} \frac{\partial^r}{\partial x^{I_1}} \smile E_{I_1}$$

Then, we have that

$$\bar{C} := C + \delta_H G = \sum_{|I_1|=r} \frac{\partial^r}{\partial x^{I_1}} \smile F_{I_1} + \tilde{C} + \sum_{|I_1|=r} \delta_H \left(\frac{\partial^r}{\partial x^{I_1}} \right) \smile E_{I_1}$$

is also a cocycle. By denoting

$$H = \tilde{C} + \sum_{|I_1|=r} \delta_H \left(\frac{\partial^r}{\partial x^{I_1}} \right) \smile E_{I_1}$$

we can see that H has only terms of order strictly less than r on the first entry. Our goal is to show that it is possible to write C as a sum of a coboundary and an element whose order of derivation on the first entry is strictly less than r. For this, write

$$H = \sum_{I_1,\dots,I_n} H_{I_1,\dots,I_n} \frac{\partial^{|I_1|}}{\partial x^{I_1}} \smile \dots \smile \frac{\partial^{|I_n|}}{\partial x^{I_n}}$$

and

$$R = \sum_{|I_1|=r} \frac{\partial^r}{\partial x^{I_1}} \smile F_{I_1} = \sum_{I_1,i_2,\dots,i_n} \frac{\partial^r}{\partial x^{I_1}} \smile \frac{\partial}{\partial x^{i_2}} \smile \dots \smile \frac{\partial}{\partial x^{i_n}}$$

Since $\delta_H(\bar{C}) = 0$ and $\bar{C} = R + H$, from the properties of the Hochschild differential it follows that

$$\sum_{I_1,i_2,\dots,i_n} \delta_H \left(\frac{\partial^r}{\partial x^{I_1}} \right) \smile \frac{\partial}{\partial x^{i_2}} \smile \dots \smile \frac{\partial}{\partial x^{i_n}} +$$

$$+ \sum_{I_1,\dots,I_n} H_{I_1,\dots,I_n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial^{|I_1|}}{\partial x^{I_1}} \smile \dots \smile \delta_H \left(\frac{\partial^{|I_k|}}{\partial x^{I_k}} \right) \smile \dots \smile \frac{\partial^{|I_n|}}{\partial x^{I_n}} = 0$$

resulting in

$$rR_{\bar{I}_1,i_1,\ldots,i_n} + 2\sum_{l=1}^{n-1} H_{\bar{I}_1,i_1,\ldots,\{i_l,i_{l+1}\},\ldots,i_n} = 0$$

where $\bar{I}_1$ denotes the multi-index arising from the reduction of the order of I_1 by one and the brackets $\{\}$ denote the symmetric indices, both effects due to the action of the Hochschild differential. The remaining terms of H do not furnish additional information for our purposes. Alternating and adding all the permutations on the last n entries, we have

$$\sum_{\sigma\in S_n} \varepsilon(\sigma) R_{\bar{I}_1,\sigma(i_1),\ldots,\sigma(i_n)} = 0$$

It follows from the anti-symmetry of R on the last $n-1$ entries that

$$R_{(I_1,i_1),i_2,\ldots,i_n} = \sum_{s=2}^{n}(-1)^s R_{(I_1,i_s),i_2,\ldots,\hat{i}_s,\ldots,i_n}$$

where (I_1, i_s) denotes the multi-index $\bar{I}_1$ plus the index i_s and $\hat{i}_s$ denotes the absence of i_s on the list. By symmetry on the first multi-index, we have

$$rR_{I_1,i_2,\ldots,i_n} = \sum_{k=1}^{r}\sum_{s=2}^{n}(-1)^s R_{(I_1,i_s),i_2,\ldots,\hat{i}_s,\ldots,i_n}$$

Taking

$$K_{(I_1,i_s),i_2,\ldots,\hat{i}_s,\ldots,i_n} = R_{I_1,i_s,i_2,\ldots,\hat{i}_s,\ldots,i_n} + \sum_{k=1}^{r} R_{(I_1;\hat{i}_r),i_r,i_2,\ldots,\hat{i}_s,\ldots,i_n}$$

where $(I_1;\hat{i}_r)$ denotes the multi-index obtained by removing i_r from I_1, we have

$$\sum_{s=2}^{n}(-1)^s K_{(I_1,i_s),i_2,\ldots,\hat{i}_s,\ldots,i_n} = (r+n-1)R_{I_1,i_2,\ldots,i_n}$$

However, we can write

$$\sum_{s=2}^{n}(-1)^s K_{(I_1,i_s),i_2,\ldots,\hat{i}_s,\ldots,i_n} = (n-1)K_{I_1,i_2,\ldots,i_n} +$$

$$+\sum_{t=3}^{n}(-1)^t(n+1-t)(K_{I_1,i_t,i_2,\ldots,\hat{i}_t,\ldots,i_n} + K_{I_1,i_{t-1},i_2,\ldots,\hat{i}_{t-1},\ldots,i_n})$$

Note that, evaluating the Hochschild differential on terms of the form

$$K_{I_1,i_2,\ldots,i_n}\frac{\partial^{|I_1|}}{\partial x^{I_1}}\frac{\partial}{\partial x^{i_2}}\smile\frac{\partial}{\partial x^{i_3}}\smile\ldots\smile\frac{\partial}{\partial x^{i_n}}$$

we get terms that correspond to terms of the form

$$K_{I_1,i_2,\ldots,i_n}\frac{\partial^{|I_1|}}{\partial x^{I_1}}\smile\frac{\partial}{\partial x^{i_2}}\smile\ldots\smile\frac{\partial}{\partial x^{i_n}}$$

and evaluating the Hochschild differential on terms of the form

$$(K_{I_1,i_t,i_2,\ldots,\hat{i}_t,\ldots,i_n}+K_{I_1,i_{t-1},i_2,\ldots,\hat{i}_{t-1},\ldots,i_n}) \\ \frac{\partial^{|I_1|}}{\partial x^{I_1}}\smile\frac{\partial}{\partial x^{i_2}}\smile\ldots\smile\frac{\partial^2}{\partial x^{i_{t-1}}\partial x^{i_t}}\smile\ldots\smile\frac{\partial}{\partial x^{i_n}}$$

we get terms that correspond to terms of the form

$$(K_{I_1,i_t,i_2,\ldots,\hat{i}_t,\ldots,i_n}+K_{I_1,i_{t-1},i_2,\ldots,\hat{i}_{t-1},\ldots,i_n}) \\ \sum_{t=3}^{n}(-1)^t\frac{\partial^{|I_1|}}{\partial x^{I_1}}\smile\ldots\smile\frac{\partial}{\partial x^{i_{t-1}}}\smile\frac{\partial}{\partial x^{t}}\ldots\smile\frac{\partial}{\partial x^{i_n}}$$

and symmetric terms. Therefore, it is possible to construct an element K such that $\delta_H K$ owns all terms of R without adding terms of order greater than or equal to r. Then, there exists a cochain G' such that $C-\delta_H(G')$ has partial derivatives of order strictly less than r on the first entry. Iterating this procedure if necessary, it is possible to construct a cochain G such that $C'=C-\delta_H G$ is of first order on the first entry. In what follows, it is enough to consider cocycles of the form

$$C=\sum_i X_i\smile D_i$$

where $X_i\in Der(A)$. By the same argument in the beginning of this proof, $\delta_H C=0$ gives $\delta_H D_i=0$, and again the induction hypothesis gives $D_i=\delta_H E_i+F_i$, with E being a cochain and F_i being a degree $n-1$ alternating multiderivation. Hence

$$\begin{aligned} C &= \sum_i X_i\smile D_i=\sum_i X_i\smile\delta_H E_i+\sum_i X_i\smile F_i= \\ &= -\delta_H\left(\sum_i X_i\smile E_i\right)+\sum_i X_i\smile F_i \end{aligned}$$

As $X_i \smile F_i \in MDer^n(A)$, the last lemma gives $X_i \smile F_i = \delta_H \bar{B}_i + \omega_i$, leading to

$$C = \delta_H \left(\sum_i (B_i - X_i \smile E_i) \right) + \omega \qquad \square$$

Thus, we have the following

Theorem A.3. *Let* **M** *be a differentiable manifold. If* $C \in D^{n,r}_{poly}(A)$ *is a cocycle, then there exist* $E \in D^{n-1,r+1}_{poly}(A)$ *and* $\omega \in MDer^n(A)$ *such that*

$$C = \delta_H E + \omega$$

Proof. Let C be an n-cocycle. Take an open covering $\{U_\alpha\}$ on **M** and a partition of unity $\{\rho_\alpha\}$ subordinated to that covering. Then we can write

$$C = \sum_\alpha \rho_\alpha C$$

with each $\rho_\alpha C$ cocycle with support on U_α. The last lemma guaranties that for each α, there exist E_α and ω_α such that

$$\rho_\alpha C = \delta_H E_\alpha + \omega_\alpha$$

which, by construction, has support on U_α. It follows that

$$E = \sum_\alpha E_\alpha \text{ and } \omega = \sum_\alpha \omega_\alpha$$

are well defined objects, $E \in D^{n-1,r+1}_{poly}(A)$, ω is antisymmetric and it is in $MDer^n(A)$, such that

$$C = \delta_H E + \omega \qquad \square$$

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