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Carleman Inequality and Null Controllability for Parabolic Equations

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Dedicated to Professor Gervásio Colares on the occasion of his 80th birthday.

Abstract

This paper is concerned with a detailed exposition on the Carleman inequality for a parabolic equation. Specifically, it represents only a part of the work of A. V. Fursikov & O. Yu Imanovilov [7] for the particular model $p_t - \Delta p + f(p) = h$ of the heat equation. Moreover, we study the null controllability employing fixe points for multi-valued mapping.

1 Introduction

Let us consider the nonlinear parabolic state equation:

$$\left| \begin{array}{l} p_t(x,t) - \Delta p(x,t) + g(p(x,t)) = \chi_w \, u(x,t) \quad \text{in} \quad Q, \\ p(x,t) = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{1.1}$$

We represent by Ω a connect open set of $\mathbb{R}^n$ with C^2 boundary $\Gamma = \partial\Omega$. For $T > 0$, real number, we consider the cylinder $Q = \Omega \times (0,T)$ of $\mathbb{R}^{n+1}$, with lateral

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boundary $\Sigma = \Gamma \times (0,T)$. The points of Ω are represented by $x = (x_1, \dots, x_n)$, $x_i \in \mathbb{R}$, $i = 1, \dots, n$ and those of Q are represented by (x,t), with $x \in \Omega$ and $0 < t < T$. By w we consider a subset of Ω, that is, $w \subset \Omega$. The real functions $p = p(x,t)$, $u = u(x,t)$ defined on Q are the state and the control respectively. All the derivatives are in the sense of the theory of distributions of Laurent-Schwartz. By p_t we represent the partial derivative $\partial p/\partial t$ and Δ is the Laplace operator, that is, $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$. With $p_0(x)$ we denote the initial data of the initial boundary value problem (1.1): χ_w is the characteristic function of w.

The function $g\colon \mathbb{R} \to \mathbb{R}$, is $C^1(\mathbb{R})$, globally Lipschitz, that is

$$|g(p_1) - g(p_2)| \le M|p_1 - p_2| \text{ for all } p_1, p_2 \in \mathbb{R} \text{ and } g(0) = 0.$$

LINEARIZED SYSTEM

We define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(p) = \begin{vmatrix} \dfrac{g(p)}{p} & \text{if} \quad |p| > 0, \\ g'(0) & \text{if} \quad p = 0. \end{vmatrix}$$

We define, employing the function f, a linearized system associated with (1.1) given by

$$\begin{vmatrix} p_t(x,t) - \Delta p(x,t) + f(\overline{p}(x,t))p(x,t) = \chi_w u(x,t) \quad \text{in} \quad Q, \\ p(x,t) = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{vmatrix} \tag{1.2}$$

DEFINITIONS

(i) The system (1.2) is said to be approximately controllable in $L^2(\Omega)$, at time $T > 0$, if for each $\varepsilon > 0$, given $p_0 \in L^2(\Omega)$ and $p_T(x) \in L^2(\Omega)$, there exists a control $u \in L^2(Q_w)$, $Q_w = w \times (0,T)$, such that the corresponding solution $p(x,t)$ of (1.2) satisfies

$$|p(x,T) - p_T(x)|_{L^2(\Omega)} < \varepsilon.$$

By $L^2(\Omega)$ we represent the Lebesgue space of square integrable functions on Ω with the inner product and norm:

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx \quad \textit{and} \quad |u|^2_{L^2(\Omega)} = \int_\Omega u(x)^2\,dx,$$

where u and v are real valued functions ■

(ii) The system (1.2) is said to be null controllable at time $T > 0$, if for each $p_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q_w)$ such that the solution p of (1.2) satisfies $p(x,T) = 0$ a.e. in Ω ■

We consider a real function $a(x,t)$ uniformly bounded in the sense

$$|a(x,t)|_{L^\infty(Q)} < M.$$

In the sequel $a(x,t) = f(\overline{p}(x,t))$. Thus, we are concerned, initially, with the *adjoint system* of (1.2) which is given by

$$\left| \begin{array}{l} w_t(x,t) + \Delta w(x,t) - a(x,t)w(x,t) = f_1(x,t) \quad \text{in} \quad Q, \\ w(x,t) = 0 \quad \text{on} \quad \Sigma, \\ w(x,T) = w_T \quad \text{in} \quad \Omega, \end{array} \right. \tag{1.3}$$

with $w_T \in L^2(\Omega)$ and $f_1 \in L^2(Q)$.

In the next section we prove the **Carleman inequality** for the adjoint system (1.3), following the method of Fursikov-Imanovilov [6]. In this methodology it is fundamental the following result:

Lemma 1.1: *Let $w_0 \subset w \subset \Omega$ a nonempty open subset. Then, there exists a function $\psi \in C^2(\overline{\Omega})$, $\overline{\Omega}$ closure of Ω, such that*

$$\left| \begin{array}{l} \psi(x) > 0 \textit{ for all } x \in \Omega, \\ \psi = 0 \textit{ for all } x \in \Gamma, \\ |\nabla\psi(x)| > 0 \textit{ for } x \in \Omega - w_0. \end{array} \right.$$

The proof of this Lemma can be found in Fursikov-Imanovilov [7]. From Lemma 1.1 we introduce the weight functions

$$\phi(x,t) = \frac{e^{\lambda\psi(x)}}{\beta(t)} \quad \text{and} \quad \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||}}{\beta(t)}, \tag{1.4}$$

with $\beta(t) = t(T-t),\ 0 < t <$T, $\lambda > 0$ a real parameter and

$$||\psi|| = \max_{x\in\overline{\Omega}} |\psi(x)|.$$

From (1.4) we verify that

$$\nabla\phi = \lambda \frac{e^{\lambda\psi}}{\beta(t)} \nabla\psi = \lambda\phi\nabla\psi = \nabla\alpha \quad ■ \tag{1.5}$$

2 Carleman inequality

All this paragraph is dedicated to prove the inequality of Carleman for solution w of the adjoint system (1.3). In the method of Fursikov-Imanovilov [7] is crucial the results of Lemma 1.1. The main result is contained in the following theorem.

Theorem 2.1. *Let ψ, ϕ, α be the functions defined above. Then, there exist positive constants λ_0, s_0 and C such that*

$$\int_Q \Big[(s\phi)^{-1}\big(|w_t|^2 + |\Delta w|^2\big) + \lambda^2 s\phi|\nabla w|^2 + \lambda^4 (s\phi)^3 |w|^2\Big] e^{2s\alpha}\, dxdt \le C\int_Q e^{2s\alpha}\,|f_1|^2\, dxdt + C\int_{Q_w} e^{2s\alpha}\lambda^4 (s\phi)^3 |w|^2\, dxdt,$$

for all $s \ge s_0$ and $\lambda \ge \lambda_0$, where $s_0 = s_1(\Omega,\omega)(T+T^2)$, $\lambda_0 = \lambda_0(\Omega,\omega)$, $C = C(\Omega,\omega)$, $w = w(x,t)$ is solution of the adjoint system (1.3), $|\cdot|$ is the absolute value of real numbers and s_1 is a suitable constant.

Remark 2.1. *Setting $p(x,t) = (s\phi)^\ell e^{s\alpha(x,t)} w(x,t)$, we get*

$$\int_Q \Big[(s\phi)^{\ell-1}\big(|w_t|^2 + |\Delta w|^2\big) + \lambda^2 (s\phi)^{\ell+1}|\nabla w|^2 + \lambda^4 (s\phi)^{\ell+3} |w|^2\Big] e^{2s\alpha}\, dxdt \le C\int_Q (s\phi)^\ell e^{2s\alpha}\,|f_1|^2\, dxdt + C\int_{Q_w} e^{2s\alpha}\lambda^4 (s\phi)^{\ell+3} |w|^2\, dxdt,$$

for all $\ell \in \mathbb{Z}$. A look at the proof of Theorem 2.1 shows that the proof of this remark can be carried out in exactly the same way.

The above inequality is called **Carleman Inequality**. The proof of Theorem 2.1 is very much technical. It will be done by steps, following Fursikov-Imanovilov [7].

Step 1. We consider a convenient change of variables to introduce in the adjoint system (1.3) by the regularization function, that is, $e^{s\lambda\alpha(x,t)}$. In fact, setting

$$w(x,t) = e^{-s\alpha(x,t)}\, p(x,t) \quad \text{or} \quad p(x,t) = e^{s\alpha(x,t)}\, w(x,t),$$

we obtain

$$w_t(x,t) = -s\alpha_t\, e^{-s\alpha}\, p + e^{-s\alpha}\, p_t\,. \tag{2.1}$$

Besides that,

$$\frac{\partial w}{\partial x_i} = -s\,\frac{\partial \alpha}{\partial x_i}\,e^{-s\alpha}\,p + e^{-s\alpha}\,\frac{\partial p}{\partial x_i},$$

$$\frac{\partial^2 w}{\partial x_i^2} = -s\,\frac{\partial^2 \alpha}{\partial x_i^2}\,e^{-s\alpha}\,p + s^2\left(\frac{\partial \alpha}{\partial x_i}\right)^2 e^{-s\alpha}\,p - 2s\,\frac{\partial \alpha}{\partial x_i}\,e^{-s\alpha}\,\frac{\partial p}{\partial x_i} + e^{-s\alpha}\,\frac{\partial^2 p}{\partial x_i^2}\,.$$

Thus, we obtain

$$\Delta w = -s\Delta\alpha\, e^{-s\alpha}\,p + s^2|\nabla\alpha|^2\,e^{-s\alpha}\,p - 2se^{-s\alpha}\,\nabla\alpha\cdot\nabla p + e^{-s\alpha}\,\Delta p.$$

From (1.5) we also have

$$\Delta\alpha = \nabla\cdot(\nabla\alpha) = \lambda\nabla\phi\cdot\nabla\psi + \lambda\phi\Delta\psi = \lambda^2\,\phi|\nabla\psi|^2 + \lambda\phi\Delta\psi,$$

and $|\nabla\alpha|^2 = \lambda^2\phi^2|\nabla\psi|^2$. Therefore, we find

$$\begin{aligned}\Delta w = -s(\lambda^2\phi|\nabla\psi|^2 + \lambda\phi\Delta\psi)e^{-s\alpha}\,p +\\ s^2\lambda^2\phi^2|\nabla\psi|^2\,e^{-s\alpha}\,p - 2s\lambda\phi e^{-s\alpha}\,\nabla\psi\cdot\nabla p + e^{-s\alpha}\,\Delta p.\end{aligned} \tag{2.2}$$

From (2.1) and (2.2) and the system (1.3) we obtain

$$\begin{aligned}&e^{-s\alpha}\,p_t - e^{-s\alpha}(s\alpha_t p) + e^{-s\alpha}(-2s\phi\nabla\phi\cdot\nabla p + s^2\lambda^2\phi^2|\nabla\psi|^2\,p -\\ &s\lambda^2\phi|\nabla\psi|^2\,p + \Delta p - s\lambda\phi\Delta\psi p) = f_1 + a(t)e^{-s\alpha}\,p.\end{aligned} \tag{2.3}$$

We also have that

$$p(x,0) = e^{s\alpha(x,0)}\,w(x,0) = 0 \quad \text{in} \quad \Omega, \tag{2.4}$$

because

$$\alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||}}{\beta(t)} < 0 \quad \text{and} \quad e^{s\alpha(x,0)} = \lim_{t\to 0+} e^{s\alpha(x,t)} = 0.$$

By similar argument, we obtain

$$p(x,T) = e^{s\alpha(x,T)}\,w(x,T) = 0 \quad \text{in} \quad \Omega.$$

Then, from (2.3) and (2.4) we re-write the state equation (1.3), in the new variables, given by

$$\left|\begin{aligned}&p_t - \alpha_t sp - 2s\lambda\phi\nabla\psi\cdot\nabla p + s^2\lambda^2\phi^2|\nabla\psi|^2\,p -\\ &\qquad s\lambda^2\phi|\nabla\psi|^2\,p + \Delta p - s\lambda\phi\Delta\psi p = e^{s\alpha}\,f_1 + a(t)p \quad \text{in} \quad Q,\\ &p(x,t) = 0 \quad \text{on} \quad \Sigma,\\ &p(x,0) = p(x,T) = 0 \quad \text{in} \quad \Omega.\end{aligned}\right. \tag{2.5}$$

Let us consider the following notation.

$$\left|\begin{array}{l} U(t)p = -2s\lambda^2\phi|\nabla\psi|^2\, p - 2s\lambda\phi\nabla\psi\cdot\nabla p, \\ V(t)p = -\Delta p - s^2\lambda^2\phi^2|\nabla\psi|^2\, p - s\lambda^2\phi|\nabla\psi|^2\, p + \alpha_t sp, \\ Z(t)p = s\lambda\phi\Delta\psi\; p + a(t)p. \end{array}\right. \tag{2.6}$$

With the notation (2.6) we re-write the equation $(2.5)_1$, as follows

$$p_t + U(t)p - V(t)p = e^{s\alpha}\, f_1 + Z(t)p. \tag{2.7}$$

Note that

$$\frac{d}{dt}\int_\Omega (V(t)p)p\,dx = \int_\Omega (V(t)p_t)p\,dx + \int_\Omega (V(t)p)p_t\,dx + \int_\Omega (V_t(t)p)p\,dx.$$

The two first integrals can be written as $2\displaystyle\int_\Omega (V(t)p)p_t\,dx$. Substituting p_t, which is given by (2.7), we obtain

$$\begin{array}{c} \dfrac{d}{dt}\displaystyle\int_\Omega (V(t)p)p\,dx = \\ 2\displaystyle\int_\Omega (V(t)p)(e^{s\alpha} f_1 + Z(t)p - U(t)p + V(t)p)dx + \int_\Omega (V_t(t)p)p\,dx. \end{array} \tag{2.8}$$

Integrating (2.8) from 0 to T and observing that $p(0) = p(x,0)$ and $p(T) = p(x,T)$ are zero on Ω, we obtain

$$\begin{array}{l} 0 = 2\displaystyle\int_Q (V(t)p)^2\,dxdt + 2\int_Q (V(t)p)(e^{s\alpha}\, f_1 + Z(t)p)dxdt + \\ \displaystyle\int_Q (V_t(t)p)p\,dxdt + 2\Big(-\int_Q (V(t)p)(U(t)p)dxdt\Big). \end{array} \tag{2.9}$$

Analysis of the terms of (2.9). Denoting by X the last integral of (2.9), we have

$$\begin{array}{rl} X = & -\displaystyle\int_Q (V(t)p)(U(t)p)dxdt\; = \\ = & -\displaystyle\int_Q (\Delta p + s^2\lambda^2\phi^2|\nabla\psi|^2\, p + s\lambda^2\phi|\nabla\psi|^2\, p - \alpha_t sp) \\ & \cdot\, (2s\lambda^2\phi|\nabla\psi|^2\, p + 2s\lambda\nabla\psi\cdot\nabla p)dxdt. \end{array} \tag{2.10}$$

Remark 2.2. *From the definition of ϕ and α, we obtain*

$$|\phi_t| = \left|\frac{\beta'(t)}{\beta^2(t)}\, e^{\lambda\psi(x)}\right| = \frac{|T-2t|}{e^{\lambda\psi(x)}}\,|\phi|^2 \le C\,\phi^2, \tag{2.11}$$

where the constant C depends on T, λ, $||\psi||$ and Ω.

$$|\alpha_t| = \left|-\frac{\beta'(t)}{\beta^2(t)}\,(e^{\lambda\psi(x)} - e^{2\lambda||\psi||})\right| \leq \frac{|T-2t|\,|e^{\lambda\psi(x)} - e^{2\lambda||\psi||}}{e^{2\lambda\psi(x)}}\,\frac{e^{2\lambda\psi(x)}}{\beta^2(t)} \leq C\,\phi^2. \tag{2.12}$$

$$\begin{aligned} |\alpha_{tt}| &= \left|\frac{-\beta''\beta^2 + 2\beta|T-2t|^2}{\beta^4}\right| \left|e^{\lambda\psi(x)} - e^{2\lambda||\psi||}\right| \leq \\ &\leq \left|\frac{2\beta^2 + 2\beta|T-2t|^2}{\beta(t)e^{3\lambda\psi(x)}}\right| \left(\frac{e^{3\lambda\psi(x)}}{\beta^3}\right) \left|e^{\lambda\psi(x)} - e^{2\lambda||\psi||}\right| = \\ &= \left|\frac{2\beta + 2|T-2t|^2}{e^{2\lambda\psi(x)}}\right| \frac{|e^{\lambda\psi(x)} - e^{2\lambda||\psi||}|}{e^{\lambda\psi(x)}}\,\phi^3 \leq C\,\phi^3. \end{aligned} \tag{2.13}$$

Note, the constant depends on λ, T and Ω.

$$\left|\alpha\,\frac{d}{dt}\,\ell n\,\beta^{-1}(t)\right| = \left|\alpha\,\frac{\beta'(t)}{\beta(t)}\right| = \left|(T-2t)\,\frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||}}{e^{2\lambda\psi(x)}}\right| \frac{e^{2\lambda\psi(x)}}{\beta^2(t)} \leq C\,\phi^2 \quad \blacksquare \tag{2.14}$$

From (2.12)–(2.14) of Remark 2.2 we return to (2.9) and set

$$\begin{aligned} X_1 &= \left|\int_Q (V_t(t)p)p\,dxdt\right| = \\ &\left|\int_0^T \int_\Omega (-\Delta p_t - s^2\lambda^2(2\phi\phi_t)|\nabla\psi|^2\,p - s\lambda^2\phi_t|\nabla\psi|^2\,p + \alpha_{tt}\,sp)p\,dxdt\right| \leq \\ &\frac{1}{2}\int_0^T \frac{d}{dt}\,|\nabla p|^2_{L^2(\Omega)}\,dt + C_1\int_Q (\lambda^2 s^2\phi^3 + s\lambda^2\phi^2 + s\phi^3)|p|^2\,dxdt, \end{aligned} \tag{2.15}$$

where C_1 depends on Ω and T. Note that the integral of the derivative of $|\nabla p|^2_{L^2(\Omega)}$ is zero. We also get from (2.9) that

$$\begin{aligned} X_2 = &\left|\int_Q 2(V(t)p)(e^{s\alpha}\,f_1 + Z(t)p)dxdt\right| \leq \\ &\int_Q |2(V(t)p)e^{s\alpha}\,f_1|\,dxdt + \int_Q |2(V(t)p)Z(t)p|\,dxdt \leq \\ &2\int_Q |V(t)p|^2\,dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + \int_Q |Z(t)p)|^2\,dxt. \end{aligned} \tag{2.16}$$

From the definition of $Z(t)p$, we obtain

$$\begin{aligned} \int_Q |Z(t)p|^2\,dxdt &= \int_Q |s\lambda\phi\Delta\psi\;p + a(t)p|^2\,dxdt \\ &\leq C\int_Q (s^2\lambda^2\phi^2 + M)|p|^2\,dxdt. \end{aligned} \tag{2.17}$$

Thus, from (2.9), (2.15), (2.16) and (2.17), we have

$$2X + 2\int_Q |V(t)p|^2\,dxdt \le$$
$$C\int_Q (\lambda^2 s^2\phi^3 + s\lambda^2\phi^2 + s\phi^3)|p|^2\,dxdt +$$
$$2\int_Q |V(t)p|^2\,dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + C\int_Q (s^2\lambda^2\phi^2 + M)|p|^2\,dxdt.$$

Then, we obtain

$$X \le C\int_Q \big[\lambda^2(s^2\phi^2 + s^2\phi^3) + s\phi^3 + 1\big]|p|^2 dxdt\; + \frac{1}{2}\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt. \quad (2.18)$$

Step 2. In this step we calculate X by another process. In fact, we have

$$X = -\int_Q (V(t)p)(U(t)p)\,dxdt.$$

Otherwise

$$X = -\int_Q \big(\Delta p + s^2\lambda^2\phi^2|\nabla\psi|^2\,p\; + \quad (2.19)$$
$$s\lambda^2\phi|\nabla\psi|^2 - \alpha_t sp\big)\big(2s\lambda^2\phi|\nabla\psi|^2\,p + 2s\lambda\phi\nabla\psi\cdot\nabla p\big)\;dxdt\; =$$
$$2\int_Q (s\lambda^2\phi|\nabla\psi|^2\,p)\Delta p\,dxdt - 2\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4\,p^2\,dxdt -$$
$$2\int_Q s^2\lambda^4\phi^2|\nabla\psi|^4\,p^2\,dxdt - 2\int_Q (s\lambda\phi\nabla\psi\cdot\nabla p)\Delta p\,dxdt -$$
$$2\int_Q (s^3\lambda^3\phi^3|\nabla\psi|^2\nabla\psi\cdot\nabla p\,p + s^2\lambda^3\phi^2|\nabla\psi|^2\nabla\psi\cdot\nabla p\,p)dxdt +$$
$$2\int_Q (s^2\lambda^2\phi\alpha_t|\nabla\psi|^2\,p + s^2\lambda\phi\alpha_t\nabla\psi\cdot\nabla p)p\,dxdt.$$

Now we employ the notation

$$M_1 = -\,2\int_Q (s\lambda^2\phi|\nabla\psi|^2\,p)\Delta p\,dxdt;$$
$$M_2 = -\,2\int_Q s^3\lambda^4\phi^3|\nabla\psi|^4\,p^2\,dxdt - 2\int_Q s^2\lambda^4\phi^2|\nabla\psi|^4\,p^2\,dxdt;$$
$$M_3 = -\,2\int_Q (s\lambda\phi\nabla\psi\cdot\nabla p)\Delta p\,dxdt;$$
$$M_4 = -\,2\int_Q (s^3\lambda^3\phi^3|\nabla\psi|^2\,\nabla\psi\cdot\nabla p\,p + s^2\lambda^3\phi^2|\nabla\psi|^2\,\nabla\psi\cdot\nabla p\,p)\,dxdt;$$
$$M_5 = 2\left(\int_Q (s^2\lambda^2\phi\alpha_t|\nabla\psi|^2\,p + s^2\lambda\alpha_t\nabla\psi\cdot\nabla p\right)p\,dxdt.$$

Thus,

$$X = M_1 + M_2 + M_3 + M_4 + M_5 \,.$$

The next steps are to calculate the integrals M_i for $i = 1, \cdots, 5$. In fact, applying Green's formula to M_1 and observing that $p = 0$ on Σ, we obtain

$$\begin{aligned} M_1 = -2\int_Q ((s\lambda^2\phi)|\nabla\psi|^2\,p)\Delta p\,dxdt \;= \\ 2\int_Q s\lambda^2\nabla\big(\phi|\nabla\psi|^2\,p\big)\cdot\nabla p\,dxdt = 2\int_Q s\lambda^3\phi|\nabla\psi|^2\,p\,\nabla\psi\cdot\nabla p\,dxdt + \\ 2\int_Q s\lambda^2\phi\nabla\big(|\nabla\psi|^2\big)\cdot\nabla p\,p\,dxdt + 2\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt. \end{aligned} \tag{2.20}$$

As $\phi(x,t) = e^{\lambda\psi(x)}/\beta(t)$, then $\nabla\phi = \lambda\phi\nabla\psi$. From this and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \left|2\int_Q s\lambda^3\phi|\nabla\psi|^2\,p\,\nabla\psi\cdot\nabla p\,dxdt\right| \;\leq \\ 4\int_Q s\lambda^4\phi|\nabla\psi|^4\,p^2\,dxdt + \frac{1}{4}\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt. \end{aligned} \tag{2.21}$$

$$\begin{aligned} \left|2\int_Q s\lambda^2\phi\nabla\big(|\nabla\psi|^2\big)\cdot\nabla p\,p\,dxdt\right| \;= \\ \left|2\int_Q s\lambda^2\phi 2|\nabla\psi|\big(\nabla|\nabla\psi|\big)\cdot\nabla p\,p\,dxdt\right| \;\leq \\ 16\int_Q s\phi\lambda^2|\nabla\big(|\nabla\psi|\big)|^2\,p^2\,dxdt + \frac{1}{4}\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt. \end{aligned} \tag{2.22}$$

Remark 2.3. *As $\phi(x,t) = e^{\lambda\psi(x)}/\beta(t)$ with $\psi \in C^2(\overline{\Omega})$, then we have $|\nabla\psi|^4 < C$ and $|\nabla\big(|\nabla\psi|\big)|^2 < C$, where C is a positive constant depending only of Ω.*

Thus, from Remark 2.3, (2.21) and (2.22) we transform (2.20) as follows.

$$M_1 \geq \frac{3}{2}\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt - C\int_Q s\phi(\lambda^4+\lambda^2)|p|^2\,dxdt. \tag{2.23}$$

Applying Green's formula in M_3, we get

$$M_3 = 2\int_Q s\lambda\nabla(\phi\nabla\psi\cdot\nabla p)\cdot\nabla p\,dxdt - 2\int_\Sigma (s\lambda\phi\nabla\psi\cdot\nabla p)\nabla p\cdot n\,d\Sigma.$$

Observe that $\nabla p\cdot n = \dfrac{\partial p}{\partial n}$, where n is the exterior unit vector normal to Σ. After same calculus, it implies

$$M_3 = 2\int_Q s\lambda^2\phi(\nabla\psi\cdot\nabla p)^2\,dxdt + 2\int_Q s\lambda\phi\psi_{x_ix_j}\,p_{x_i}\,p_{x_j}\,dxdt + \\ 2\int_Q s\lambda\phi\psi_{x_i}\,p_{x_ix_j}\,p_{x_j}\,dxdt - 2\int_\Sigma (s\lambda\phi\nabla\psi\cdot\nabla p)\nabla p\cdot n\,d\Sigma.$$

Setting

$$N_1 = 2\int_Q s\lambda\phi\psi_{x_i}\,p_{x_ix_j}\,p_{x_j}\,dxdt,$$

and observing that $p_{x_ix_j}\,p_{x_j} = \dfrac{1}{2}\Big(\big(p_{x_j}\big)^2\Big)_{x_i}$, we obtain

$$N_1 = 2\int_Q s\lambda\phi\psi_{x_i}\frac{1}{2}\Big(\big(p_{x_j}\big)^2\Big)_{x_i}\,dxdt.$$

Auxiliary Computations. By Gauss Lemma, we obtain

$$\int_Q \frac{\partial}{\partial x_i}\Big(s\lambda\phi\psi_{x_i}\big(p_{x_j}\big)^2\Big)\,dxdt = \int_\Sigma s\lambda\phi\psi_{x_i}\big(p_{x_j}\big)^2 n_i\,d\Sigma.$$

Applying the derivative $\dfrac{\partial}{\partial x_i}$, we find

$$\int_Q s\lambda\phi\psi_{x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_j}\right)^2 dxdt + \int_Q s\lambda\phi_{x_i}\psi_{x_i}\big(p_{x_j}\big)^2\,dxdt + \\ \int_Q s\lambda\phi\psi_{x_ix_i}\big(p_{x_j}\big)^2\,dxdt = \int_\Sigma s\lambda\phi\psi_{x_i}\big(p_{x_j}\big)^2 n_i\,d\Sigma.$$

Inserting the above equality in the integral N_1, it yields

$$N_1 = -\int_Q s\lambda\phi_{x_i}\psi_{x_i}\big(p_{x_j}\big)^2\,dxdt - \int_Q s\lambda\phi\psi_{x_ix_i}\big(p_{x_j}\big)^2\,dxdt \\ + \int_\Sigma s\lambda\phi\big(p_{x_j}\big)^2\psi_{x_i}\,n_i\,d\Sigma.$$

Since $\phi(x,t) = e^{\lambda\psi(x)}/\beta(t)$ then $\phi_{x_i} = \lambda\phi\psi_{x_i}$. Thus, $s\lambda\phi_{x_i}\psi_{x_i} = s\lambda\phi(\psi_{x_i})^2$. Therefore, we obtain

$$N_1 = -\int_Q s\lambda^2\phi(\psi_{x_i})^2(p_{x_j})^2\,dxdt - \int_Q s\lambda\phi\psi_{x_ix_i}(p_{x_j})^2\,dxdt \\ + \int_\Sigma s\lambda\phi(p_{x_j})^2\,\psi_{x_i}\,n_i\,d\Sigma.$$

Thus,

$$N_1 = -\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt - \int_Q s\lambda\phi\Delta\psi|\nabla p|^2\,dxdt + \int_\Sigma s\lambda\phi\,\frac{\partial\psi}{\partial n}\,|\nabla p|^2\,d\Sigma.$$

Inserting this identity in M_3, we obtain

$$M_3 = 2\int_Q s\lambda^2\phi(\nabla\psi\cdot\nabla p)^2\,dxdt + 2\int_Q s\lambda\phi\psi_{x_ix_j}p_{x_i}p_{x_j}\,dxdt - 2\int_\Sigma s\lambda\phi(\nabla\psi\cdot\nabla p)\,\nabla p\cdot n\,d\Sigma - \int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt - \int_Q s\phi\Delta\psi|\nabla p|^2\,dxdt + \int_\Sigma s\lambda\phi\,\frac{\partial\psi}{\partial n}\,|\nabla p|^2\,d\Sigma. \tag{2.24}$$

Remark 2.4. *The two surface integrals of (2.24) satisfy*

$$\int_0^T\int_\Gamma s\lambda\phi\,\frac{\partial\psi}{\partial n}\,|\nabla p|^2\,d\Gamma\,dt \le 0 \quad \textit{and} \quad -2\int_0^T\int_\Gamma s\lambda\phi(\nabla\psi\cdot\nabla p)\,\nabla p\cdot n\,d\Sigma \ge 0.$$

In fact, since ψ satisfies the conditions of Lemma 1.1, that is, $\psi\in C^2(\overline{\Omega})$; $\psi=0$ on $\Gamma=\partial\Omega$ and $\psi>0$ in Ω, thus, by definition

$$\frac{\partial\psi}{\partial n}(x) = \lim_{k\to 0-}\frac{\psi(x+kn)-\psi(x)}{k} = \lim_{k\to 0-}\frac{\psi(x+kn)}{k} < 0,$$

because $\psi(x)=0$ on Γ and $\psi(x)>0$ for $x\in\Omega$. If $x\in\Gamma$, $k<0$, then $x+kn\in\Omega$, where n exterior unit normal to Γ. Thus, $\dfrac{\partial\psi}{\partial n}<0$ on Γ and

$$\int_\Sigma s\lambda\phi\,\frac{\partial\psi}{\partial n}\,|\nabla p|^2\,d\Sigma \le 0. \tag{2.25}$$

For the second surface integral, $\nabla\psi = n\,\dfrac{\partial\psi}{\partial n}$ and $\dfrac{\partial\psi}{\partial n}<0$ on Γ, then

$$-2\int_\Sigma s\phi\lambda\left(\frac{\partial\psi}{\partial n}\,n\cdot\nabla p\right)\frac{\partial p}{\partial n}\,d\Sigma = \int_\Sigma s\phi\lambda\,\frac{\partial\psi}{\partial n}\left(\frac{\partial p}{\partial n}\right)^2 d\Sigma \ge 0 \quad \blacksquare \tag{2.26}$$

Thus, using (2.25) in (2.24), we obtain

$$M_3 - N_2 \le 2\int_Q s\lambda^2\phi(\nabla\psi\cdot\nabla p)^2\,dxdt - \int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt + 2\int_Q s\lambda\phi\psi_{x_ix_j}p_{x_i}p_{x_j}\,dxdt - \int_Q s\phi\Delta\psi\,|\nabla p|^2\,dxdt, \tag{2.27}$$

where $N_2 = -2\displaystyle\int_\Sigma s\lambda\phi\nabla\psi\cdot\nabla p\,\nabla p\cdot n\,d\Sigma.$

$$|M_3 - N_2| \leq \int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt + C\int_Q s\lambda\phi|\nabla p|^2\,dxdt, \tag{2.28}$$

where we have used that $|\Delta\psi|$ and $|\psi_{x_ix_j}|$ are bounded in Ω, due to hypótese on ψ of Lemma 1.1. Thus, from (2.28), we obtain

$$M_3 - N_2 \geq -\int_Q s\lambda^2|\nabla\psi|^2|\nabla p|^2\,dxdt - C\int_Q s\lambda\phi|\nabla p|^2\,dxdt$$

and consequently

$$M_3 \geq -\int_Q s\lambda^2\phi|\nabla\psi|^2|\nabla p|^2\,dxdt - C\int_Q s\lambda\phi|\nabla p|^2\,dxdt. \tag{2.29}$$

Thus, from (2.23) and (2.29), we have

$$\begin{aligned} M_1 + M_3 \geq \frac{1}{2}\int_Q s\lambda^2\phi|\nabla\psi|^2\,|\nabla p|^2\,dxdt - \\ C\int_Q s\phi(\lambda^4+\lambda^2)|\nabla p|^2\,dxdt - C\int_Q s\lambda\phi|\nabla p|^2\,dxdt. \end{aligned} \tag{2.30}$$

where $C > 0$ is a constant that depends on Ω.

Now, we are working with M_4. In fact,

$$\begin{aligned} M_4 = -2\int_Q (s^3\lambda^3\phi^3|\nabla\psi|^2\,\nabla\psi\cdot\nabla p\,p + s^2\lambda^3\phi^2|\nabla\psi|^2\,\nabla\psi\cdot\nabla p\,p)dxdt = \\ -\int_Q (s^3\lambda^3\phi^3|\nabla\psi|^2\,\nabla\psi\,\nabla p^2 + s^2\lambda^3\phi^2|\nabla\psi|^2\,\nabla\psi\cdot\nabla p^2)dxdt = \\ \int_Q (s^3\lambda^3\,\nabla\cdot(\phi^3|\nabla\psi|^2\,\nabla\psi)|p|^2 + s^2\lambda^3\,\nabla\cdot(\phi^2|\nabla\psi|^2\,\nabla\psi)|p|^2)dxdt = \\ 3\int_Q (s^3\lambda^4\phi^3|\nabla\psi|^4|p|^2)dxdt + \int_Q s^3\lambda^3\phi^3\,\nabla\cdot(|\nabla\psi|^2\,\nabla\psi)|p|^2\,dxdt + \\ 2\int_Q s^2\lambda^4\phi^2|\nabla\psi|^4\,|p|^2x + \int_Q s^2\lambda^3\phi^2\,\nabla\cdot(|\nabla\psi|^2\,\nabla\psi)|p|^2\,dxdt. \end{aligned} \tag{2.31}$$

Note that $\left|\nabla\left(|\nabla\psi|^2\,\nabla\psi\right)\right| < C$ and $\nabla\phi^3 = 3\phi^2\nabla\phi = 3\phi^2(\lambda\phi\nabla\psi) = 3\lambda\phi^3\nabla\psi$. Thus, $\lambda\phi^3\nabla\psi = \dfrac{1}{3}\,\nabla\phi^3$. Also we have $\nabla\phi^2 = 2\phi\nabla\phi = 2\lambda\phi^2\nabla\psi$. Therefore, we modify (2.31) to find

$$\begin{aligned} M_4 \geq \int_Q \lambda^4(3s^3\phi^3 + 2s^2\phi^2)|\nabla\psi|^4|p|^2\,dxdt \\ -C\int_Q (\lambda^3s^3\phi^3 + \lambda^3s^2\phi^2)|p|^2\,dxdt. \end{aligned} \tag{2.32}$$

Thus, form (2.32) and the definition of M_2, we obtain

$$M_2 + M_4 \geq \int_Q s^3\lambda^4\phi^3|\nabla\psi|^4|p|^2\,dxdt$$
$$-C\int_Q (\lambda^3 s^3\phi^3 + \lambda^2 s^2\phi^2)|p|^2\,dxdt. \tag{2.33}$$

Finally, we will modify M_4. From (2.11) and (2.14), we obtain

$$M_5 = 2\int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2|p|^2\,dxdt + \int_Q s^2\lambda\phi\alpha_t\nabla\psi\cdot\nabla(p^2)\,dxdt =$$
$$2\int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2|p|^2\,dxdt - \int_Q s^2\lambda\nabla(\phi\alpha_t\nabla\psi)|p|^2\,dxdt.$$

Applying Gauss Lemma in the second integral and as $p = 0$ on Σ, we get

$$M_5 = 2\int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2|p|^2\,dxdt - \int_Q s^2\lambda^2\phi\alpha_t|\nabla\psi|^2|p|^2\,dxdt -$$
$$\int_Q s^2\lambda\phi\alpha_t(\lambda\phi_t|\nabla\psi|^2)|p|^2\,dxdt - \int_Q s^2\lambda\phi\alpha_t\Delta\psi|p|^2\,dxdt.$$

Hence, we obtain

$$|M_5| \leq C\left(\int_Q s^2\lambda^2\phi^3|\nabla\psi|^2|p|^2\,dxdt + \int_Q s^2\lambda\phi^3|\Delta\psi||p|^2\,dxdt\right).$$

Thus,

$$M_5 \geq -C\int_Q (s^2\lambda^2 + s^2\lambda)\phi^3 p^2\,dxdt. \tag{2.34}$$

From (2.18), (2.19), (2.30), (2.33) and (2.34) we have

$$\frac{1}{2}\int_Q s\lambda^2\phi|\nabla\psi|^2|\nabla p|^2\,dxdt + \int_Q s^3\lambda^4\phi^3|\nabla\psi|^4|p|^2\,dxdt \leq \tag{2.35}$$
$$C\int_Q s\phi(\lambda^4 + \lambda^2)|p|^2\,dxdt + C\int_Q s\lambda\phi|\nabla p|^2\,dxdt +$$
$$C\int_Q (\lambda^3 s^3\phi^3 + \lambda^3 s^2\phi^2)|p|^2\,dxdt + C\int_Q (s^2\lambda^2 + s^2\lambda)\phi^3|p|^2\,dxdt +$$
$$C\int_Q (\lambda^2(s^2\phi^2 + s^2\phi^3) + s\phi^3 + 1)|p|^2\,dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\,dxdt \leq$$
$$C\int_Q (s\lambda\phi|\nabla p|^2 + e^{2s\alpha}\,|f_1|^2)dxdt + \int_Q \phi^3(s^2\lambda^4 + s^3\lambda^3 + 1)|p|^2\,dxdt.$$

Since $\lambda \geq \lambda_0 \geq 1$, $s \geq s_0(\lambda) \geq 1$ and $C < |\phi|$, we have

$$s\phi(\lambda^4 + \lambda^2)|p|^2 \leq \lambda^4 s^2\phi^3|p|^2 \quad \text{and} \quad \lambda^3 s^3\phi^2 p^2 \leq \lambda^3 s^3\phi^3|p|^2.$$

Besides, since $|\nabla\psi| > 0$ in $\partial\Omega$, then $|\nabla\psi| > 0$ in the compact set $\partial\Omega \cup (\Omega - w_0)$. Therefore, there exists $\gamma > 0$ such that $0 < \gamma < |\nabla\psi|$ for all $x \in \partial\Omega \cup (\Omega - \omega_0)$ and for all $x \in \Omega - \omega_0$. About the open sets ω and ω_0 look at the Lemma 1.1. With this in mind we get from (2.35), that

$$\int_{Q-Q_{\omega_0}} (\gamma^4 s^3\lambda^4\phi^3 p^2 + \gamma^2 s\lambda^2\phi|\nabla p|^2)dxdt \leq \tag{2.36}$$
$$C\Big(\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + \int_Q (s\lambda\phi|\nabla p|^2 + (s^3\lambda^3 + s^2\lambda^4 + 1)\phi^3 p^2)dxdt\Big).$$

Hence,

$$\int_{Q-Q_{\omega_0}} (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)\,dxdt \leq \tag{2.37}$$
$$C\Big(\int_Q e^{2s\alpha}|f_1|^2 dxdt + \int_Q \big[s\lambda\phi|\nabla p|^2 + (s^3\lambda^3 + s^2\lambda^4 + 1)\phi^3 p^2\big]\,dxdt\Big).$$

For $\lambda > \lambda_0$ and $s \geq s_0(\lambda)$ sufficiently large we get from (2.37) that

$$\frac{1}{2}\int_{Q-Q_{\omega_0}} (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)dxdt \leq \tag{2.38}$$
$$C\Big(\int_Q e^{2s\alpha}|f_1|^2\,dxdt + \int_{Q_{\omega_0}} (s\lambda\phi|\nabla p|^2 + s^3\lambda^3\phi^3 p^2)dxdt\Big),$$

for s such that $s^2\lambda^4 \leq s^3\lambda^3$ (for example $\lambda = s$). Therefore,

$$\int_Q (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)dxdt \leq \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)dxdt + \tag{2.39}$$
$$2C\Big(\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + \int_{Q_{\omega_0}} (s\phi\lambda|\nabla p|^2 + s^3\lambda^3\phi^3 p^2)dxdt\Big) \leq$$
$$(1+2C)\Big(\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + \int_{Q_{\omega_0}} (s^3\lambda^3\phi^3 p^2 + s\lambda\phi|\nabla p|^2)\Big)dxdt \leq$$
$$C_1\Big(\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt + \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)dxdt\Big).$$

Therefore, we have

$$\int_Q (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)\,dxdt \leq \tag{2.40}$$
$$C_1\Big(\int_Q e^{2s\alpha}|f_1|^2\,dxdt + \int_{Q_{\omega_0}} (s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi|\nabla p|^2)dxdt\Big).$$

Step 3. (**Return to original variables**)

As $p = e^{s\alpha}w$, then $\nabla p = s\,\nabla\alpha e^{s\alpha}w + e^{s\alpha}\nabla w$, and it implies $|\nabla p|^2 \le 2s^2e^{2s\alpha}\,|\nabla\alpha|^2\,|w|^2 + 2e^{2s\alpha}\,|\nabla w|^2$ and $|\nabla\alpha|^2 \le \lambda^2\phi^2|\nabla\psi|^2$. Thus, we obtain

$$|\nabla p|^2 \le C\,e^{2s\alpha}(s^2\lambda^2\phi^2|w|^2 + |\nabla w|^2). \tag{2.41}$$

Besides, we also have

$$\nabla p = s\,\nabla\alpha p + e^{s\alpha}\,\nabla w, \tag{2.42}$$

and so

$$|\nabla p|^2 = s^2|\nabla\alpha|^2\,p^2 + e^{2s\alpha}\,|\nabla w|^2 + 2se^{s\alpha}(\nabla\alpha\cdot\nabla w)p.$$

Using (2.41) and (2.42) in (2.40), we have

$$\int_Q [e^{2s\alpha}\,\lambda^4 s^3\phi^3|w|^2 + s\lambda^2\phi(s^2|\nabla\alpha|^2\,p^2 +$$
$$e^{2s\alpha}\,|\nabla w|^2 + 2se^{s\alpha}(\nabla\alpha\cdot\nabla w)p)]dxdt \le C\Big[\int_{Q_{\omega_0}} (s\lambda^2\phi Ce^{2s\alpha}(s^2\lambda^2\phi^2|w|^2 + |\nabla w|^2) +$$
$$e^{2s\alpha}\,s^3\lambda^4\phi^3|w|^2)dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\,dxdt\Big] \le$$
$$C\Big[\int_{Q_{\omega_0}} (e^{2s\alpha}\,s^3\lambda^4\phi^3|w|^2 + e^{2s\alpha}\,s\lambda^2\phi|\nabla w|^2)dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\,dxdt\Big].$$

Re-write this inequality in a convenient form, we have

$$\int_Q e^{2s\alpha}\,s^3\lambda^4\phi^3|w|^2\,dxdt + \int_Q s^3\lambda^2\phi|\nabla\alpha|^2\,p^2\,dxdt + \tag{2.43}$$
$$\int_Q s\lambda^2\phi\,e^{2s\alpha}\,|\nabla w|^2\,dxdt + 2\int_Q s^2\lambda^2\phi e^{s\alpha}(\nabla\alpha\cdot\nabla w)p\,dxdt \le$$
$$C\int_{Q_{\omega_0}} s^3\lambda^4\phi^3\,e^{2s\alpha}|w|^2\,dxdt + C\int_{Q_{\omega_0}} s\lambda^2\phi\,e^{2s\alpha}|\nabla w|^2\,dxdt$$
$$+C\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt.$$

From (2.43) if

$$N_3 = C\int_{Q_{\omega_0}} s^3\lambda^4\phi^3\,e^{2s\alpha}\,|w|^2\,dxdt + C\int_{Q_{\omega_0}} s\lambda^2\phi\,e^{2s\alpha}|\nabla w|^2\,dxdt$$
$$+\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt,$$

then we can see directly

$$\int_Q s^3\lambda^2\phi|\nabla\alpha|^2\, p^2\, dxdt \le N_3\,. \tag{2.44}$$

Moreover,

$$\left|2\int_Q s^2\lambda^2\phi\, e^{s\alpha}\, |\nabla\alpha\cdot\nabla w| p\, dxdt\right| \le \tag{2.45}$$

$$2\int_Q s^3\lambda^2\phi|\nabla\alpha|^2\, p^2\, dxdt + \frac{1}{2}\int_Q s\lambda^2\phi\, e^{2s\alpha}|\nabla w|^2\, dxdt.$$

Thus, from (2.43), (2.44), (2.45), we obtain

$$\int_Q (s^3\lambda^4\phi^3 e^{2s\alpha}|w|^2 + s\lambda^2\phi e^{2s\alpha}|\nabla w|^2) dxdt \le \tag{2.46}$$

$$N_3 + N_3 + 2N_3 + \frac{1}{2}\int_Q s\lambda^2\phi\, e^{2s\alpha}\, |\nabla w|^2\, dxdt.$$

Hence,

$$\int_Q \left(s^3\lambda^4\phi^3 e^{2s\alpha}|w|^2 + s\lambda^2\phi e^{2s\alpha}|\nabla w|^2\right) dxdt \le \tag{2.47}$$

$$C\Big[\int_{Q_{\omega_0}} \left(s^3\lambda^4\phi^3 e^{2s\alpha}|w|^2 + s\lambda^2\phi e^{2s\alpha}|\nabla w|^2\right) dxdt + \int_Q e^{2s\alpha}\, |f_1|^2\, dxdt\Big] \quad \blacksquare$$

Step 4. (**Carleman Estimates: Conclusion of the proof of Theorem 2.1**)

Let us consider the square of both sides of the state equation $(1.3)_1$. After, multiply both sides by $(s\phi)^{-1}\, e^{2s\alpha}$ and integrate on Q, we obtain

$$\int_Q e^{2s\alpha}(s\phi)^{-1}\left(|w_t|^2 + |\Delta w|^2\right) dxdt = \tag{2.48}$$

$$\int_Q e^{2s\alpha}(s\phi)^{-1}|f_1|^2\, dxdt + \int_Q e^{2s\alpha}(s\phi)^{-1}|w|^2\, |a(t)|^2\, dxdt +$$

$$2\int_Q e^{2s\alpha}(s\phi)^{-1}\, f_1\, a(t) w\, dxdt - 2\int_Q e^{2s\alpha}(s\phi)^{-1}\, w_t\Delta w\, dxdt.$$

Now we will examine each term of (2.48). First, note that

$$(s\phi)^{-1} \le C \quad \text{and} \quad (s\phi)^{-1} \le C(s\phi)^3. \tag{2.49}$$

From this and (2.49), we obtain

$$\left|\int_Q e^{2s\alpha}(s\phi)^{-1}\, |f_1|^2\, dxdt\right| \le C\int_Q e^{2s\alpha}\, |f_1|^2\, dxdt; \tag{2.50}$$

$$\left|\int_Q (s\phi)^{-1}\, e^{2s\alpha}|w|^2\, |a(t)|^2\, dxdt\right| \le MC_1 \int_Q e^{2s\alpha}\, s^3\phi^3|w|^2\, dxdt. \qquad (2.51)$$

As $\alpha(0) = \alpha(T) = -\infty$, then $e^{2s\alpha(0)} = e^{2s\alpha(T)} = 0$. Thus, by Gauss Lemma, we get

$$\begin{aligned}
&2\int_Q e^{2s\alpha}(s\phi)^{-1}\, w_t\Delta w\, dxdt = \qquad (2.52)\\
&-2\int_Q \nabla(e^{2s\alpha}(s\phi)^{-1}\, w_t)\cdot\nabla w\, dxdt =\\
&-2\int_Q 2s\,\nabla\alpha\, e^{2s\alpha}(s\phi)^{-1}\, w_t\nabla w\, dxdt +\\
&+2\int_Q e^{2s\alpha}\, s^{-1}\phi^{-2}\nabla\phi\cdot\nabla w\, w_t\, dxdt - \int_Q e^{2s\alpha}(s\phi)^{-1}\,\frac{d}{dt}\,|\nabla w|^2\, dxdt =\\
&-4\int_Q \lambda\, e^{2s\alpha}\, w_t\nabla\psi\cdot\nabla w\, dxdt + 2\int_Q e^{2s\alpha}(s\phi)^{-1}\,\lambda\, w_t\nabla\psi\cdot\nabla w\, dxdt +\\
&\int_Q (2s\alpha_t\, e^{2s\alpha}(s\phi)^{-1} - e^{2s\alpha}\, s^{-1}\phi^{-2}\phi_t|\nabla w|^2\, dxdt \le\\
&C\int_Q e^{2s\alpha}|\nabla\psi|\,|\nabla w|\,|w_t|\, dxdt + C\int_Q e^{2s\alpha}(s\phi)|\nabla w|^2\, dxdt \le \quad (C < s\phi)\\
&C\int_Q e^{2s\alpha}(s\phi)|\nabla w|^2\, dxdt + \frac{1}{2}\int_Q e^{2s\alpha}(s\phi)^{-1}\,|w_t|^2\, dxdt.
\end{aligned}$$

Substituting (2.49), (2.50), (2.51) and (2.52) in (2.48), we obtain

$$\begin{aligned}
&\int_Q e^{2s\alpha}(s\phi)^{-1}\left(\frac{1}{2}\,|w_t|^2 + |\Delta w|^2\right) dxdt \le \qquad (2.53)\\
&C\Big[\int_Q e^{2s\alpha}|f_1|^2\, dxdt + \int_Q e^{2s\alpha}\, s^3\phi^3|w|^2\, dxdt + \int_Q e^{2s\alpha}\, s\phi|\nabla w|^2\, dxdt\Big] \le\\
&C_1\Big[\int_Q e^{2s\alpha}|f_1|^2\, dxdt + \int_{Q_{\omega_0}} e^{2s\alpha}\, s^3\phi^3|w|^2\, dxdt +\\
&\int_{Q_{\omega_0}} e^{2s\alpha}\, s\phi|\nabla w|^2\, dxdt\Big].
\end{aligned}$$

Note that Q_{ω_0} and $\frac{1}{2}\,|w_t|$ comes from (2.47) and (2.52), (2.48) respectively.

Let us consider a function $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in $\overline{\omega}_0$, closure of ω_0, and $\chi = 0$ on $\Omega - \omega$. Multiplying the adjoint system (1.3) by $e^{2s\alpha}\,\chi\, s\phi w$ and integrating on Q, we obtain

$$\int_Q e^{2s\alpha}\,\chi\,s\phi\,w\,w_t\,dxdt+\int_Q e^{2s\alpha}\,\chi\,s\phi\,w\,\Delta w\,dxdt-\tag{2.54}$$
$$\int_Q e^{2s\alpha}\,\chi\,s\phi a(t)w^2\,dxdt=\int_Q e^{2s\alpha}\,\chi\,s\phi\,f_1\,dxdt.$$

Analysis of the terms of (2.54). We have

$$\int_Q e^{2s\alpha}\,\chi\,s\phi\,w\,w_t\,dxdt=\frac{1}{2}\int_{Q_\omega}(e^{2s\alpha}\chi\,s\phi)\,\frac{d}{dt}\,w^2\,dxdt=\tag{2.55}$$
$$-\frac{1}{2}\int_{Q_\omega}(e^{2s\alpha}s\phi)_t|w|^2\,dxdt\le\frac{1}{2}\int_{Q_\omega}\chi(2s\alpha_t\,e^{2s\alpha}\,s\phi+e^{2s\alpha}\,\phi_t)w^2\,dxdt\ \le$$
$$C\int_{Q_\omega}e^{2s\alpha}\,s^3\phi^3w^2\,dxdt,\quad\text{for}\quad s\ge s_1>1.$$

Applying the Green's formula and observing that $w=0$ on Σ, we get

$$\int_Q e^{2s\alpha}\,\chi\,s\phi|\nabla w|^2\,dxdt\ =\tag{2.56}$$
$$-\int_Q e^{2s\alpha}\,\chi\,s\phi w\Delta w\,dxdt-\int_Q sw\nabla(e^{2s\alpha}\chi\phi)\nabla w\,.$$

On the other hand,

$$\left|\int_Q sw\,\nabla(e^{2s\alpha}\chi\phi)\cdot\nabla w\,dxdt\right|\ =\tag{2.57}$$
$$\left|\int_{Q_\omega}sw\Big[\chi(2s\nabla\alpha\,e^{2s\alpha}\phi+e^{2s\alpha}\alpha\phi\nabla\psi)+e^{2s\alpha}\phi\nabla\chi\Big](\nabla w)dxdt\right|\ \le$$
$$C\varepsilon\int_{Q_\omega}s\,\phi\,e^{2s\alpha}w^2\,dxdt+\frac{1}{2\varepsilon}\int_{Q_\omega}e^{2s\alpha}\chi s\phi|\nabla w|^2\,dxdt+$$
$$\frac{1}{2C_1}\int_Q e^{2s\alpha}(s\phi)^{-1}\,|\Delta w|^2\,dxdt.$$

Now adding (2.47) with (2.53), we get

$$\int_Q e^{2s\alpha}(s\phi)^{-1}\left(\frac{1}{2}\,|w_t|^2+|\Delta w|^2\right)dxdt+\tag{2.58}$$
$$\int_Q\big(e^{2s\alpha}\,s^3\lambda^4\phi^3|w|^2+e^{2s\alpha}\,s\lambda^2\phi\,e^{2s\alpha}\,|\nabla w|^2\big)dxdt\ \le$$
$$C\Big[\int_{Q_{\omega_0}}e^{2s\alpha}\,s^3\lambda^4\phi^3\,|w|^2\,dxdt+$$
$$\int_{Q_{\omega_0}}e^{2s\alpha}\,s\lambda^2\phi|\nabla w|^2\,dxdt+\int_Q e^{2s\alpha}\,|f_1|^2\,dxdt\Big].$$

From (2.56), we obtain

$$\int_{Q_{\omega_0}} e^{2s\alpha}\, s\lambda^2\phi|\nabla w|^2\, dxdt \le \int_Q e^{2s\alpha}\,\chi\, s\lambda^2\phi|\nabla w|^2\, dxdt \le \tag{2.59}$$
$$\left|\int_Q e^{2s\alpha}\,\chi(s\phi)w\,\Delta w\, dxdt\right| + C\varepsilon\int_{Q_{\omega_0}} e^{2s\alpha}(s\phi)^3\, w^2\, dxdt +$$
$$\frac{1}{\varepsilon}\int_{Q_{\omega_0}} e^{2s\alpha}(s\phi)|\nabla w|^2\, dxdt \le$$
$$\frac{1}{2C_1}\int_{Q_\omega} e^{2s\alpha}\,(s\phi)^{-1}|\Delta w|^2\, dxdt + C\varepsilon\int_{Q_\omega} e^{2s\alpha}\, s^3\lambda^4\phi^3\,|w|^2\, dxdt +$$
$$\frac{1}{\varepsilon}\int_{Q_\omega} e^{2s\alpha}\, s\lambda^2\phi|\nabla w|^2\, dxdt.$$

Finally, from (2.58), (2.59) and choosing $\varepsilon > 0$, such that $C_2(\lambda)/\varepsilon = 1/2$ (to bring the term in $|\nabla w|^2$ to the left-hand side), we get

$$\int_Q \Big[(s\phi)^{-1}\big(|w_t|^2 + |\Delta w|^2\big) + s^3\lambda^4\phi^3|w|^2 + s\lambda^2\phi|\nabla w|^2\Big]e^{2s\alpha}\, dxdt \le$$
$$C\Big[\int_{Q_\omega} e^{2s\alpha}\, s^2\lambda^4\phi^2|w|^2\, dxdt + \int_Q e^{2s\alpha}\,|f_1|^2\, dxdt\Big],$$

where $w = w(x,t)$ is weak solutions of the adjoint system (1.3). It was proved by Fursikov-Imanovilov [7], see also Fursikov [6].

3 Observability inequality

In the present section we will prove the observability inequality for weak solutions of the adjoint system (1.3). Observe that it is a consequence of the Carleman Inequality proved in Section 2.

Theorem 3.1. *Suppose α, ϕ as in Theorem 2.1. Then, for $\lambda > \lambda_0 > 0$ and $s > s(\lambda) > 1$, we have*

$$\int_\Omega |w(x,0)|^2\, dx \le C\int_Q e^{2s\alpha}|f_1|^2\, dxdt + \int_{Q_\omega} e^{2s\alpha}\,\phi^3|w|^2\, dxdt, \tag{3.1}$$

where C is a positive constant that depends only of Ω and T.

This inequality is called *Observability Inequality* for the adjoint system (1.3).

Proof. Multiply both sides of $(1.3)_1$ by w and integrate on Ω, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\, dxdt + \int_\Omega |\nabla w|^2\, dx = \int_\Omega w\, f_1\, dxdt - \int_\Omega a(x,t)w^2\, dx. \tag{3.2}$$

Multiplying both sides of (3.2) by $e^{2(M+1)t}$ and observing that $|a(t)|_{L^\infty(\Omega)} < M$ a. e. in $[0,T[$, we obtain

$$-\frac{d}{dt}\left(e^{2(M+1)t}\int_\Omega w^2\,dxdt\right) \le \frac{e^{2(M+1)t}}{2}\int_\Omega f_1^2\,dx. \tag{3.3}$$

Integrating (3.3) from 0 to t, we get:

$$|w(x,0)|^2_{L^2(\Omega)} \le e^{2(M+1)t}|w(t)|^2_{L^2(\Omega)} + \int_0^t \left(\frac{e^{2(M+1)y}}{2}\int_\Omega f_1(x,y)^2\,dx\right)dy. \tag{3.4}$$

Setting

$$\theta(t) = \sup\left\{e^{-2s\alpha(x,t)};\ x\in\Omega\right\},$$

we have

$$\theta(t) \le e^{\left(2s\,e^{2\lambda||\psi||}\right)/\beta(t)}, \tag{3.5}$$

since $-\alpha \le \dfrac{e^{2\lambda||\psi||}}{\beta(t)}\cdot$ Then $e^{-2s\alpha(x,t)} \le \theta(t)$ and

$$1 \le \theta(t)\,e^{2s\alpha(x,t)}. \tag{i}$$

We know that $\phi(x,t) = e^{\lambda\psi(x)}/\beta(t)$, $\beta(t) = t(T-t)$, $0<t<T$, for $x\in\Omega$. The function ψ as in Lemma 1.1. Thus, $\dfrac{1}{\phi^3} \le C$, for $0\le t\le T$, $x\in\overline{\Omega}$, or

$$1 \le C\,\phi^3(x,t),\ \ 0\le t\le T,\ \ x\in\overline{\Omega}. \tag{ii}$$

If $1\le s^3$, then by (3.5), (i) and (ii) we obtain from (3.4), that

$$\begin{aligned}|w(x,0)|^2_{L^2(\Omega)} &\le C_T\int_\Omega |w(x,t)|^2\,dx + C_T\int_0^t\left(\frac{1}{2}\int_\Omega |f_1(x,y)|^2\,dx\right)dy\\ &\le C\,\theta(t)\int_\Omega e^{2s\alpha}\,|w(x,t)|^2\,dx + C\,\theta(t)\int_0^t\left(\int_\Omega e^{2s\alpha(x,t)}\,|f_1(x,y)|^2\,dx\right)dy.\end{aligned}$$

Thus,

$$\frac{1}{\theta(t)}\,|w(x,0)|^2 \le C\int_\Omega e^{2s\alpha}\,|w(x,t)|^2\,dx + C\int_0^t\left(\int_\Omega e^{2s\alpha}\,|f_1(x,y)|^2\,dx\right)dy. \tag{3.6}$$

From (3.5), we obtain

$$0 < k_T < e^{-\frac{2s\,e^{2\lambda||\psi||}}{\beta(t)}} \le \frac{1}{\theta(t)},\quad 0<t<T.$$

Now, we fixe $t_1 < t_2$ in $(0,T)$ and integrate (3.6) on (t_1,t_2) with respect to t, to get

$$|w(x,0)|^2_{L^2(\Omega)} \int_{t_1}^{t_2} e^{-\frac{2s\, e^{2\lambda||\psi||}}{\beta(t)}}\, dt \leq \tag{3.7}$$
$$C \int_{t_1}^{t_2} \left(\int_\Omega e^{2s\alpha(x,t)}\, |w(x,t)|^2\, dx \right) dt +$$
$$C \int_{t_1}^{t_2} \left[\int_0^t \left(\int_\Omega e^{2s\alpha(x,t)}\, |f_1(x,y)|^2\, dx \right) dy \right] dt.$$

In the Appendix, established to follow, we prove

$$\int_\Omega \int_0^T \int_0^t e^{2s\alpha}\, |f_1(x,y)|^2\, dxdy\, dt \leq C \int_0^T \int_\Omega |f_1(x,y)|^2\, dxdy.$$

Since

$$\inf_{t\in[0,T]} e^{-\frac{2s\, e^{2\lambda||\psi||}}{\beta(t)}} \geq C_T\,,$$

from (3.7) and from the Appendix above cited, we obtain

$$|w(x,0)|^2_{L^2(\Omega)} \leq C \int_0^T \int_\Omega e^{2s\alpha}\, |w(x,t)|^2\, dxdt + \tag{3.8}$$
$$C \int_0^T \int_\Omega |f_1(x,t)|^2\, dxdt.$$

By (ii), $1 \leq C\, \phi^3(x,t)$, for $x \in \overline{\Omega}$ and $0 < t < T$. Then, we have from (3.8):

$$|w(x,0)|^2_{L^2(\Omega)} \leq C \int_0^T \int_\Omega e^{2s\alpha}\, \phi^3 |w(x,t)|^2\, dxdt + C \int_0^T \int_\Omega |f_1(x,y)|^2\, dxdy. \tag{3.9}$$

By Carleman inequality we obtain:

$$\int_Q e^{2s\alpha}\, s^3 \phi^3 |w(x,t)|^2\, dxdt \leq$$
$$C \int_Q e^{2s\alpha}\, |f_1(x,y)|^2\, dxdy + \int_{Q_\omega} e^{2s\alpha}\, \phi^3 |w(x,t)|^2\, dxdt.$$

Then,

$$\int_Q e^{2s\alpha}\, \phi^3 |w(x,t)|^2\, dxdt \leq C \int_{Q_\omega} e^{2s\alpha}\, \phi^3 |w(x,t)|^2\, dxdt +$$
$$C \int_Q e^{2s\alpha}\, |f_1(x,t)|^2\, dxdt.$$

Substituting the last inequality in (3.9) we obtain the **Observability Inequality** for the weak solution of the adjoint state (1.3). Note that $C > 0$ is a constant that depends on Ω and T ■

4 Null controllability: Linear State Equation

We consider the linear state equation:

$$\left| \begin{array}{l} p_t(x,t) - \Delta p(x,t) + a(x,t)p(x,t) = \chi_\omega u(x,t) \quad \text{in} \quad Q, \\ p(x,t) = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{4.1}$$

When we have $\chi_\omega u \in L^2(Q)$, $p_0 \in L^2(\Omega)$, the weak solution p of (4.1) has the following regularity, see Brezis [3],

$$p \in H^1\big(0,T;H^{-1}(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big).$$

Thus, $p \in C^0\big([0,T];L^2(\Omega)\big)$. The null controllability for (4.1) consist in to obtain a control $u \in L^2(Q)$, such that

$$p(x,T) = 0 \quad \text{a. e. in} \quad \Omega.$$

Observe that $a(t)$ is bounded in Q and $|a|_{L^\infty(Q)} < M$, as in Section 1.

We have the following regularity

$$\begin{gathered} p \in W^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \\ = \big\{p \in L^2(0,T;H_0^1(\Omega), p_t \in L^2(0,T;H^{-1}(\Omega))\big\} \subset C^0([0,T];L^2(\Omega)). \end{gathered}$$

Theorem 4.1. *For $p_0 \in L^2(\Omega)$, there exists a control $u \in L^2(Q)$ such that the weak solution $p = p(x,t)$ of the state equation (4.1) satisfies $p(x,T) = 0$ in Ω.*

Proof. The proof of Theorem 4.1 is done by a variational method and an application of the observability inequality, cf. Section 3. The control u picked up in $L^2(Q, e^{-2s\alpha}\,\phi^{-3})$ satisfies the inequality.

$$\int_{Q_\omega} e^{-2s\alpha}\,\phi^{-3}\,u^2\,dxdt \le C \int_\Omega p_0^2\,dx. \tag{4.2}$$

Note that $e^{-2s\alpha}\,\phi^{-3} \ge C_0$. For each $\varepsilon > 0$, we define the functional

$$N_\varepsilon(p,u) = \int_Q e^{-2s\alpha}\,\phi^{-3}\,u^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega p(x,T)^2\,dx, \tag{4.3}$$

for $u \in L^2(Q, e^{-2s\alpha}\,\phi^{-3})$ and p is the weak solution of (4.1), with $p_0 \in L^2(\Omega)$. Observe that $N_\varepsilon(p,u)$ is lower semi-continuous, strictly convex and coercive in $L^2(Q)$. Then, the variational problem

$$\min\, N_\varepsilon(p,u),$$

has a unique solution $u_\varepsilon \in L^2(Q)$.

We suppose that $u_\varepsilon \in L^2(Q)$ is the minimizer of $N_\varepsilon(p,u)$. Thus, by mean of the state equation (4.1) we find the weak solution p_ε. The next step consists to prove the convergence

$$\lim_{\varepsilon\to 0} u_\varepsilon = u \quad \text{and} \quad \lim_{\varepsilon\to 0} p_\varepsilon = p.$$

Then, we have to prove that p is the weak solution of (4.1) corresponding to the control u and that

$$p(x,T) = 0 \quad \text{a.e. in} \quad \Omega.$$

Lemma 4.1. *We have that*

$$u_\varepsilon = e^{2s\alpha}\,\phi^3\,\chi_\omega\,w_\varepsilon \quad \textit{a. e. in} \quad Q,$$

with

$$w_\varepsilon \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

is the weak solution of the parabolic problem

$$\left|\begin{array}{l} w_{\varepsilon t} + \Delta w_\varepsilon - a(t) w_\varepsilon = 0 \quad \textit{in} \quad Q, \\ w_\varepsilon = 0 \quad \textit{on} \quad \Sigma, \\ w_\varepsilon(x,T) = -\dfrac{1}{\varepsilon}\,p(x,T) \quad \textit{in} \quad \Omega, \end{array}\right. \tag{4.4}$$

being $p(x,t)$ the weak solution of (4.1), that is

$$\left|\begin{array}{l} p_t - \Delta p + a(t) p = \chi_\omega u_\varepsilon \quad \textit{in} \quad Q, \\ p = 0 \quad \textit{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \textit{in} \quad \Omega, \end{array}\right. \tag{4.5}$$

$p_0 \in L^2(\Omega)$, $u \in L^2(Q)$.

Proof. We write $p = \widehat{p} + \overline{p}$ with $\widehat{p}$ and $\overline{p}$ weak solutions of the systems

$$\left|\begin{array}{l} \widehat{p}_t - \Delta\widehat{p} + a(t)\widehat{p} = 0 \quad \text{in} \quad Q, \\ \widehat{p} = 0 \quad \text{on} \quad \Sigma, \\ \widehat{p}(x,0) = \widehat{p}(x) \quad \text{in} \quad \Omega, \end{array}\right. \tag{4.6}$$

and

$$\left| \begin{array}{l} \overline{p}_t - \Delta \overline{p} + a(t)\overline{p} = \chi_\omega u \quad \text{in} \quad Q, \\ \overline{p} = 0 \quad \text{on} \quad \Sigma, \\ \overline{p}(x,0) = 0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{4.7}$$

We observe that in (4.7) we have a linear dependence of the solution $\overline{p}$ from the control u, which we denote by:

$$Lu = \overline{p}(x,T). \tag{4.8}$$

Note that $L\colon L^2(Q) \to L^2(\Omega)$ is linear and bounded, because p belongs to $C^0([0,T];L^2(\Omega))$. Thus, we re-write $N_\varepsilon(p,u) = J_\varepsilon(u)$, with

$$J_\varepsilon(u) = \int_Q e^{-2s\alpha}\,\phi^{-3}\,u^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega (\widehat{p}(T) + Lu)^2\,dx. \tag{4.9}$$

The stationary value $u_\varepsilon \in L^2(Q)$, for the functional $J_\varepsilon(u)$, defined by (4.9), is that in which the Gateaux derivative is null in all direction $\widetilde{\omega} \in L^2(Q)$. It means,

$$J'_\varepsilon(u_\varepsilon)\cdot\widetilde{\omega} = 0 \quad \text{for all} \quad \widetilde{\omega} \in L^2(Q),$$

that is,

$$\frac{d}{d\lambda}\,J_\varepsilon(u_\varepsilon + \lambda\widetilde{\omega})\Big|_{\lambda=0} = 0 \quad \text{for all} \quad \widetilde{\omega} \in L^2(Q).$$

Now, we will do the computation with the weight $e^{-2s\alpha}\,\phi^{-3}$ in the functional J_ε observing that $p(T) = \widehat{p}(T) + \overline{p}(T) = \widehat{p}(T) + Lu$. In fact, we obtain

$$\frac{d}{d\lambda}\,J_\varepsilon(u_\varepsilon + \lambda\widetilde{\omega})\Big|_{\lambda=0} = 2\int_Q e^{-2s\alpha}\,\phi^{-3}\,u_\varepsilon\,\widetilde{\omega}\,dxdt + \frac{2}{\varepsilon}\int_\Omega p_\varepsilon(T)L\widetilde{\omega}\,dx,$$

for all $\widetilde{\omega} \in L^2(Q)$.

By the condition of u_ε to be stationary point of $J_\varepsilon(u)$, we must have

$$2\int_Q e^{-2s\alpha}\,\phi^{-3}\,u_\varepsilon\,\widetilde{\omega}\,dxdt + \frac{2}{\varepsilon}\int_\Omega p_\varepsilon(T)z(T)\,dx = 0, \tag{4.10}$$

for all $\widetilde{\omega} \in L^2(Q)$ and z weak solution of

$$\left| \begin{array}{l} z_t - \Delta z + a(t)z = \chi_\omega \widetilde{w} \quad \text{in} \quad Q, \\ z = 0 \quad \text{on} \quad \Sigma, \\ z(0) = 0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{4.11}$$

Observe that $z = L\widetilde{\omega}$ and L is a bounded linear function of $\widetilde{\omega} \in L^2(Q)$.

Remark 4.1. *As* $\widetilde{\omega} \in L^2(Q)$ *and* $z(x,t) = L\widetilde{\omega}(x,t)$*, then* $z \in C^0([0,T];L^2(\Omega))$*. Thus, makes sense* $z(x,T) = L\widetilde{\omega}(x,T)$ ■

Multiply (4.11) by w_ε and integrate in Q. We obtain:

$$\int_Q (-w_{\varepsilon t} - \Delta w_\varepsilon + a(t) w_\varepsilon) z \, dxdt \tag{4.12}$$
$$+ \int_\Omega z(T) w_\varepsilon(T)\, dx - \int_\Omega z(0) w_\varepsilon(0)\, dx = \int_Q \chi_\omega \widetilde{w}\, w_\varepsilon \, dxdt.$$

Therefore, if w_ε is the weak solution of the problem:

$$\left| \begin{aligned} & w_{\varepsilon t} + \Delta w_\varepsilon - a(t) w_\varepsilon = 0 \quad \text{in} \quad Q, \\ & w_\varepsilon = 0 \quad \text{on} \quad \Sigma, \\ & w_\varepsilon(x,T) = -\frac{1}{\varepsilon}\, p_\varepsilon(x,T) \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{4.13}$$

We obtain, from (4.11), (4.12) and (4.13):

$$-\frac{1}{\varepsilon} \int_\Omega p(x,T) z(x,T)\, dx = \int_Q \chi_\omega \widetilde{w}\, w_\varepsilon \, dxdt, \tag{4.14}$$

because $z(x,0) = 0$ in Ω by (4.11). From (4.10) we modify (4.14) obtaining

$$\int_Q e^{-2s\alpha}\, \phi^{-3}\, u_\varepsilon\, \widetilde{\omega}\, dxdt = \int_Q \chi_\omega \widetilde{w}\, w_\varepsilon\, dxdt,$$

or

$$\int_Q \big(e^{-2s\alpha}\, \phi^{-3}\, u_\varepsilon - \chi_\omega w_\varepsilon\big) \widetilde{\omega}\, dxdt = 0,$$

for all $\widetilde{\omega} \in L^2(Q)$. It implies that

$$u_\varepsilon = e^{2s\alpha}\, \phi^3\, \chi_\omega\, w_\varepsilon\,, \quad \text{a.e. in} \quad Q,$$

with w_ε weak solution of (4.13), i.e., (4.4). This proves Lemma 4.1 ■

Now, we will return to the proof of Theorem 4.1. In fact, the first step, still technic, is to obtain as application of Lemma 4.1 estimates for u_ε and p_ε to get convergence in order to obtain our objective, which is $p(x,T) = 0$. To this, we multiply both sides of (4.13) by $p_\varepsilon(x,t)$ and integrate in Q.

The second one, we multiply both sides of the system (4.15) below by w_ε and integrate in Q.

$$\left| \begin{aligned} & p_{\varepsilon t} - \Delta p_\varepsilon + a(t) p_\varepsilon = \chi_\omega w_\varepsilon\, e^{2s\alpha} \phi^3 \quad \text{in} \quad Q, \\ & p_\varepsilon = 0 \quad \text{on} \quad \Sigma, \\ & p_\varepsilon(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{4.15}$$

Adding both results, we obtain:

$$\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt = \int_\Omega p_\varepsilon(t)w_\varepsilon(t)\,dx\Big|_0^T =$$
$$-\frac{1}{\varepsilon}\int_\Omega (p_\varepsilon(x,T))^2\,dx - \int_\Omega p_\varepsilon(x,0)w_\varepsilon(x,0)\,dx.$$

After computations we obtain:

$$\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon\,dxdt + \frac{1}{\varepsilon}\int_\Omega p_\varepsilon(x,T)^2\,dx = -\int_\Omega p_0(x)w_\varepsilon(x,0)\,dx \tag{i}$$

$$\left|\int_\Omega p_0(x)w_\varepsilon(x,0)\,dx\right|_{\mathbb{R}} \le |p_0|_{L^2(\Omega)}|w_\varepsilon(x,0)|_{L^2(\Omega)}. \tag{ii}$$

By inequality of observability for $w_\varepsilon(x,0)$, cf. Theorem 3.1, Section 3, we obtain from (ii)

$$\left|\int_\Omega p_0(x)w_\varepsilon(x,0)\,dx\right|_{\mathbb{R}} \le \tag{4.16}$$
$$C|p_0|_{L^2(\Omega)}\left(\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2(x,t)\,dxdt + \frac{1}{\varepsilon}\int_\Omega p_\varepsilon(x,T)^2\,dx\right)^{1/2} \le$$
$$\frac{C^2}{2}\,|p_0|^2_{L^2(\Omega)} + \frac{1}{2}\left(\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega p_\varepsilon(x,T)^2\,dx\right).$$

By (i), (ii) and (4.16), we have

$$\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega p_\varepsilon(x,T)^2\,dx \le$$
$$\frac{C^2}{2}\,|p_0|^2_{L^2(\Omega)} + \frac{1}{2}\left(\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega p_\varepsilon(x,T)^2\,dx\right).$$

From this last inequality $\left(A \le \dfrac{C^2}{2}\,|p_0|^2 + \dfrac{1}{2}\,A\right)$, we obtain

$$\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt + \frac{1}{\varepsilon}\int_\Omega p(x,T)^2\,dx \le C|p_0|^2_{L^2(\Omega)} = \text{ constant.} \tag{4.17}$$

Thus from (4.17), we get

$$\int_\Omega p_\varepsilon(x,T)^2\,dx \le C\varepsilon. \tag{4.18}$$

From (4.18), we have

$$p_\varepsilon(x,T) \to 0 \quad \text{strongly} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0.$$

From (4.17), we obtain

$$\int_{Q_\omega} e^{2s\alpha}\,\phi^3\,w_\varepsilon^2\,dxdt < \text{ constant.}$$

By Lemma 4.1,

$$u_\varepsilon = e^{2s\alpha}\,\phi^3\,\chi_\omega\,w_\varepsilon \quad \text{a.e. in} \quad Q,$$

or

$$\int_Q u_\varepsilon^2\,dxdt \le C_0 \int_Q e^{2s\alpha}\,\phi^3\,\chi_\omega\,w_\varepsilon^2\,dxdt < C.$$

Thus, we have

$$u_\varepsilon \rightharpoonup u \quad \text{weaky in} \quad L^2(Q).$$

From (4.15), we obtain

$$p_\varepsilon \rightharpoonup p \quad \text{weakly} \quad H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

then

$$p_\varepsilon \rightharpoonup p \quad \text{strongly} \quad C^0([0,T];L^2(\Omega)). \tag{4.19}$$

From (4.18), we obtain

$$p_\varepsilon(x,T) \to 0 \quad \text{strongly} \quad L^2(\Omega),$$

whence there exists a subsequence of $p_\varepsilon(x,T)$ such that

$$p_\varepsilon(x,T) \to 0 \quad \text{a.e. in} \quad \Omega. \tag{4.20}$$

From (4.19), we have

$$p_\varepsilon(x,t) \to p(x,t) \quad \text{a.e. in} \quad \Omega,$$

for $0 \le t \le T$. Then

$$p_\varepsilon(x,T) \to p(x,T) \quad \text{a.e. in} \quad \Omega.$$

By (4.20) we have $p(x,T) = 0$ a.e. in Ω. Observe that $u_\varepsilon = w_\varepsilon\,e^{2s\alpha}\,\phi^3$ in the system (4.15). Thus, when $\varepsilon \to 0$ in (4.15) we obtain a control $u \in L^2(Q)$ and a function

$$p \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

solution, in weak sense, of

$$\left| \begin{array}{l} p_t - \Delta p + a(t)p = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0 \quad \text{in} \quad \Omega, \end{array} \right.$$

such that

$$p(x,T) = 0 \quad \text{a.e. in} \quad \Omega.$$

This conclusion proves Theorem 4.1 ■

5 Null controllability: Nonlinear State Equation

We now investigate null controllability for nonlinear state equation:

$$\left| \begin{array}{l} p_t - \Delta p + g(p) = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{5.1}$$

We suppose $g\colon \mathbb{R} \to \mathbb{R}$, globally Lipschitz and $g(0) = 0$. It means,

$$|g(p_1) - g(p_2)| \leq M|p_1 - p_2| \quad \text{for all} \quad p_1,\, p_2 \in \mathbb{R} \quad \text{and} \quad M \quad \text{constant.}$$

We define the function $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(p) = \left| \begin{array}{l} \dfrac{g(p)}{p} \quad \text{if} \quad |p| > 0, \\ \lim\limits_{\varepsilon \to 0} \dfrac{g(\varepsilon)}{\varepsilon} \quad \text{if} \quad p = 0. \end{array} \right.$$

We introduce the Hilbert space

$$W^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) = \{p \in L^2(0,T;H_0^1(\Omega)), p_t \in L^2(0,T;H^{-1}(\Omega))\},$$

with the norm

$$||p||^2_{W^1} = ||p||^2_{L^2(0,T;H_0^1(\Omega))} + ||p_t||^2_{L^2(0,T;H^{-1}(\Omega))}\,.$$

We have

$$W^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \subset C^0([0,T];L^2(\Omega)) \subset L^2(Q).$$

We consider the subset B of $L^2(Q)$ defined by

$$B = \{b \in W^1(0,T; H_0^1(\Omega), H^{-1}(\Omega)); ||b||_{W^1} < M_1\}.$$

For $\overline{p} \in B$, $p_0 \in L^2(\Omega)$, $u \in L^2(Q)$, we consider the linear state equation

$$\left| \begin{array}{l} p_t - \Delta p + f(\overline{p})p = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{5.2}$$

Observe that (5.2) is a linearization of (5.1). Also note that $a(x,t) = f(\overline{p}(x,t))$ with $\overline{p} \in B$ a ball of W^1. We have $|a(x,t)| < M$. Thus, for $\overline{p} \in W^1$, for $T > 0$, there exists $u \in L^2(Q, e^{-2s\alpha}\,\phi^{-3})$ such that the weak solution p of (5.2) is satisfies $p(x,T) = 0$ in Ω, that is, we have null controllability for (5.2).

Theorem 5.1. *Suppose $g\colon \mathbb{R} \to \mathbb{R}$, globally Lipschitz and $g(0) = 0$, $p_0 \in L^2(\Omega)$ and $T > 0$. There exists $u \in L^2(Q, e^{-2s\alpha}\,\phi^{-3})$ and $p \in W^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$, weak solution of (5.1) such that $p(x,T) = 0$ in Ω.*

Proof. We apply fixed point method as is usually done. As we will work with multi-valued mapping we need a infinity dimensional version of Shizuo Kakutani fixe point theorem. Among many generalization we employ Glicksberg [9] version, see also Browder [4], which is the following.

"Let B be a non-empty convex, compact subset of a locally convex topological vector space X and Φ a mapping which takes $p \in B$ into a non-empty subset $\Phi(p)$ of X, such that is convex, compact and has closed graphic. Then the set of fixe point of Φ is non-empty and compact.""

In our case we have $X = L^2(Q)$ and

$$B \;=\; \{b \in W^1(0,T; H_0^1(\Omega), H^{-1}(\Omega)); ||b||_{W^1} < M_1\} \subset C^0([0,T]; L^2(\Omega)) \subset L^2(Q),$$

the constant $M_1 > 0$ is obtained in (5.6). Observe that B is a convex set of $L^2(Q)$. Let us prove that B is a compact set of $L^2(Q)$. In fact, let $(b_n)_{n\in\mathbb{N}}$ be a sequence of $b_n \in B$. Then $||b||_{W^1} < M$, that is, $(b_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T; H_0^1(\Omega))$ and $\frac{db_n}{dt}$ is bounded in $L^2(0,T; H^{-1}(\Omega))$. By the theorem of compacticity of Aubin [1], it follows

$$b_n \to b \quad \text{strongly in} \quad L^2(Q).$$

We define the mapping Φ in B as follows: for $\overline{p} \in B$, we set

$$\Phi(\overline{p}) = \Big\{ p \in W^1(0,T; H_0^1(\Omega), H^{-1}(\Omega)), \text{ weak solution of (5.2) for } u \in L^2(Q, e^{-2s\alpha}\phi^{-3}) \text{ with } \int_Q e^{-2s\alpha}\,\phi^{-3}\, u^2\,dxdt \leq C \int_\Omega p_0^2\,dx \text{ such that } p(x,T) = 0 \text{ in } \Omega \Big\}.$$

Remark 5.1. *Observe that for* $\overline{p} \in B$, $a(x,t) = f(\overline{p}(x,t))$ *is bounded by definition of* f, *because* g *is Lipschitz and* $g(0) = 0$. *Thus for* $u \in L^2(Q, e^{-2s\alpha}\,\phi^{-3})$, $p_0 \in L^2(\Omega)$ *there exists* $p \in W^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ *solution of (5.2) such that* $p(x,T) = 0$, *cf. Section 4. Thus, when* $\overline{p} \in B$, $\Phi(\overline{p})$ *is not empty in* $L^2(Q)$ ■

Thus, when $u \in L^2(Q, e^{-2s\alpha}\,\phi^{-3})$, $p_0 \in L^2(\Omega)$, $\overline{p} \in B$, $\Phi(\overline{p})$ is a subset of $L^2(Q)$, that is, Φ is a multi-valued mapping. A fixe point of $\Phi(\overline{p})$ is a vector $\overline{p} \in B$ such that $\overline{p} \in \Phi(\overline{p})$. Thus, this fixed point $\overline{p}$ is solution of (5.2) with $\overline{p}(x,T) = 0$, that is, $\overline{p}$ is solution of (5.1) with $\overline{p}(x,T) = 0$, which implies null controllability for (5.1).

Thus, $\Phi\colon B \to 2^B$ and we prove that it has a fixed point. We must prove, see Glicksberg [9], that $\Phi(\overline{p})$ is non-empty, $\Phi(B) \subset B$ and Φ is closed.

(i) $\Phi(\overline{p})$ is non-empty for $\overline{p} \in B$, already proved.

(ii) $\Phi(B) \subset B$. In fact, for all $\overline{p} \in B$ if $p \in \Phi(\overline{p})$, by definiton $\Phi(\overline{p})$, p is weak solution of (5.2). Multiplying both sides of $(5.2)_1$ by p and integrate in Ω, we get

$$\frac{d}{dt}\,|p(t)|^2_{L^2(\Omega)} + ||p(t)||_{H_0^1(\Omega)} \leq M|p(t)|^2_{L^2(\Omega)} + |\chi_\omega|_{L^2(Q)}\,|u|_{L^2(Q)}\,.$$

Integrating on $[0,t)$, we have

$$|p(t)|^2_{L^2(\Omega)} + \int_0^T ||p(t)||^2_{H_0^1(\Omega)}\,dt \leq |p_0|^2_{L^2(\Omega)} + |u|^2_{L^2(Q)} + \left(M + \frac{1}{4}\right)\int_0^t |p(s)|^2_{L^2(\Omega)}\,ds.$$

By Gronwall inequality, it yields

$$|p(t)|^2_{L^2(\Omega)} + \int_0^T ||p(t)||^2_{H_0^1(\Omega)}\,dt \leq \left(|p_0|^2_{L^2(\Omega)} + C_1\,|p_0|^2_{L^2(\Omega)}\right) e^{(M+\frac{1}{4})T} = C_2\,. \tag{5.3}$$

We also have, for all $v \in H_0^1(\Omega)$ with $||v||_{H_0^1(\Omega)} \leq 1$, that

$$
\begin{gathered}
|\langle p_t, v\rangle| = |\langle +\Delta p - f(\overline{p})p + \chi_\omega u, v\rangle| \leq \\
||p(t)||_{H_0^1(\Omega)}||v||_{H_0^1(\Omega)} + M|p(t)|_{L^2(\Omega)}||v||_{L^2(\Omega)} + |u|_{L^2(Q)}|v(t)|_{L^2(\Omega)} \leq \\
\left(||p(t)||_{H_0^1(\Omega)} + M\, C_2^{1/2}\, C_3 + C_1^{1/2}\, |p_0|_{L^2(\Omega)}\, C_3\right) ||v||_{H_0^1(\Omega)}\,,
\end{gathered} \tag{5.4}
$$

where C_3 is the constant of immersion of $H_0^1(\Omega)$ into $L^2(\Omega)$. Thus, we obtain

$$
||p_t(t)||_{H^{-1}(\Omega)} \leq ||p(t)||_{H_0^1(\Omega)} + C_4\,,
$$

with $C_4 = M\, C_2^{1/2}\, C_3 + C_1^{1/2}\, |p_0|_{L^2(\Omega)}\, C_3$. It follows that

$$
||p_t||^2_{L^2(0,T;H^{-1}(\Omega))} \leq 2||p||^2_{L^2(0,T;H_0^1(\Omega))} + 2C_4^2 T = 2C_2 + 2C_4^2 T. \tag{5.5}
$$

From (5.3) and (5.5), we obtain

$$
\int_0^T ||p(t)||^2_{H_0^1(\Omega)}\, dt + \int_0^T ||p_t(t)||_{H^{-1}(\Omega)} \leq M_1^2\,. \tag{5.6}
$$

Thus,

$$
||p||_{W^1} \leq M_1\,, \text{ with } M_1 = 2C_2 + 2C_4^2\, T^{1/2}.
$$

Therefore, if $\overline{p} \in B$ then $\Phi(\overline{p}) \subset B$. Thus, $\Phi\colon B \to 2^B$.

(iii) $\Phi(\overline{p})$ is closed in $L^2(Q)$. In fact, let $\overline{p}$ be in B fixed and $p_n \in \Phi(\overline{p})$ such that

$$
p_n \to p \quad \text{strongly in} \quad L^2(Q).
$$

By definition of $\Phi(\overline{p})$, we have

$$
\left|
\begin{array}{l}
p_{nt} - \Delta p_n + f(\overline{p})p_n = \chi_\omega u_n \quad \text{in} \quad Q, \\
p_n = 0 \quad \text{on} \quad \Sigma, \\
p_n(0) = p_0 \quad \text{in} \quad \Omega,
\end{array}
\right. \tag{5.7}
$$

with

$$
\int_Q e^{-2s\alpha}\phi^{-3}\, u_n^2(x,t)\, dxdt \leq C \int_\Omega p_0^2(x)\, dx,
$$

which implies $|u_n|^2_{L^2(Q)} \leq C|p_0|^2_{L^2(\Omega)}$. Thus, we extract a subsequence of $(u_n)_{n\in\mathbb{N}}$, which will also be denote by $(u_n)_{n\in\mathbb{N}}$, such that

$$
u_n \rightharpoonup u \quad \text{weakly in} \quad L^2(Q). \tag{5.8}
$$

By the same argument to obtain (5.6) from (5.4) and (5.5), we get

$$||p_n||^2_{L^2(0,T;H^1_0(\Omega))} + ||p_{nt}||^2_{L^2(0,T;H^{-1}(\Omega))} \leq M_1^2 . \tag{5.9}$$

From (5.9) we extract a subsequence $(p_n)_{n\in\mathbb{N}}$, such that

$$\left| \begin{array}{l} p_n \rightharpoonup p \quad \text{weakly} \quad L^2(0,T;H^1_0(\Omega)), \\ p_{nt} \rightharpoonup p_t \quad \text{weakly} \quad L^2(0,T;H^{-1}(\Omega)), \\ p_n \rightharpoonup p \quad \text{strongly} \quad L^2(Q). \end{array} \right. \tag{5.10}$$

The last convergence has been obtained compactness theorem, cf Aubin [1]. From (5.10) we pass to the limits in (5.7), as $n \to \infty$, to obtain

$$\left| \begin{array}{l} p_t - \Delta p + f(\overline{p})p = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma. \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega, \end{array} \right. \tag{5.11}$$

and

$$\int_Q e^{-2s\alpha}\, \phi^{-3}\, u^2(x,t)\, dxdt \leq C \int_\Omega p_0(x)^2\, dx.$$

Thus, $p \in \Phi(\overline{p})$ and $\Phi(\overline{p})$ is closed. Therefore, since B is compact of $L^2(Q)$ and $\Phi(\overline{p}) \subset B$ is closed, it implies that $\Phi(\overline{p})$ is a compact of $L^2(Q)$.

(iv) Φ has the closed graph in $L^2(Q) \times L^2(Q)$.

Remark 5.2. *Let X be a locally convex topological vector space and $B \subset X$ and the mapping*

$$\Phi\colon B \to X$$

which for each $\overline{p} \in B$ corresponds a non void convex set $\Phi(\overline{p})$ of X.

We say that Φ is closed if its graph

$$\bigcup_{\overline{p}\in B} (\overline{p}, \Phi(\overline{p}))$$

is a closed subset of the Cartesian product $X \times X$.

In terms of direct sets it may be stated as follows:

$$x_\varepsilon \to x \quad \textit{in} \quad X, \quad y_\varepsilon \in \Phi(x_\varepsilon) \quad \textit{and} \quad y_\varepsilon \to y. \quad \textit{Then} \quad y \in \Phi(x).$$

This argument generalize the terminology for closed operator A with A a function with domain $D(X)$ dense in X. In fact, we say that $A\colon D(A) \to X$ is a closed operator, when

$$x_n \to x \quad \textit{and} \quad Ax_n \to y,$$

then $x \in D(A)$ and $y = Ax$. This means that the graph of A is closed in $X \times X$ ■

Let us prove (iv). In fact, let $\overline{p}_n$, p_n be such that

$$\overline{p}_n \to \overline{p},\ p_n \to p \quad \text{strongly} \quad L^2(Q), \tag{5.12}$$

and $p_n \in \Phi(\overline{p}_n)$. We must prove that $p \in \Phi(\overline{p})$. In fact, from $p_n \in \Phi(\overline{p}_n)$, it follows that p_n is weak solution of:

$$\left| \begin{array}{l} p_{nt} - \Delta p_n + f(\overline{p}_n)p_n = \chi_\omega u_n \quad \text{in} \quad Q, \\ p_n = 0 \quad \text{on} \quad \Sigma, \\ p_n(x,0) = p_0(x) \quad \text{in} \quad \Omega, \end{array} \right. \tag{5.13}$$

and

$$\int_Q e^{-2s\alpha}\,\phi^{-3}\,u_n^2(x,t)\,dxdt \le C\int_\Omega p_0(x)^2\,dx.$$

See definition of $\Phi(\overline{p})$. By the same argument to obtain (5.9), applied to (5.13), we obtain

$$||p_n||^2_{L^2(0,T;H_0^1(\Omega))} + ||p_{nt}||^2_{L^2(0,T;H^{-1}(\Omega))} \le M_1^2\,. \tag{5.14}$$

From (5.14) and the estimate for u_n in $L^2(Q)$, since $e^{-2s\alpha}\,\phi^{-3} \ge C$, we obtain subsequences $(u_n)_{n\in\mathbb{N}}$ and $(p_{nt})_{n\in\mathbb{N}}$ such that

$$\left| \begin{array}{l} u_n \rightharpoonup u \quad \text{weakly} \quad L^2(\Omega), \\ p_n \rightharpoonup p \quad \text{weakly} \quad L^2(0,T;H_0^1(\Omega)), \\ p_{nt} \rightharpoonup p_t \quad \text{weakly} \quad L^2(0,T;H^{-1}(\Omega)). \end{array} \right. \tag{5.15}$$

From (5.12) we obtain subsequences $(\overline{p}_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ such that

$$\left| \begin{array}{l} \overline{p}_n \to \overline{p} \quad \text{a.e. in} \quad Q, \\ p_n \to p \quad \text{a.e. in} \quad Q. \end{array} \right. \tag{5.16}$$

By continuity of $f\colon \mathbb{R} \to \mathbb{R}$ we obtain $f(\overline{p}_n) \to f(\overline{p})$ a.e. in Q, then by (5.16), we have

$$f(\overline{p}_n)p_n \to f(\overline{p})p \quad \text{a.e. in} \quad Q. \tag{5.17}$$

We also have

$$\int_Q |f(\overline{p}_n)p_n|^2\,dxdt \le M\int_Q |p_n|^2\,dxdt \le C\,M. \tag{5.18}$$

Thus, by Lions [12]- Lemma 3 we obtain from (5.17) and (5.18), that

$$f(\overline{p}_n)p_n \rightharpoonup f(\overline{p})p \quad \text{weakly} \quad L^2(Q). \tag{5.19}$$

Thus, passing to the limits in (5.13) as $n \to \infty$, we obtain:

$$\left| \begin{array}{l} p_t - \Delta p + f(\overline{p})p = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0(x) \quad \text{in} \quad \Omega, \end{array} \right. \tag{5.20}$$

and

$$\int_Q e^{-2s\alpha}\, \phi^{-3}\, u^2(x,t)\, dxdt \leq C \int_\Omega p_0(x)^2\, dx,$$

what proves that $p \in \Phi(\overline{p})$.

Conclusion. The multi-valued mapping $\Phi\colon B \to 2^B$ satisfies the conditions of the infinity dimensional version of Shizuo Kakutani [10], cf. Glicksberg [9], thus it has a fixe point, that is, there exists $\overline{p} \in \Phi(\overline{p})$. It proves null controllability for the nonlinear state equation (5.1). The proof of Theorem 5.1 is complete ■

6 Approximate controllability

We consider the linear parabolic state system

$$\left| \begin{array}{l} p_t - \Delta p + a(t)p = \chi_\omega u \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(x,0) = p_0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{6.1}$$

As we have defined, in the Introduction, Section 1, (ii), we say that (6.1) is approximate controllable in $L^2(\Omega)$, at time $T > 0$, if for each $\varepsilon > 0$, given $p_0 \in L^2(\Omega)$ and $p_T \in L^2(\Omega)$, there exists a control $u \in L^2(Q_\omega)$, $Q_\omega = \omega \times (0,T)$ such that the corresponding solution of $p(x,t)$ of (6.1) satisfies:

$$|p(x,T) - p_T(x)|_{L^2(\Omega)} < \varepsilon. \tag{6.2}$$

This concept of approximate controllability was introduced by J. L. Lions [11] employing a theorem of continuation by Mizohata, see also Cara-Guerreiro [5], Fabre-Puel-Zuazua [8], Zuazua [13].

In this section we prove the same result as an application of Carleman Inequalities.

Theorem 6.1. *Fixe $T > 0$ and given $\varepsilon > 0$ and $p_0, p_T \in L^2(\Omega)$. Then, there exists a control $u \in L^2(Q_\omega)$ such that the solution p of the state equation (6.1) satisfies (6.2).*

Proof. As the system is linear we can suppose $p_0 = 0$. In fact, with $p_0 \in L^2(\Omega)$ we solve the problem:

$$\left| \begin{aligned} &\widehat{p}_t - \Delta \widehat{p} + a\widehat{p} = 0 \quad \text{in} \quad Q, \\ &\widehat{p} = 0 \quad \text{on} \quad \Sigma, \\ &\widehat{p}(0) = p_0 \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{6.3}$$

Thus, if $w = p - \widehat{p}$, w is solution of (6.1) with $w(0) = 0$. Therefore, we consider (6.1) but with $p_0 = 0$. To prove approximate controllability we define the set:

$$R_L(T) = \left\{ p(x,T); p \text{ solution of (6.1), with } u \in L^2(Q_\omega) \right\}.$$

This set is called reacheble set and the index L means for linear problem. To prove approximate controllability it is sufficient to prove that $R_L(T)$ is dense in $L^2(\Omega)$. We will prove reasoning by contradiction.

Suppose $R_L(T)$ is not dense in $L^2(\Omega)$. Thus, there exists a non null vector w_T in the orthogonal complement $R_L(T)^\perp$ in $L^2(\Omega)$. With w_T we consider the adjoint state:

$$\left| \begin{aligned} &w_t + \Delta w - a(t)w = 0 \quad \text{in} \quad Q, \\ &w = 0 \quad \text{on} \quad \Sigma, \\ &w(x,T) = w_T \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{6.4}$$

Multiply (6.4) by p, solution of (6.1) with $u \in L^2(Q_\omega)$ and integrate in Q. We obtain:

$$\int_Q (w_t - \Delta w + a(t)w) p \, dxdt = 0.$$

Then,

$$\int_Q (p_t - \Delta p + a(t)p) w \, dxdt + \int_\Omega w(T)p(T) dx - \int_\Omega w(0)p(0) dx = 0.$$

Observe that p satisfies $(6.1)_1$, $w(T) = w_T$ and $p(0) = p_0 = 0$. Thus, we obtain:

$$-\int_{Q_\omega} u(x,t)w(x,t) \, dxdt + \int_\Omega w_T(x)p(x,T) \, dx = 0.$$

To analyse the second integral above, observe that $w_T(x)$ belongs to the orthogonal $R_L(T)^\perp$ and $p(x,T)$ belongs to $R_L(T)$. Thus, the second integral is zero.

We obtain

$$\int_{Q_\omega} u(x,t)w(x,t)\,dxdt = 0 \quad \text{for all} \quad u \in L^2(Q_\omega),$$

what implies $w(x,t) = 0$ a.e. in Q_ω. By Carleman Inequality, cf. Section 2, with $f = 0$, we obtain

$$\int_Q (s^3\phi^3\, w(x,t)^2)e^{2s\alpha}\,dxdt \leq 0. \tag{6.5}$$

We have $s^3\phi^3 \geq C > 0$, $e^{2s\alpha} > 0$ in $\Omega \times (0,T)$. Then it implies $w = 0$ a.e. in Q, by (6.5). We then have $w(x,T) = w_T(x) = 0$, which is a contradiction. Thus $R_L(T)$ is dense in $L^2(\Omega)$ ■

7 APPENDIX

Observe that $\varphi = e^{\lambda||\psi||}$ is constant for $x \in \overline{\Omega}$ and

$$e^{2s\alpha(x,t)} \leq e^{-\frac{2s\varphi}{\beta(t)}}, \ \ 0 < t < T.$$

Then, we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[\int_0^t\left(\int_\Omega e^{2s\alpha(x,t)}\,|f_1(x,y)|^2\,dx\right)dy\right]dt \leq \\
&\int_{t_1}^{t_2}\left[\int_0^t\left(\int_\Omega e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2|\,dx\right)dy\right]dt \leq \\
&\int_0^T\left[\int_0^t\left(\int_\Omega e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2|\,dx\right)dy\right]dt = \\
&\int_\Omega\left[\int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2|\,dy\right)dt\right]dx.
\end{aligned}$$

Now we will prove the following inequality.

$$\int_\Omega\left[\int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2\,dy\right)dt\right]dx \leq C\int_0^T\int_\Omega |f_1(x,y)|^2\,dxdy.$$

Analysis of the integral

$$I = \int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2\,dy\right)dt.$$

In the integral I, we have

$$0 < y < t \qquad \text{and} \qquad 0 < t < T.$$

Let us consider the change of variables in I, defined by the linear mapping $\sigma(t,y)=(y,t),$ an involution, from $\mathbb{R}^2$ into $\mathbb{R}^2$. It is given by

$$(at+by, ct+dy)=(y,t),$$

with $a=0$, $b=1$, $c=1$, $d=0$. The matrix of σ is $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ with $|\det\sigma|=1$.

Let us consider the domain K of $\mathbb{R}^2$ defined by:

$$K=\{(t,y);\, 0<t<T,\, y<t\}$$

and $\widehat{K}=\sigma(K)$ is defined by:

$$\widehat{K}=\{(y,t);\, 0<y<T,\, y>t\}.$$

We have

$$I=\int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2\,dy\right)dt=\int_0^T\left(\int_y^T e^{-2s\varphi/\beta(y)}\,|f_1(x,t)|^2\,dt\right)dy.$$

We have regularity for $e^{-2s\varphi/\beta(t)}$. Then:

$$\int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,y)|^2\,dy\right)dt\le\int_0^T\left(\int_0^T e^{-2s\varphi/\beta(y)}\,|f_1(x,t)|\,dt\right)dy$$

$$=\left(\int_0^T |f_1(x,t)|^2\,dt\right)\left(\int_0^T e^{-2s\varphi/\beta(y)}\,dy\right)=C\int_0^T |f_1(x,t)|\,dt,$$

with $C>0$ depending of Ω and T. Integrating on Ω, we have

$$\int_\Omega\left(\int_0^T\left(\int_0^t e^{-2s\varphi/\beta(t)}\,|f_1(x,t)|^2\,dy\right)\right)dtdx\le C\int_0^T\int_\Omega |f_1(x,t)|^2\,dxdt \quad\blacksquare$$

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"Porque de feitos tais, por mais que diga,
Mais me há de ficar ainda por dizer"

Camões (Lusíadas-Canto III)

References

[1] Aubin, J. P., *Un theorème de compacité,* C. R. Acad. Sci. Paris, 256 (1963), pp. 5042-5044.

[2] Border, K. C, *Fixe point theorems with applications to economics and game theory*, Cambridge University Press, 2003.

[3] Brezis, H., *Analyse Fonctionelle (Théorie et Applications)*, Dunod, Paris (1999).

[4] Browder, F. E., *Proceeding fo Symposia in Pure Mathematics,* vol. XVIII, Part 2, AMS, pp. 71-79.

[5] Fernádez-Cara, E. & Guerreiro, S., *Global Carleman Inequalities for parabolic systems and applications to controllability,* SIAM Control Optm., V. 45, nº 4 (2006) pp. 1395-1446.

[6] Fursikov, A. V., *Optimal Control of Distributed Systems: Theory and Applications,* Translations of Mathematical Monographs, Vol. 87, AMS, 1999, Providence - RI, USA.

[7] Fursikov, A. V., & Yu Imanuvilov, O., *Controllability of evolution equation. Lectures Notes Series,* Vol. 34, National University, RIM, Seoul, South Korea, 1996.

[8] Fabre, C., Puel, J. P. & Zuazua, E., *Controlabilité approchée de l'équation de la chaleur linear avec controles de norm L^∞ minimale,* C. R. Acad. Sci. Paris (316) (1993), pp. 679-684;

[9] Glicksberg, I. L., *A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points,* Proc. of the AMS, serie 3, (9) (1952), pp. 170-174.

[10] Kakutani, S., *A generalization of Brower fixed point theorem,* Duke Math. Journal, vol. 8 (1941), pp. 457-459.

[11] Lions, J. L., *Remarques sur la controllabilité approché,* Jornadas Hispano Francesas sobre Control de Systemes Distribuidos, Univ. Malaga - Spain (1991), pp. 77-81.

[12] Lions, J. L., *Quelques métodes de résolution des problèmes aux limites non linéaires,* Dunod, Paris, 1969.

[13] Zuazua, E., *Approximate controllability for nonlinear heat equations with globally Lipschitz nonlinearity,* Control and Cybernetic, 28,, 3 (1999), pp. 665-683.

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