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A superlinear type problem for a p-laplacian perturbation

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Dedicated to Antonio Gervásio Colares on the occasion of his 80th birthday.

Abstract

In this work we investigate existence and multiplicity of positive solutions for the superlinear type problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = f(x)u^{q-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $q > p > 1$ and f changes sign.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $2 < q < 2^*$ and $f : \Omega \to \mathbb{R}$ be a bounded sign-changing function. Unless otherwise stated we assume that $\partial\Omega$ is C^2. It is a standard fact that the semilinear equation

$$(I) \qquad \begin{cases} -\Delta u = f(x)u^{q-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

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has a positive solution. Such existence result persists if (I) is linearly perturbed, giving rise to:

$$(II) \qquad \begin{cases} -\Delta u - \lambda u = f(x)u^{q-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The perturbation has to be accorded with λ_1, the first eigenvalue of $-\Delta$, in the following sense:

- if $\lambda < \lambda_1$ then the positive definiteness of the left-hand side is preserved and (II) has a positive solution.
- if $\lambda > \lambda_1$ then the functional $\int_\Omega(|\nabla u|^2 - \lambda|u|^2)$ is no longer coercive, but (II) keeps the positive solution originated from the coercive case provided $\lambda < \lambda^*$, for some $\lambda^* > \lambda_1$. Furthermore, a second positive solution arises if $\int_\Omega f\varphi_1^q < 0$.
- if $\lambda > \lambda^*$ then (II) has no positive solution.

The result described above was established in several works devoted to semilinear equations involving indefinite superlinearities, starting from Ouyang [13] and passing by Alama-Tarantello [1], Del Pino [9], Berestycki-Capuzzo-Dolcetta-Nirenberg [4], Terhani [14] and Chabrowski-Marcos do Ó [6].
One may then consider the quasilinear version of (II), namely,

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u - \lambda u^{p-1} = f(x)u^{q-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $1 < p < q \leq p^*$. This problem was investigated by Ilyasov [12] and Birindelli-Demengel [5] (see also Drabek-Huang [10] for a similar problem with $\Omega = \mathbb{R}^N$). Both works extended partially the above result to (P_λ). Here we complete the result, see Theorem 1.7. A formulation for the optimal value of λ^* was also given by Ilyasov.

Our purpose here is to treat the 'no parameter' version of (P_λ):

$$(P) \qquad \begin{cases} -\Delta_p u + V(x)u^{p-1} = f(x)u^{q-1} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We assume that $1 < p < q < p^*$, $f \in L^\infty(\Omega)$ and $V \in L^r(\Omega)$ with $r > \frac{N}{p}$ if $p < N$ and $r > 1$ if $p \geq N$. We aim at obtaining existence and multiplicity of solutions for (P) according to the sign of

$$\lambda_1(V) := \min\left\{\int_\Omega (|\nabla u|^p + V|u|^p)\,;\ \|u\|_p = 1\right\},$$

the first eigenvalue of $-\Delta_p + V$, and $\int_\Omega f\varphi_V^q$, where φ_V is the first positive eigenfunction L^p-normalized. Notice that when $V \equiv -\lambda$ one has $\varphi_V \equiv \varphi_1$, but, in general, φ_V depends on V.

We describe now our approach and results.

Let the functionals E_V and F be defined on $W_0^{1,p}(\Omega)$ by

$$E_V(u) = \int_\Omega (|\nabla u|^p + V(x)|u|^p)\,, \quad F(u) = \int_\Omega f(x)|u|^q.$$

It is straightforward that F is weakly continuous and that E_V is weakly lower semi-continuous. Moreover, E_V is coercive if and only if $\lambda_1(V) > 0$.

Our approach is prompted by [7], where the indefinite eigenvalue problem

$$(P_{V,m}) \qquad -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega)$$

is studied. When m is sign-changing, a natural way to obtain positive solutions for this equation is to consider the minimization problems

$$\min\{E_V(u);\ \int_\Omega m|u|^p = \pm 1\}.$$

It is shown that these minima are achieved if and only if $\alpha_1(V,m) \geq 0$, where

$$\alpha_1(V,m) := \min\{E_V(u);\ \int m|u|^p = 0 \text{ and } \|u\|_p = 1\}.$$

This condition comes from the relation

$$\alpha_1(V,m) = \max_{t\in\mathbb{R}} \mu_1(t)$$

where $\mu_1(t) = \min\{E_V(u) - t\int m|u|^p;\ \|u\|_p = 1\}$. One can easily see that the zeros of μ_1 provide all the principal eigenvalues of $(P_{V,m})$. Properties of μ_1 such as concavity and decaying to $-\infty$ when $t \to \pm\infty$ yield then the existence of principal eigenvalues, given by

$$\lambda_1(V,m) := \inf\left\{E_V(u);\ \int_\Omega m\,|u|^p\ = 1\right\}$$

and

$$\lambda_{-1}(V,m) := \inf\left\{E_V(u);\ \int_\Omega m\,|u|^p\ = -1\right\}.$$

The lack of homogeneity in (P) prevents us from setting $\mu(t) := \min\{E_V(u) - t\int f|u|^q;\ \|u\|_p = 1\}$. However, several conclusions similar to those holding for $(P_{V,m})$ can be deduced by dealing with

$$\alpha(V) = \alpha(V,f) := \min\left\{E_V(u);\ \int f|u|^q = 0 \text{ and } \|u\|_p = 1\right\}.$$

Indeed, one is naturally led to consider

$$c_1 := \inf\left\{E_V(u);\ \int f|u|^q = -1\right\} \text{ and } c_2 := \inf\left\{E_V(u);\ \int f|u|^q = 1\right\}.$$

We can prove that these infima are achieved and that at least one of them provides a solution when $\int f\varphi_V^q < 0$ and $\alpha(V,f) > 0$. More generally:

Theorem 1.1. *Problem (P) has a solution if either $\lambda_1(V) > 0$ or $\lambda_1(V) \leq 0$ and $F(\varphi_V) < 0 < \alpha(V,f)$. Moreover, if $\lambda_1(V) < 0$ and $F(\varphi_V) < 0 < \alpha(V,f)$ then (P) has at least two solutions. If $F(\varphi_V) \geq 0 \geq \lambda_1(V)$ then (P) has no solution.*

This theorem summarizes the following results:

Theorem 1.2. *Assume that $\alpha(V) > 0$.*

1. *If $\lambda_1(V) > 0$ then (P) has at least one solution.*
2. *If $\lambda_1(V) < 0$ and $F(\varphi_V) < 0$ then (P) has at least two solutions.*

Theorem 1.2 is the analogue version of the classical statement that holds for the semilinear case. The assumption $\alpha(V) > 0$, similar to $\lambda < \lambda^*$, guarantees that c_1 and c_2 are well-defined.

When dropping the assumption $\alpha(V) > 0$, the second statement in Theorem 1.2 can be proved provided $\lambda_1(V)$ is close to zero. To this end, we need to assume a L^∞ *a priori* bound on V and that $F(\varphi_V)$ is negative and away from zero.

Corollary 1.3. *Let $\delta, R > 0$ be fixed. Then there exists $\varepsilon = \varepsilon(\delta, R) > 0$ such that (P) has at least two solutions if $||V||_r \leq R$, $-\varepsilon < \lambda_1(V) < 0$, and $F(\varphi_V) \leq -\delta$.*

The 'borderline case' $\alpha(V) = 0$ stands for the case $\lambda = \lambda^*$ in (II):

Proposition 1.4. *If $f \neq 0$ a.e., $\alpha(V) = 0$, and $\int f\varphi_V^q < 0$ then (P) has at least one solution.*

The non-existence for (P) holds as follows:

Theorem 1.5. *If* $\int f\varphi_V^q = 0 = \lambda_1(V)$ *then* (P) *has no solution.*

Theorem 1.6. *Given* $\delta, R > 0$ *there exists* $\varepsilon = \varepsilon(\delta, R) > 0$ *such that* (P) *has no solution if* $\|V\|_\infty \leq R$, $\lambda_1(V) \leq -\delta$ *and* $\int f|\varphi_V|^q \geq \varepsilon$.

Finally, going back to the parameter version of (P), it is possible to get a better description of the solution set:

Theorem 1.7. *Assume that* $p < q < p^*$, $\int f\varphi_1^q < 0$ *and either that* Ω *has a* C^2 *boundary or that* $p \geq 2$. *Then there exists* $\lambda^* > \lambda_1$ *such that* (P_λ) *has at least two solutions if* $\lambda_1 < \lambda < \lambda^*$. *If* $\lambda > \lambda^*$ *then* (P_λ) *has no solution.*

The paper is organized as follows: in section 2, we prove the existence statements for (P). Section 3 is devoted to the non-existence results. In Section 4, we focus on the parameter version of (P).

2 Existence of solutions

We collect now some results in order to prove Theorem 1.2.

Some Lemmata

Lemma 2.1. $\alpha(V)$ *is achieved.*

Proof. We make use of the following inequality (see [7]):

$$\|u\|^p \leq C_1 E_V(u) + C_2\|u\|_p^p \tag{2.1}$$

for every $u \in W_0^{1,p}(\Omega)$ and some $C_1, C_2 > 0$ depending only on V. Hence if u_n is a minimizing sequence for $\alpha(V, f)$, it is bounded and, up to a subsequence, it converges weakly to some u_0. By a standard compacity argument, u_0 realizes $\alpha(V, f)$. □

From its definition it is clear that $\alpha(V) \geq \lambda_1(V)$ and that $\alpha(V) > \lambda_1(V)$ if $\int f|\varphi_V|^q < 0$.

The next lemma gives a necessary relation between the first eigenpair $(\lambda_1(V), \varphi_V)$ and the solutions of (P).

Lemma 2.2. *If* u *solves* (P) *then* $\int f\varphi_V^q \leq \lambda_1(V)\int \varphi_V^q u^{p-q}$.

Proof. We proceed as in [5, proof of Theorem 1.2]. Let us assume that u is a solution of $(P_{V,f})$. Hence, by the strong maximum principle in [17], we have $u > 0$ on Ω. Moreover, by the Hopf lemma, $\frac{\partial u}{\partial \nu} < 0$ and $\frac{\partial \varphi_V}{\partial \nu} < 0$ on $\partial\Omega$. It follows that $\frac{\varphi_V^q}{u^{q-1}}$ and $\frac{\varphi_V^{q-p+1}}{u^{q-p}}$ can be tested in (P) and the eigenvalue problem, respectively. Thus we get:

$$\int f|\varphi_V|^q = q\int (\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u \nabla \varphi_V \tag{2.2}$$

$$-(q-1)\int (\frac{\varphi_V}{u})^{q}|\nabla u|^{p} + \int V u^{p-q}\varphi_V^q$$

and

$$\lambda_1(V)\int u^{p-q}\varphi_V^q = (q-p+1)\int (\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^p \tag{2.3}$$

$$-(q-p)\int (\frac{\varphi_V}{u})^{q-p+1}|\nabla \varphi_V|^{p-2}\nabla \varphi_V \nabla u + \int V u^{p-q}\varphi_V^q$$

On the other hand, from Picone's identity, one has

$$(\frac{\varphi_V}{u})^{q}|\nabla u|^{p} - p(\frac{\varphi_V}{u})^{q-p+1}|\nabla \varphi_V|^{p-2}\nabla \varphi_V \nabla u \tag{2.4}$$

$$+(p-1)(\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^{p} \geq 0$$

and also

$$(\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^{p} - p(\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u \nabla \varphi_V \tag{2.5}$$

$$+(p-1)(\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u \nabla \varphi_V \geq 0.$$

Adding up (4) multiplied by $(\frac{q}{p}-1)$ and (5) multiplied by $\frac{q}{p}$ and integrating over Ω we get the same result as substracting (2) from (3), which yields the conclusion. □

Proof of Theorem 1.2

Consider the following infima:

$$c_1 = \inf\left\{E_V(u);\ \int f|u|^q = -1\right\} \text{ and } c_2 = \inf\left\{E_V(u);\ \int f|u|^q = 1\right\}.$$

We will show that these infima are achieved. Moreover, we will prove that $c_1 < 0$ and $c_2 > 0$, so that the associated minimizers correspond to positive solutions of (P).

Step 1: $c_1 < 0$.

Let w be defined by

$$w = \frac{\varphi_V}{\left(-\int f\varphi_V^q\right)^{\frac{1}{q}}};$$

then $\int fw^q = -1$, so that

$$c_1 \le E_V(w) = \frac{E_V(\varphi_V)}{\left(-\int f\varphi_V^q\right)^{\frac{p}{q}}} = \frac{\lambda_1(V)}{\left(-\int f\varphi_V^q\right)^{\frac{p}{q}}} < 0.$$

Step 2: c_1 is well defined, i.e., $c_1 > -\infty$.

Assume by contradiction that there is a sequence of nonnegative function u_n such that $\int fu_n^q = -1$ and

$$E_V(u_n) = \int |\nabla u_n|^p + \int V^+ u_n^p - \int V^- u_n^p \to -\infty.$$

It follows that $\int V^- u_n^p \to \infty$. Define w_n by

$$w_n = \frac{u_n}{\left(\int V^- u_n^p\right)^{\frac{1}{p}}},$$

so that $\int fw_n^q \to 0$ and $\int V^- w_n^p = 1$. Moreover, for n large,

$$\int |\nabla w_n|^p + \int V^+ w_n^p - 1 \le 0,$$

thus $||w_n||$ is bounded. We can assume that $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, and so $\int fw^q = 0$, $\int V^- w^p = 1$ and $E_V(w) \le 0$. Define $w_0 = \frac{w}{\|w\|_p}$, then

$$E_V(w_0) \le 0, \quad \int fw_0^q = 0, \quad \text{and} \quad \|w_0\|_p = 1.$$

This is in contradiction with $\alpha(V) > 0$.

Step 3: c_1 is achieved.

Let u_n be a nonnegative minimizing sequence, i.e. $\int fu_n^q = -1$ and $E_V(u_n) \to c_1$. We claim that $||u_n||$ is bounded. If not, from

$$E_V(u_n) = \int |\nabla u_n|^p + \int V^+ u_n^p - \int V^- u_n^p \to c_1,$$

we have that $\int V^- u_n^p \to \infty$. Again, define

$$w_n = \frac{u_n}{\left(\int V^- u_n^p\right)^{\frac{1}{p}}},$$

so that $||w_n||$ is bounded. We can assume that $w_n \rightharpoonup w$ in W_0^1, and so $\int f w^q = 0$, $\int V^- w^2 = 1$ and $E_V(w) = 0$. As in Step 2, we have a contradiction with the assumption $\alpha(V) > 0$.

Now, we can assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, so that $\int f|u|^q = -1$ and

$$c_1 \leq E_V(u) \leq \liminf E_V(u_n) = c_1.$$

Thus u is a minimizer for c_1.

Step 4 c_2 is well defined, is achieved and is positive.

The first two assertions can be proved as for c_1. In order to prove that $c_2 > 0$, let u be a minimizer for c_2. Then, up to a multiplicative constant, u is a solution of

$$-\Delta_p u + V(x) u^{p-1} = c_2 f(x) u^q, \quad u \in W_0^{1,p}(\Omega).$$

From Lemma 2, there holds

$$c_2 \int f \varphi_V^q \leq \lambda_1(V) \int \varphi_V^q u^{p-q} < 0.$$

Thus $c_2 > 0$. This finishes the proof of Theorem 1.2.

□

Next we prove Corollary 1, as a consequence of the next lemma, which shows that the condition $\alpha(V) > 0$ can be obtained if we control the sign of $\lambda_1(V)$ and $\int f \varphi_V^q$ in an appropriate way:

Lemma 2.3. *Given $R, \delta > 0$ there is $\varepsilon_0 > 0$ such that for all V with $||V||_r \leq R$, $\int f \varphi_V^q < -\delta$ and $\lambda_1(V) > -\varepsilon_0$, we have $\alpha(V) > 0$.*

Proof. Let ε_0 be given by the previous lemma. We have that $\lambda_1(V - \lambda_1(V)) = 0$ and it is achieved by φ_V, so that $\alpha(V - \lambda_1(V)) \geq \varepsilon_0$. Then, for $u \in W_0^{1,p}(\Omega)$, we have

$$E_{V+\lambda_1(V)}(u) \geq \epsilon \ \text{ if } \ \|u\|_p = 1 \ \text{ and } \ \int f|u|^q = 0,$$

and so

$$E_V(u) \geq \lambda_1(V) + \epsilon \ \text{ if } \ |u|_p = 1 \ \text{ and } \ \int f|u|^q = 0.$$

Thus $\alpha(V) \geq \lambda_1(V) + \epsilon > 0$. □

Now we turn to the proof of Proposition 1.4:

Proof of Proposition 1.4. We set $F(u) = \int_\Omega f|u|^q$ and $G(u) = \|u\|_p^p$ for $u \in W_0^{1,p}(\Omega)$. Let us show that $F'(u_0)$ and $G'(u_0)$ are linearly independent if u_0 realizes $\tilde{\alpha}(V)$. Assume that $a\langle F'(u_0), v\rangle + b\langle G'(u_0), v\rangle = 0$ for every $v \in W_0^{1,p}(\Omega)$. Taking $v = u_0$ we get $b = 0$, so that $a\langle F'(u_0), v\rangle = 0$ for every $v \in W_0^{1,p}(\Omega)$. If $a \neq 0$ then $f|u_0|^{q-1} \equiv 0$, a contradiction. Therefore one can apply Lagrange multipliers rule to infer that u_0 is a solution of $-\Delta_p u + Vu^{p-1} = t_0 f u_0^{q-1} + s_0 u_0^{p-1}$. Multiplying this equation by u_0 we get $s_0 = \alpha(V) = 0$. Furthermore, from Lemma 2, we have

$$t_0 \int f\varphi_V^q \leq \lambda_1(V) \int \varphi_V^q u^{p-q} < 0,$$

so that $t_0 > 0$. Therefore, after rescaling, u_0 provides a solution for $P_{V,f}$.

□

3 Nonexistence of solutions

3.1 Proof of Theorem 1.5

Repeating the proof of Lemma 2 we obtain

$$0 = q\int (\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u\nabla\varphi_V \; - \tag{3.1}$$
$$(q-1)\int (\frac{\varphi_V}{u})^{q}|\nabla u|^{p} + \int V u^{p-q}\varphi_V^q$$

and

$$0 = (q-p+1)\int (\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^{p} \; - \tag{3.2}$$
$$(q-p)\int (\frac{\varphi_V}{u})^{q-p+1}|\nabla \varphi_V|^{p-2}\nabla\varphi_V\nabla u + \int V u^{p-q}\varphi_V^q,$$

so that

$$(\frac{\varphi_V}{u})^{q}|\nabla u|^{p} - p(\frac{\varphi_V}{u})^{q-p+1}|\nabla \varphi_V|^{p-2}\nabla\varphi_V\nabla u \; + \tag{3.3}$$
$$(p-1)(\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^{p} \equiv 0$$

and

$$(\frac{\varphi_V}{u})^{q-p}|\nabla \varphi_V|^{p} - p(\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u\nabla\varphi_V \; + \tag{3.4}$$

$$(p-1)(\frac{\varphi_V}{u})^{q-1}|\nabla u|^{p-2}\nabla u\nabla\varphi_V \equiv 0.$$

Therefore Picone's identity states that u and φ_V are proportional, so that, up to a multiplicative constant, one has $f \equiv \frac{\lambda_1(V)}{\varphi_V^{q-p}}$, which is impossible.

□

3.2 Proof of Theorem 1.6

Assume the existence of $R_0 > 0$ and a sequence $V_n \in B$ such that $\lambda_1(V_n) \leq -R_0$, $\int f|\varphi_n|^q \geq -\frac{1}{n}$ (where $\varphi_n = \varphi_{V_n}$) and that, for $V = V_n$, (P) has a solution u_n. We may assume that $V_n \rightharpoonup V_0$ in $L^\infty(\Omega)$ so that, by weak continuity, $\lambda_1(V_0) \leq -R_0$ and $\int f|\varphi_0|^q \geq 0$. Let us now show that u_n converges to a solution of (P), for $V = V_0$. Lemma 2 will then provide a contradiction. From [3, Lemma 3.2] we have that the sequence u_n is bounded in $L^\infty(\Omega)$ and therefore, by (2.1), it is bounded in $W_0^{1,p}(\Omega)$. Hence $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, strongly in $L^{qs'}(\Omega)$. Multiplying (P), with $V = V_n$, by $u_n - u_0$ one gets that $\langle E'_{V_n}(u_n), u_n - u_0\rangle \to 0$ so that, by the (S^+) property of the p-laplacian, $u_n \to u_0$ in $W_0^{1,p}(\Omega)$. Therefore u_0 is a solution of (P) for $V = V_0$. □

4 The parameter dependent problem

Let $\lambda > 0$ a parameter and consider the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda u^{p-1} + f(x)u^{q-1} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where we assume that $1 < p < q < p^*$. Here f is a bounded function which changes sign in Ω. More precisely, we will assume that $\Omega^+ = \{x \in \Omega : f(x) > 0\}$ and $\Omega^- = \{x \in \Omega : f(x) < 0\}$ are open and nonempty sets.

The solutions of (P_λ) are the critical points of the following C^1-functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega (u^+)^p - \frac{1}{q}\int_\Omega f(x)(u^+)^q, \quad u \in W_0^{1,p}(\Omega).$$

Define

$$\Lambda = \{\lambda : (P_\lambda) \text{ has a nontrivial solution}\} \quad \text{and} \quad \lambda^* = \sup\Lambda.$$

It follows from Theorem 1.2 that $\Lambda \neq \emptyset$. In fact $(-\infty, \lambda_1) \subset \Lambda$. Here λ_1 denotes the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, it is a consequence of Corollary 1.3 that if $\int f\varphi_1^q < 0$, where $\varphi_1 > 0$ is the L^p-normalized eigenfunction associated to λ_1, then $\lambda^* > \lambda_1$. Also, we know that $\lambda^* \leq \lambda_1(\Omega \setminus \overline{\Omega^-})$, see the begin of the proof of Lemma 4.2 in Appendix.

4.1 Proof of Theorem 1.7

The proof will be divided in several steps.

Step 1. There is a nontrivial solution for all $\lambda < \lambda^*$.

Pick $\lambda < \lambda^*$ and $\overline{\lambda} > \lambda$ with $\overline{\lambda} \in \Lambda$. Let $\overline{u}$ be a nontrivial solution of $(P_{\overline{\lambda}})$. Then

$$-\Delta_p \overline{u} = \overline{\lambda}\,\overline{u}^{p-1} + f(x)\overline{u}^{q-1} \geq \lambda \overline{u}^{p-1} + f(x)\overline{u}^{q-1},$$

so that $\overline{u}$ is a supersolution of (P_λ). Now consider

$$M = \{u \in W_0^{1,p}(\Omega); 0 \leq u \leq \overline{u}\}.$$

Let $u_1 \in M$ such that $I(u_1) = \inf_M I$, then u_1 is a solution of (P_λ), see for instance [16, Theorem I.2.4]. In order to prove that u_1 is nontrivial, let $0 \leq \phi \in C_0^1(\Omega)$ with support compact and such that

$$\frac{\int_\Omega |\nabla \phi|^p}{\int_\Omega \phi^p} < \lambda. \tag{4.1}$$

The existence of ϕ holds since $\lambda_1 < \lambda$. There is $s_0 > 0$ such that $s\phi \leq \overline{u}$ for $0 < s < s_0$. Moreover, by (4.1) we have

$$I(s\phi) \leq \frac{s^p}{p}\left(\int_\Omega |\nabla \phi|^p - \lambda \int_\Omega \phi^p\right) - \frac{s^q}{q}\int_\Omega f(x)\phi^q < 0$$

if $s > 0$ is small enough (recall that $p < q$). Then $I(u_1) < 0$, and so u_1 is a nontrivial solution.

Step 2. Let u_1 be the solution of (P_λ) constructed in Step 1. We claim that u_1 can be assumed to be a local minimizer of I in $W_0^{1,p}(\Omega)$.

Case 1. Assume $\partial\Omega$ is C^2. By classical regularity results, $u \in C_0^1(\overline{\Omega})$. Then, by the strong maximum principle, we have

$$u_1 > 0 \text{ in } \Omega \quad \text{ and } \quad \partial_\nu u_1 < 0 \text{ on } \partial\Omega,$$

where ∂_ν denotes the normal exterior derivative. Moreover, we have that $u_1 \leq \overline{u}$, where $\overline{u}$ is a nontrivial solution of $(P_{\overline{\lambda}})$ with $\lambda < \overline{\lambda} < \lambda^*$. Applying the comparison principle, see [8, Proposition 3.4], we can conclude that

$$\overline{u} > u_1 > 0 \text{ in } \Omega \quad \text{ and } \quad \partial_\nu \overline{u} < \partial_\nu u_1 < 0 \text{ on } \partial\Omega.$$

That means that M contains a $C_0^1(\Omega)$ neighborhood of u_1 and so u_1 is a local minimum in C_0^1. By classical results, we can conclude that u_1 is a local minimum in $W_0^{1,p}(\Omega)$.

Case 2. Assume that $p \geq 2$. Set $\overline{u}$ and $\overline{\lambda}$ as before. We have that $u_1 \leq \overline{u}$ a.e. in Ω. Set $\lambda' = (\lambda + \overline{\lambda})/2$ and let $\delta > 1$ such that

$$\left(\delta^{q-p} - 1\right) b(x)\overline{u}^{q-p} < \overline{\lambda} - \lambda'.$$

Hence, multiplying the above inequality by $\delta^{p-1}\overline{u}^{p-1}$, we obtain

$$-\Delta_p(\delta\overline{u}) = \delta^{p-1}\overline{\lambda}\overline{u}^{p-1} + \delta^{p-1}f(x)\overline{u}^{q-1} \geq \delta^{p-1}\lambda'\overline{u}^{p-1} + \delta^{q-1}f(x)\overline{u}^{q-1}.$$

Thus $\delta\overline{u}$ is a supersolution for (P_λ) and moreover $u_1 < \delta\overline{u}$ a.e. in Ω. Without loss of generality, we can assume that

$$I(u_1) = \min\{F(u) : 0 \leq u(x) \leq \delta\overline{u}(x) \;\text{ a.e. }\; x \in \Omega\}.$$

It follows by Lemma 4.1 (see Appendix) that u_1 can be assumed to be a local minimizer of I in $W_0^{1,p}$.

Step 3. We will apply the mountain pass theorem to obtain the second nontrivial solution to (P_λ).

Consider now the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla(u_1 + u)|^p - \int_\Omega l(x,u) - \int_\Omega G(x,u),$$

where

$$l(x,s) = \frac{\lambda}{p}((u_1 + s)^+)^p - \frac{\lambda}{p}u_1^p.$$

and

$$G(x,s) = \frac{1}{q} f(x)((u_1+s)^+)^q - \frac{1}{q} f(x) u_1^q.$$

It is clear that if u is a nontrivial critical point of J, then $u_1 + u \geq 0$, so that $u_1 + u$ is a solution of (P_λ) different from u_1.

Since u_1 is a local minimizer of I and $J(u) = I(u_1+u) - I(u_1) + ||u_1||^2$, it follows that there is $r > 0$ such that

$$J(u) \geq J(0) \quad \text{for all } u \in W_0^{1,p}(\Omega) \text{ with } ||u|| \leq r.$$

Now, fix $0 \leq v \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega f(x) v^q > 0.$$

Then

$$\begin{aligned} J(sv) &= \frac{s^p}{p} \int_\Omega |\nabla (v + \frac{u_1}{s})|^p - \frac{s^p \lambda}{p} \int_\Omega m(x)(v + \frac{u_1}{s})^p \\ &+ \frac{\lambda}{p} \int_\Omega m(x) u_1^p - \frac{s^q}{q} \int_\Omega f(x)(v + \frac{u_1}{s})^q + \frac{1}{q} \int_\Omega f(x) u_1^q \\ &\to -\infty \quad \text{as} \quad s \to \infty. \end{aligned}$$

Finally, if u_n is a $(PS)_c$ sequence for J, then it follows that $u_1 + u_n$ is a $(PS)_{c+I(u_1)-||u_1||^2}$ for I. Thus J satisfies the (PS) condition since I satisfies the (PS) condition if $\lambda < \lambda^*$, see Lemma 4.2 in Appendix.

□

4.2 Appendix

Here we will denote by $\overline{w} := \delta \overline{u}$.

Lemma 4.1. *Assume that $p \geq 2$. Suppose that u_1 is the unique minimizer of I restricted to $M = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq \overline{w}(x) \text{ a.e. } x \in \Omega\}$. Then u_1 is a local minimizer of I_λ in $W_0^{1,p}(\Omega)$.*

Proof. Consider the set

$$M_n = \left\{ u \in W_0^{1,p}(\Omega) : dist(u, M) \leq \frac{1}{n} \right\}.$$

It is easy to verify that M_n is weakly closed, and I is coercive and weakly lower semi-continuous on M_n with respect to the norm of $W_0^{1,p}$. Then, by [16, Theorem I.1.2], there is $u_n \in M_n$ such that

$$I(u_n) = \min_{M_n} I.$$

It follows that

$$I'(u_n) \cdot (u_n - \overline{w})^+ \leq 0,$$

i.e.,

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla (u_n - \overline{w})^+ dx &\leq \lambda \int_\Omega u_n^{p-1}(u_n - \overline{w})^+ dx \\ &+ \int_\Omega f(x) u_n^{q-1}(u_n - \overline{w})^+ dx.\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\int_\Omega |\nabla \overline{w}|^{p-2}\nabla \overline{w} \nabla (u_n - \overline{w})^+ dx &\geq \lambda' \int_\Omega \overline{w}^{p-1}(u_n - \overline{w})^+ dx \\ &+ \int_\Omega f(x) \overline{w}^{q-1}(u_n - \overline{w})^+ dx.\end{aligned}$$

Now, fix $\epsilon > 0$ such that

$$\lambda' \geq (1+\epsilon)\lambda + \epsilon f(x)\overline{w}^{q-p}.$$

Multiplying this inequality by $\overline{w}^{p-1}$ we get

$$\lambda' \overline{w}^{p-1} \geq (1+\epsilon)\lambda \overline{w}^{p-1} + \epsilon f(x)(\overline{w})^{q-1}.$$

It follows that

$$\begin{aligned}\int_\Omega |\nabla \overline{w}|^{p-2}\nabla \overline{w} \nabla (u_n - \overline{w})^+ dx &\geq \lambda \int_\Omega (1+\epsilon)\overline{w}^{p-1}(u_n - \overline{w})^+ dx \\ &+ \int_\Omega f(x)(1+\epsilon) \overline{w}^{q-1}(u_n - \overline{w})^+ dx.\end{aligned}$$

Remark that for $p \geq 2$ the p-Laplacian has the strongly monotone property, so we can conclude that

$$\int_\Omega |\nabla (u_n - \overline{w})^+|^p dx \leq \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla \overline{w}|^{p-2}\nabla \overline{w} \right) \nabla (u_n - \overline{w})^+ dx.$$

Using the above inequalities, we get that

$$\begin{aligned}\int_\Omega |\nabla (u_n - \overline{w})^+|^p dx &\leq \lambda \int_\Omega (u_n^{p-1} - (1+\epsilon)\overline{w}^{p-1})(u_n - \overline{w})^+ dx \\ &+ \int_\Omega f(x)(u_n^{q-1} - (1+\epsilon)\overline{w}^{q-1})(u_n - \overline{w})^+ dx.\end{aligned}$$

There is a constant $C = C(\epsilon) > 0$ such that, for $a > b \geq 0$,

$$a^{p-1} - b^{p-1} \leq \epsilon b^{p-1} + C(a-b)^{p-1}$$

and

$$a^{q-1} - b^{q-1} \leq \epsilon b^{q-1} + C(a-b)^{q-1}.$$

Therefore

$$\begin{aligned} ||(u_n - \overline{w})^+||^p &\leq \lambda C \int_\Omega [(u_n - \overline{w})^+]^p dx \\ &+ C \int_\Omega f(x)[(u_n - \overline{w})^+]^q dx \\ &\leq C|\{x : u_n(x) > \overline{w}(x)\}|^{\frac{p}{N}} ||(u_n - \overline{w})^+||^p \\ &+ C||(u_n - \overline{w})^+||^{q-p}||(u_n - \overline{w})^+||^p. \end{aligned}$$

Note that u_n converges to a minimizer of I in M, so from the uniqueness of u_0 it follows that $||u_n - u_0|| \to 0$ as $n \to \infty$. Moreover, since $u_0 < \overline{w}$, it follows that $|\{x : u_n(x) > \overline{w}(x)\}| \to 0$ as $n \to \infty$. Thus

$$||(u_n - \overline{w})^+||^p \leq o(1)||(u_n - \overline{w})^+||^p,$$

so that there is n_0 such that $(u_n - \overline{w})^+ = 0$ for $n \geq n_0$, and so $u_n \leq \overline{w}$. As a consequence, we have that $u_n^+ \in M$ and so $I(u_n^+) \geq I(u_1)$. Now, if $n \geq n_0$, then

$$I(u_1) \geq I(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n^-|^p dx + I(u_n^+) \geq \frac{1}{p}\int_\Omega |\nabla u_n^-|^p dx + I(u_1).$$

We infer that $u_n^- = 0$, and so $u_n \in M$ for $n \geq n_0$. Therefore u_1 is a local minimizer of I in $W_0^{1,p}$.

□

Lemma 4.2. *The functional I_λ satisfies the (PS) condition if $\lambda < \lambda^*$.*

Proof. First, note that $\lambda^* \leq \lambda_1(\Omega^*)$, where $\Omega^* = \Omega \setminus \overline{\Omega^-}$. Actually, let u be a solution of (P_λ) and ψ be the first eigenfunction associated to $\lambda_1(\Omega^*)$. By Picone's identity (c.f. [2]),

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(\phi^p/u^{p-1})dx \leq \int_\Omega |\nabla \phi|^p dx,$$

It follows that, using the equation (P_λ),

$$\lambda \int_\Omega \phi^p + \int_\Omega f^+(x)u^{p-q}\phi^p \leq \lambda_1(\Omega^*)\int_\Omega \phi^p,$$

and consequently $\lambda \leq \lambda_1(\Omega^*)$. Thus $\lambda_1(\Omega^*)$ is an upper bound for Λ, and so $\lambda^* \leq \lambda_1(\Omega^*)$.

Now, consider a sequence $u_n \in W_0^{1,p}$ such that

$$I(u_n) \leq c \quad \text{and} \quad I'(u_n) \to 0.$$

We need to show that u_n has a convergent subsequence. But since $q < p^* - 1$, it is enough to show that u_n is bounded in $W_0^{1,p}(\Omega)$. Note that

$$I(u_n) - \frac{1}{q} I'(u_n)u_n = \left(\frac{1}{p} - \frac{1}{q}\right)(||u_n||^p - \lambda ||u_n^+||_p^p) \leq c + c||u_n||, \tag{4.2}$$

so that the lemma follows from the next claim:

Claim:. The sequence u_n is bounded in $L^p(\Omega)$.

Suppose by contradiction that $||u_n||_p \to \infty$ as $n \to \infty$. Set $v_n = u_n/||u_n||_p$. From (4.2), we have that v_n is bounded. So we can assume that $v_n \rightharpoonup v_0$ in $W_0^{1,p}$ with $||v_0||_p = 1$. For any $w \in W_0^{1,p}$ we know that $I'(u_n)w \to 0$ as $n \to \infty$. In particular

$$||u_n||_2^{q-p} \int_\Omega f(x)(v_n^+)^{q-1} w dx = \int_\Omega [|\nabla v_n|^{p-2} \nabla v_n \nabla w - \lambda (v_n^+)^{p-1} w] dx \tag{4.3}$$
$$+ o(1).$$

Consequently

$$\int_\Omega f(x)(v_0^+)^{q-1} w dx = 0 \quad \text{for all} \quad w \in W_0^{1,p}(\Omega). \tag{4.4}$$

If $\Omega_0 := \Omega \setminus (\overline{\Omega^+ \cup \Omega^-})$ is empty, then (4.4) implies that $v_0 = 0$, but it is a contractions since $||v_0|| = 1$.

On the other hand, if $\Omega_0 \neq \emptyset$, then (4.4) implies that $v_0 \in W_0^{1,p}(\Omega_0)$. Now, passing a subsequence, we can assume that

$$\int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla w \to \int_\Omega |\nabla v_0|^{p-2} \nabla v_0 \nabla w,$$

see for instance [11, 18]. From (4.3) we have that

$$\int_\Omega [|\nabla v_0|^{p-2} \nabla v_0 \nabla w - \lambda (v_0^+)^{p-1} w] dx = 0 \quad \text{for all} \quad w \in W_0^{1,p}(\Omega_0).$$

It follows that $-\Delta_p v_0 = \lambda (v_0^+)^{p-1}$, and so $v_0 \geq 0$ and $\lambda = \lambda_1(\Omega_0)$ since $||v_0||_p = 1$. It is a contradiction with the assumption $\lambda < \lambda^*$, since $\lambda^* \leq \lambda_1(\Omega^*) < \lambda_1(\Omega_0)$.

□

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