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Immersion of almost Ricci solitons into a Riemannian manifold

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Dedicated to Professor Gervásio Colares on the occasion of his 80th birthday.

Abstract

The principal aim of this short paper is to study immersions of an almost Ricci soliton or a Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$. First we shall present some obstruction results in order to obtain a minimal immersion under conditions on the sectional curvature of $\widetilde{M}^{n+p}$. When $\widetilde{M}^{n+p}$ is a space form $\widetilde{M}_c^{n+p}$ of sectional curvature c, the pinching $\lambda \geq (n-1)(c+H^2)$ gives that such an immersion is umbilical. Finally, concerning to Ricci solitons we shall show that a shrinking Ricci soliton immersed into a space form with constant mean curvature must be the Gaussian soliton or its traceless tensor associated to the second fundamental form has supremum strictly positive.

1 Introduction and statement of the main results

Ricci solitons play a remarkable role in the study of the Ricci flow. Among their properties we detach that they are stationary points of the Ricci flow in

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the space of metrics on M^n modulo diffeomorphisms and scalings of M^n. It usually serves as a dilation limit of solutions to the Ricci flow. Therefore, it is very important to classify Ricci solitons or to understand their geometry. In addition, when M^n is compact, Perelman [18] reduced the study of such manifold to gradient case of a smooth function f on M^n called Perelman's potential. On the other hand, in [19] Pigola et al. modified the definition of a gradient Ricci soliton by adding the condition on the parameter λ to be a variable function. In [4], the following general definition of an almost Ricci soliton was considered.

Definition 1.1. *An almost Ricci soliton is a Riemannian manifold M^n endowed with a metric* g*, a vector field X and a soliton function $\lambda : M^n \to \mathbb{R}$ satisfying*

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_X g$ stands for the Lie derivative of the metric g in the direction of X.

When X is a gradient vector field of a smooth function $f : M^n \to \mathbb{R}$ this definition agrees with that one given in [19]. In this case the previous equation turns out

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f.

Following the terminology of Ricci solitons, an almost Ricci soliton will be *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. If λ has no constant sign it will be called *indefinite*.

We point out that if λ is constant, equation (1.1) reduces to that associated to a Ricci soliton. Under this point of view an almost Ricci soliton generalizes a Ricci soliton. Moreover, when either the vector field X is trivial, or the potential f is constant, an almost Ricci soliton will be called *trivial*, while for a nontrivial almost Ricci soliton its associated potential vector field X or its function are not trivial. We notice that when $n \geq 3$ and X is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, which implies λ constant. Therefore, the study of almost Ricci solitons will be interesting when the field X is not a Killing field.

Taking into account Perelman's potential for Ricci soliton it was proved in [2], that up to constant, this potential is the function which appears on the Hodge-de Rham decomposition associated to the 1-form $X^\flat$. In the noncompact case, there exist non gradient Ricci solitons, see Bair and Danielo [3] and Lott [15]. In [17] it was proved that every noncompact shrinking soliton is a gradient soliton.

We point out that an ancient solution of the Ricci flow has nonnegative scalar curvature, see [10]. For more details about Ricci solitons we recommend [8]. Moreover, in order to complete our ingredients we also recall that the Gaussian soliton is the Euclidean space $\mathbb{R}^n$ endowed with its standard metric $|| \, ||$ and the potential $f(x) = \frac{\lambda}{2}||x||^2$.

Into the direction of understand the geometry of almost Ricci solitons it was proved in [19] some conditions to existence of gradient almost Ricci solitons. Moreover, in [4] it was proved some structural equations and rigidity theorems to almost Ricci solitons. For more details about the geometry of almost Ricci solitons see [4] and [19]. For a locally conformally flat gradient almost Ricci soliton, Catino proved in [9], that around any regular point of f, such manifold is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature. Recently, in [16] it was proved that under some analytic conditions a steady or shrinking Ricci soliton minimally immersed into a Euclidean space is totally geodesic.

In order to proceed we remember a result due to Yau [21] which is a generalization of Hopf's maximum principle: a subharmonic function $f : M^n \to \mathbb{R}$ defined over a complete noncompact Riemannian manifold is constant, provided its gradient belongs to $\mathrm{L}^1(M^n)$. Recently this result was extended by Camargo et al. [7] for a vector field X. With the aid of this extension we derive our first result. This result will give conditions for nonexistence of minimal immersion of an almost Ricci soliton into a Riemannian manifold. More precisely, we have the following theorem.

Theorem 1.2. *Let $\varphi : M^n \looparrowright \widetilde{M}^{n+p}$ be an isometric immersion of an almost Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$ of sectional curvature $\widetilde{k}$. Then the following conditions hold.*

1. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} \leq 0$ and $\lambda > 0$, then φ can not be minimal.*
2. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} < 0$ and $\lambda \geq 0$, then φ can not be minimal.*
3. *If $|X| \in \mathcal{L}^1(M)$, $\widetilde{k} \leq 0$, $\lambda \geq 0$ and φ is minimal, then M^n is flat and totally geodesic.*
4. *If $\sup_M |X| < \infty$, $\widetilde{k} \leq 0$ and $\lambda \geq c > 0$, where $c \in \mathbb{R}$, then φ can not be minimal.*

As a consequence of Theorem 1.2 we shall show a condition for nonexistence of minimal immersion of shrinking Ricci soliton into a Riemannian manifold of non positive sectional curvature. More precisely, we derive the following corollary.

Corollary 1.3. *Let $\varphi : M^n \looparrowright \widetilde{M}^{n+p}$ be an isometric immersion of a shrinking Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$ of sectional curvature $\widetilde{k} \leq 0$. If $|X| \in \mathcal{L}^1(M)$, then φ can not be minimal.*

Now we recall that if $\sup_M |X| < \infty$ and $(M^n,\, g,\, X\,, \lambda)$ is a shrinking Ricci soliton, Theorem 1 of [22] gives that M is compact. Therefore, we have the following corollary.

Corollary 1.4. *Let $\varphi : M^n \looparrowright \mathbb{M}_c^{n+p}$ be an isometric immersion of a shrinking Ricci soliton $(M^n,\, g,\, X\,, \lambda)$ into a space form $\mathbb{M}_c^{n+p}$ of sectional curvature c. If $\sup_M |X| < \infty$ and $c \leq 0$, then φ can not be minimal.*

One notices that when M^n is compact the assumption of $|X| \in \mathcal{L}^1(M)$ is clearly satisfied in Theorem 1.2. Moreover, under compactness assumption M^n can not be simply connected, since by Kuiper [13], it will be conformal to a Euclidean sphere, which gives a contradiction with flatness.

Now we shall consider an almost Ricci soliton immersed into a Riemannian manifold of constant sectional curvature to obtain the following result.

Theorem 1.5. *Let (M^n, g, X, λ) be an almost Ricci soliton immersed into a Riemannian manifold $\widetilde{M}_c^{n+p}$ of constant sectional curvature c. Then we get:*

1. *If $|X| \in \mathcal{L}^1(M)$ and $\lambda \geq (n-1)c + n|H|^2$, then (M^n, g) is totally geodesic, with $\lambda = (n-1)c$ and scalar curvature $R = n(n-1)c$.*
2. *If M^n is compact and $\lambda \geq (n-1)c + n|H|^2$, then M^n is isometric to a Euclidean sphere.*
3. *If $|X| \in \mathcal{L}^1(M)$, $p = 1$ and $\lambda \geq (n-1)(c + H^2)$, then M^n is totally umbilical. In particular, the scalar curvature $R = n(n-1)k$ is constant, where $k = \frac{\lambda}{n-1}$ is the sectional curvature of (M^n, g).*

In [4] it was proved that a non trivial compact almost Ricci soliton is isometric to a Euclidean sphere $\mathbb{S}^n$ provided (M^n, g) has constant scalar curvature. Using this result we will obtain the following theorem.

Theorem 1.6. *Let $(M^n, g, \nabla f, \lambda)$ be a non trivial gradient compact almost Ricci soliton, minimally immersed into a unit Euclidean sphere $\mathbb{S}^{n+1}$. Suppose that $R \geq n(n-2)$, then M^n is isometric to a Euclidean sphere. Moreover, $f + \lambda$ is constant and λ satisfies the following partial differential equation:*

$$\Delta\lambda + n\lambda = n(n-1). \tag{1.3}$$

On the other hand, when M^n is a hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$ of constant sectional curvature c, it is useful to introduce the operator $\Phi = A - HI$, where A and I denote, respectively, the shape operator of the immersion and the identity operator on TM. Finally, we have the following characterization for a gradient shrinking Ricci soliton immersed into a space form.

Theorem 1.7. *Let $(M^n, g, \nabla f, \lambda)$ be a gradient shrinking Ricci soliton immersed with constant mean curvature H into a space form $\mathbb{M}_c^{n+1}$. Then we have:*

1. *either $(M^n, g, \nabla f, \lambda)$ is the Gaussian soliton, with $c \leq 0$,*
2. *or* $\sup|\Phi| \geq \frac{\sqrt{n}}{2(n-1)}\{\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\} > 0$.

2 Preliminaries and basic equations

In this section we shall present some preliminaries that will be used to obtain our results. First of all, we consider $M^n \looparrowright \widetilde{M}^{n+p}$ immersed as an oriented submanifold into a Riemannian manifold $\widetilde{M}^{n+p}$ and we recall Gauss equation, which is given by

$$\begin{aligned} \langle \mathrm{R}(\mathrm{X},\mathrm{Y})\mathrm{Z},\mathrm{W}\rangle =& \langle \widetilde{\mathrm{R}}(\mathrm{X},\mathrm{Y})\mathrm{Z},\mathrm{W}\rangle + \langle \alpha(\mathrm{X},\mathrm{W}),\alpha(\mathrm{Y},\mathrm{Z})\rangle \\ &- \langle \alpha(\mathrm{X},\mathrm{Z}),\alpha(\mathrm{Y},\mathrm{W})\rangle, \end{aligned} \tag{2.1}$$

where $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$ stands for the second fundamental form. We also recall that the mean curvature vector $H(x)$ of such an immersion at $x \in M^n$ is defined by

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} \alpha(e_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of T_xM. In particular, taking trace of Gauss equation we obtain

$$Ric(X,Y) = c(n-1)\langle X,Y\rangle + nH\langle AX,Y\rangle - \langle AX,AY\rangle, \tag{2.2}$$

for $X, Y \in \mathfrak{X}(M)$. Hence, if $AX = HX$ we deduce

$$Ric(X,Y) = (n-1)\big(c + H^2\big)\langle X,Y\rangle. \tag{2.3}$$

Furthermore, it follows from Gauss equation that the scalar curvature R of M^n satisfies

$$R = \sum_{i,j}^{n} \langle \widetilde{R}(e_i,e_j)e_j,e_i\rangle + n^2|H|^2 - \sum_{i,j}^{n} |\alpha(e_i,e_j)|^2. \tag{2.4}$$

Therefore, when $\widetilde{M}^{n+1}$ is a space form of sectional curvature c we have the next identity for the scalar curvature

$$R = n(n-1)c + n^2H^2 - |A|^2. \tag{2.5}$$

In order to finish our preliminaries we recall the following results in [4].

Lemma 2.1. *Let (M^n, g, X, λ) be an almost Ricci soliton. Then we have*

1. $\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - Ric(X,X) - (n-2)g(\nabla\lambda, X)$.
2. *If $X = \nabla f$, then*
$$\Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2 = -\frac{1}{2}\Delta R + (n-1)\Delta\lambda + \frac{R}{n}\Delta f + \frac{1}{2}\langle\nabla R, \nabla f\rangle.$$
3. *In particular, if M^n is compact and $X = \nabla f$, then*
$$\int_M \Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2 dM = \frac{(n-2)}{2n}\int_M \langle\nabla R, \nabla f\rangle dM.$$

3 Proof of the results

3.1 Proof of Theorem 1.2

Proof. If (M^n, g, X, λ), $\lambda > 0$ is minimally immersed into a Riemannian manifold of sectional curvature $\widetilde{k} \leq 0$, then we conclude from equation (2.4) that $R \leq 0$. Consequently, contracting equation (1.1) we have $divX = n\lambda - R > 0$, which contradicts Proposition 1 in [7], since $|X| \in \mathcal{L}^1(M)$. On the other hand, if $\widetilde{k} < 0$ and $\lambda \geq 0$, then we get $divX = n\lambda - R > 0$, again we derive a contradiction and this completes the proof of the two first assertions.

For the third assertion initially we notice that, under the assumptions, it follows from the previous assertions that $\widetilde{k}$, λ and R must vanish at some point $p \in M^n$, otherwise there is no such immersion. Actually, we shall show that these functions are null. To do that, pick $x \in M^n$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_xM. By using that the ambient space has sectional curvature $\widetilde{k} \leq 0$ and the immersion is minimal, we deduce from equation (2.4) that

$$R = \sum_{i,j}^{n} \widetilde{k}(e_i, e_j) - \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2 \leq 0.$$

On the other hand, as $\lambda \geq 0$, we have that $divX = \lambda n - R \geq 0$. Since $|X| \in \mathcal{L}^1(M)$, we have, once more, from Proposition 1 in [7] that $divX = 0$ in M^n.

Hence, we deduce $0 \geq R = \lambda n \geq 0$, i.e., $R = \lambda = 0$. This implies that, for $i, j = 1, \ldots, n$, $\widetilde{k}(e_i, e_j) = |\alpha(e_i, e_j)| = 0$ in M^n. Therefore, we conclude that M^n is totally geodesic and flat, which proves the first part of this assertion. Moreover, from Lemma 1, $\Delta|X|^2 = 2|\nabla X|^2 \geq 0$, so, if M^n is compact, we conclude by Hopf's maximum principle that $\nabla X = 0$, then (M^n, g) is Einstein, which concludes the proof of the third assertion. The assumption of the last item implies that (M^n, g) is compact and has finite first fundamental group, confront with the proof of Theorem 1.1 in [22]. Now, we assume that there exists a compact almost Ricci soliton with $\lambda \geq c > 0$ minimally immersed into a complete Riemannian manifold of non-positive sectional curvature. By using a result due to Frankel in [12] we conclude that the almost Ricci soliton must have infinite first fundamental group. Hence we obtain a contradiction and this completes the proof of the theorem. □

3.2 Proof of Theorem 1.5

Proof. Since the ambient space has constant sectional curvature equal to c, we use once more equation (2.4) to obtain

$$R = n(n-1)c + n^2|H|^2 - \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2. \tag{3.1}$$

This, jointly with the hypothesis on λ, imply that

$$\begin{aligned} divX = & n\lambda - R \;=\; n\{\lambda - \big((n-1)c + n|H|^2\big)\} \\ & + \sum_{i,j}^{n} |\alpha(e_i, e_j)|^2 \;\geq\; 0. \end{aligned} \tag{3.2}$$

Thus, we can apply again Proposition 1 in [7] to obtain $divX = 0$ in M^n. So, from equation (3.2) we conclude that M^n is totally geodesic and $\lambda = (n-1)c$. Then $H = 0$ and finally we use (3.1) to deduce $R = n(n-1)c$. If M^n is compact, as it is totally geodesic, then the ambient space is a sphere $\mathbb{S}^{n+p}$ and M^n is isometric the a Euclidean sphere $\mathbb{S}^n$, which proves the first two assertions.

For the third assertion, we can use $|A|^2 = |\Phi|^2 + nH^2$ jointly with (2.5) to infer

$$div\, X = n[\lambda - (n-1)(c + H^2)] + |\Phi|^2. \tag{3.3}$$

Hence, under the assumptions of the assertion, we can apply once more Proposition 1 in [7] to obtain $divX = 0$ on M^n. Using (3.3) we conclude $\lambda =$

$(n-1)(c+H^2)$ and $|\Phi|^2 = 0$, which gives that M^n is totally umbilical. Thus, if we denote the principal curvatures of M^n by μ, we use Gauss equation to deduce $k - c = \mu^2$, where k is the sectional curvature of M^n. Now, a straightforward computation gives $k = c + H^2 = \frac{\lambda}{n-1}$ and $R = n\lambda = n(n-1)k$. Thus R is constant and this finishes the proof of the theorem. □

3.3 Proof of Theorem 1.6

Proof. Taking into account that the immersion is minimal we have from (2.5) that $R = n(n-1) - |A|^2$. Since we are supposing $R \geq n(n-2)$ we deduce $|A|^2 \leq n$ in M^n. Thus, it follows from Simons [20] that either $|A|^2 = 0$ or $|A|^2 = n$. Therefore, R will be constant and we can apply Corollary 1 in [4] to conclude that M^n is isometric to a standard sphere. Now we can use Chern et al. [11] or Lawson [14] to conclude that $|A|^2 = 0$. Next we use Lemma 2.1 to obtain

$$0 = \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 = (n-1)\Delta(\lambda + f),$$

which enables us to apply Hopf's maximum principle to deduce that $\lambda + f$ is constant. Moreover, we also have

$$\Delta\lambda = -\Delta f = R - n\lambda = n(n-1) - n\lambda = n(n-1-\lambda).$$

Taking into account that $\lambda = -\langle x, a\rangle + (n-1)$ is a solution of this equation, where x is the position vector of the sphere, while a is a fixed vector in $\mathbb{R}^{n+1}$, we conclude that λ is the solution of the quoted equation and we complete the proof of the theorem. □

3.4 Proof of Theorem 1.7

Proof. First we recall that a result due to [10] gives that a shrinking gradient Ricci soliton has non-negative scalar curvature. So we deduce from equation (2.5) that

$$n(n-1)(c+H^2) \geq |\Phi|^2. \tag{3.4}$$

Whence, we obtain $c + H^2 \geq 0$ occurring equality if and only if $AX = HX$ for all $X \in \mathfrak{X}(M)$. Therefore, if this equality occurs we use (2.3) to conclude that M^n is an Einstein manifold. This enables us to use Theorem 3 in [2] to conclude that $(M^n, g, \nabla f, \lambda)$ is the Gaussian soliton.

Next we consider the case $c+H^2>0$. Using again equation (3.4) we deduce the next inequality $\sup_M |\Phi| < \infty$. Now we recall the following inequality obtained in [1]

$$\frac{1}{2}\Delta|\Phi|^2 \quad \geq \quad -|\Phi|^2(P_H(|\Phi|)), \tag{3.5}$$

where

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(c+H^2).$$

Since $H^2+c>0$ we have that $P_H(x)$ has a unique positive root given by

$$B_H = \frac{\sqrt{n}}{2\sqrt{n-1}}(\sqrt{n^2H^2+4(n-1)c} - (n-2)|H|).$$

Recently, in [6], it was proved that on every gradient Ricci soliton the full Omori-Yau maximum principle holds for the Laplacian. Therefore, since $\sup_M |\Phi| < \infty$, we may apply Omori-Yau principle to $|\Phi|$. Thus, we deduce the existence of a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^n such that

$$\lim_{k\to\infty} |\Phi|(p_k) = \sup_M |\Phi|, \quad \nabla|\Phi|(p_k) < \frac{1}{k} \quad \text{and} \quad \Delta|\Phi|(p_k) < \frac{1}{k}.$$

On the other hand, we have

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2(p_k) \quad &= \quad |\Phi|(p_k)\Delta|\Phi|(p_k) + |\nabla|\Phi|(p_k)|^2 \\ &< \quad |\Phi|(p_k)\frac{1}{k} + \frac{1}{k}. \end{aligned}$$

Using this in (3.5) we obtain

$$|\Phi|(p_k)\frac{1}{k} + \frac{1}{k} > \frac{1}{2}\Delta|\Phi|^2(p_k) \geq -|\Phi|^2(p_k)P_H(|\Phi|(p_k)).$$

Taking limits we infer

$$0 \geq -(\sup|\Phi|)^2 P_H(\sup|\Phi|),$$

that is

$$(\sup|\Phi|)^2 P_H(\sup|\Phi|) \geq 0,$$

which implies that either $\sup|\Phi| = 0$, from which we have $AX = HX$ for any $X \in \mathfrak{X}(M)$, or $\sup|\Phi| > 0$. Then we deduce $P_H(\sup|\Phi|) \geq 0$ and $\sup|\Phi| \geq B_H$. First let us consider $|\Phi| = 0$. Using (2.5) we have $R = n(n-1)(c+H^2) \geq 0$. Now we claim that $R = 0$. Indeed, since we have a Ricci soliton $Ric(\nabla f, X) = \frac{1}{2}\langle \nabla R, X\rangle$, see e.g. [2]. On the other hand, equation (2.3) gives $Ric(\nabla f, X) = \frac{R}{n}\langle \nabla f, X\rangle$. Since f is non trivial and R is constant we compare the last two identities to conclude that $R = 0$ as we wish. Since $c + H^2 > 0$ we arrive at a contradiction. Therefore, we have $\sup|\Phi| > 0$ which gives $\sup|\Phi| \geq B_H$ and we complete the proof of the theorem. □

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