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On total coloring of snark products*

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Abstract

Snarks are cubic bridgeless graphs of chromatic index 4 which had their origin in the search of counterexamples to the Four Color Conjecture. In 2003, Cavicchioli et al. proved that for snarks with less than 30 vertices, the total chromatic number is 4, and proposed the problem of finding (if any) the smallest snark which is not 4-total colorable. Several families of snarks have had their total chromatic number determined to be 4, such as the Flower Snark family, the Goldberg family and the Loupekhine family.

We show properties of 4-total colorings of the dot product, of the square product, and of the star product, known operations for constructing snarks. We consider subfamilies of snarks using the star product and obtain a 4-total coloring for each family. Moreover, we prove a property about a specific 4-total coloring of both Blanusa subfamilies, two specific families constructed by the dot product of Petersen graphs.

1 Introduction

Let $G = (V, E)$ be a connected graph where V is the set of vertices of G and E is the set of edges of G. A *k-edge coloring* of a graph G is an assignment of k colors to the edges of G so that adjacent edges have different colors. A graph is *k-edge colorable* if it admits a k-edge coloring. The *chromatic index* $\chi'(G)$ of G is the smallest k such that G is k-edge colorable.

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A *k-total coloring* of a graph G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. A graph is *k-total colorable* if it admits a k-total coloring. The *total chromatic number* $\chi_T(G)$ of G is the smallest k such that G is k-total colorable. Clearly, this number is greater than or equal to $\Delta + 1$, where Δ is the maximum degree of the vertices of G. The Total Coloring Conjecture (TCC) [1, 11] states that the total chromatic number is less than or equal to $\Delta+2$, and it has been proved for cubic graphs [9]. Thus, the total chromatic number of snarks is either $\Delta + 1 = 4$ or $\Delta + 2 = 5$.

Coloring is a challenging problem that models many real situations where the adjacencies represent conflicts. Snarks were defined in the context of the famous Four Color Theorem for coloring maps, trying to find counterexamples to the Four Color Conjecture. This search yielded cubic bridgeless graphs that are not 3-edge colorable, the *snark* graphs. The importance of snarks arises partly from the fact that several conjectures would have snarks as minimal counterexamples. This applies to the following three conjectures: Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture [4].

The name snark was given by Gardner [5] and the earliest and smallest snark to appear was the Petersen graph which has 10 vertices. Isaacs introduced an operation to construct snarks, the dot product [7], and Blanusa snarks are construct by using the dot product of two copies of the Petersen graph. Preissmann proved that there are just two nontrivial Blanusa snarks of order 18. Isaacs also determined the snark family formed by the recursive dot product of Petersen graphs, which he called the BDS family, whose name represents Blanusa, Descartes, and Szekeres, who had discovered some snarks of the family. We shall use the name Blanusa families which has been more widely used than the original name BDS family. In the same paper, Isaacs defined the Flower Snark family.

Later on, the Loupekhine and the Goldberg families have been introduced as well as other operations to construct snarks, namely the square product and the star product. For papers on snarks we refer to [4, 6, 8, 12] and for a list of the well-known snarks, we refer to Table 1.

More recently, Cavicchioli et al. [4] proved that for snarks with less than 30 vertices, the total chromatic number is 4, and proposed the problem of finding, if one exists, the smallest snark which is not 4-total colorable. Since then, several families of snarks have had their total chromatic number determined to be 4, such as the Flower Snark family, the Goldberg family, and the Loupekhine family [2, 10].

In this work, we consider the dot product, the square product, and the star product, operations to construct snarks and we study the behavior of the total coloring with respect to these operations.

We consider additional families of snarks constructed using the star product and we determine that the total chromatic number of infinite subfamilies of these families is 4. Due to space, in this extended abstract we omit the details of most proofs.

2 Products of snarks

Snarks are cubic bridgeless graphs that are not 3-edge colorable. We consider three known operations to construct snarks: the dot product, the square product, and the star product and we prove properties of these products with respect to 4-total coloring.

2.1 The dot product

The *dot product* is an operation that generates a snark G from two snarks G_1 and G_2 and it is denoted by $G_1 \cdot G_2$ [7]. The dot product operates as follows: remove two nonadjacent edges ab and cd of G_1 and remove two adjacent vertices x and y of G_2. Now, letting r, s be the two vertices adjacent to x in G_2, $r \neq y$ and $s \neq y$; and t, u be the two vertices adjacent to y in G_2, $t \neq x$ and $u \neq x$; include edges ar, bs, ct, du, see Figure 1. Note that depending on the choice of the set of vertices and edges of graphs G_1 and G_2 in the process, the dot product $G_1 \cdot G_2$ may produce different snarks.

From now on, we represent by dashed vertices and edges those elements that have been removed from the graph.

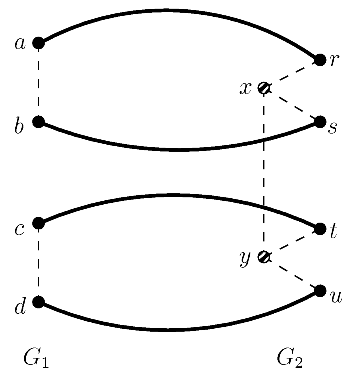


Figure 1: $G_1 \cdot G_2$, we depict only the relevant elements: two edges ab and cd of G_1; and two vertices x and y of G_2, and their respective neighbors r, s, t and u.

Lemma 2.1 defines a parity property about 4-total coloring for general cubic graphs obtained by the dot product. This property helps us to obtain the next 4-total colorings of the dot product of snarks. A *k-matching-cutset* is a k-cutset which is an independent set of edges.

Lemma 2.1. *Let G be a cubic graph with a 4-matching-cutset M and with a 4-total coloring. Let X be a connected component defined by $G - M$. The number of vertices of X with the same color c is even if and only if the number of edges in M with color c is even.*

Proof. Let A be the vertex set of X. Given a color c, we can partition A into three subsets A_1, A_2 and A_3: The subset A_1 is the set of vertices that are endvertices of edges that have color c, the subset A_2 is the set of vertices that have color c, and the subset A_3 is the set of vertices that are endvertices of edges of M that have color c. The number of vertices in A is even, because there is an even number of vertices of degree 3 and four vertices of degree 2. On the other hand, A_1 has an even number of vertices, since two vertices are endvertices of a common edge. Therefore, A_2 is even if and only if A_3 is also even. □

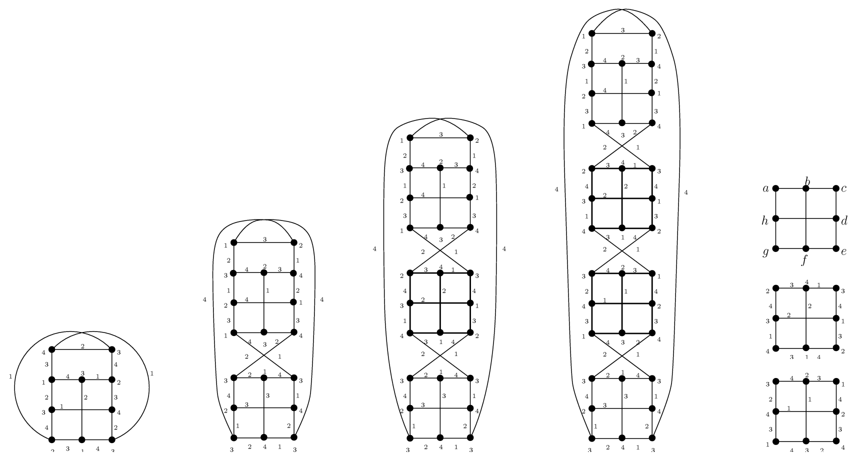
2.1.1 The Blanusa First and the Blanusa Second subfamilies

The first members of both Blanusa families is the Petersen graph that we call $P = BF_0 = BS_0$. Regarding the second members, Preissmann has proved that there exist only two Blanusa snarks of order 18 that we call BF_1 and BS_1, which yields the two families Blanusa First and Blanusa Second [8]. The subsequent members of Blanusa families are formed by all possible applications of the dot product of the previous snark and the Petersen graph [12]. We have proved that the Blanusa subfamilies are 4-total colorable using: the recursive construction of two precise subfamilies of the Blanusa families, and appropriate 4-total colorings of the first member (Figure 2(a)), of the second member (Figure 2(b)) and of the link graph L (Figure 2(e)) (the Petersen graph with two adjacent vertices removed) [10]. See Figure 2 for the obtained 4-total coloring of the first four members of the considered Blanusa First subfamily. Note that two alternating fixed 4-total colorings of the link graph are needed.

Theorem 1 says that, unlike the case of other families of snarks, such as the Loupekhine family and its link graph, any 4-total coloring of the considered Blanusa subfamilies does not have only one fixed 4-total coloring of the link graph L.

Theorem 1. *Let L be the link graph of the considered Blanusa subfamilies. It is not possible to use only one fixed 4-total coloring of L in 4-total colorings of these subfamilies.*

Sketch of proof. Color c is represented at element z when either z has color c or it is incident or adjacent to an element which has color c. Remark that the pairs of nonadjacent vertices e, a and g, c of the link graph L will be adjacent in each snark constructed by the dot product of the previous snark and the Petersen graph, starting from BF_3 (BS_3, resp.). Thus, these vertices would have different colors. Furthermore, note that the pair of vertices e, a cannot have color 1 represented and the pair of vertices g, c cannot have color 2



(a) BF_0 (b) BF_1 (c) BF_2 (d) BF_3 (e) L

Figure 2: The first four members of the Blanusa First subfamily with a 4-total coloring and the labeled link graph with two 4-total colorings.

represented in this possible 4-total coloring. It is not possible to obtain only one 4-total coloring of the link graph with all these properties. Similar properties would hold for the specific Blanusa Second subfamily which we have considered. □

2.2 The square product

The *square product* is an operation that generates a cubic graph G from two cubic graphs G_1 and G_2 and it is denoted by $G_1 \diamond G_2$ [3]. The square product operates as follows: remove one vertex v and one edge de of G_1 ($de \notin \{va, vb, vc\}$) and remove three vertices x, y and z that induce a path of G_2. Now, letting a, b and c be the three vertices adjacent to v in G_1; r, s be the two vertices adjacent to x in G_2; t, u be the two vertices adjacent to z in G_2 and w the other vertex adjacent to y in G_2, include edges ar, bs, ct, du and ew, see Figure 3. As in the dot product, the square product $G_1 \diamond G_2$ may produce different graphs.

Let G_1 and G_2 be two snarks. The square product $G_1 \diamond G_2$ may produce a cubic graph that is not a snark. For example, Figure 4 shows a 3-edge

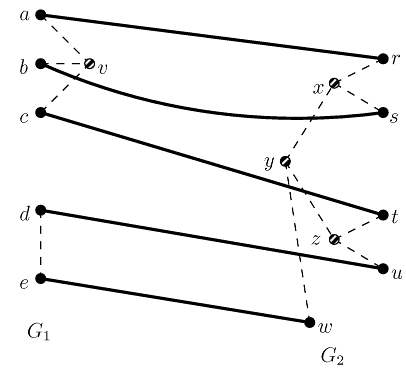


Figure 3: $G_1 \diamond G_2$, we depict only the relevant elements: vertex v and edge de of G_1 and vertices x, y and z of G_2.

coloring of $P \diamond P$.

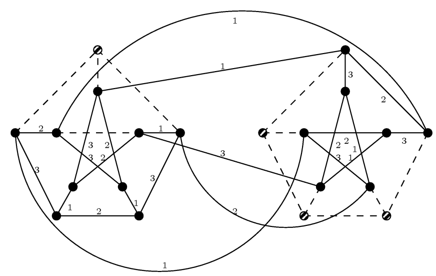


Figure 4: $P \diamond P$ with a 3-edge coloring.

We study the square product with respect to total coloring in order to obtain graphs that are 4-total colorable from graphs that are 4-total colorable. Moreover, we give properties about this 4-total coloring.

Lemma 2.2. *Let π_i be a 4-total coloring of snark G_i, $i = 1, 2$. It is not possible to obtain a 4-total coloring of $G_1 \diamond G_2$ such that corresponding elements (vertices and edges) preserve the colors used in π_i, $i = 1, 2$.*

Sketch of proof. We refer to Figure 3. We analyze two cases. Case 1: edges av, bv, cv and de of G_1 have different colors in π_1. Case 2: edges av, bv and cv of G_1 have different colors in π_1 and edge de of G_1 has the same color as one of the edges av or bv of G_1 in π_1. In any case, to obtain a 4-total coloring of $G_1 \diamond G_2$ preserving a 4-total coloring π_1, specific elements of G_2 would have specific colors, but there is no color available for vertex y of G_2 in π_2. □

The next Lemma defines a parity property about 4-total coloring for general cubic graphs obtained by the square product. This proof is analogous to the proof of Lemma 2.1.

Lemma 2.3. *Let G be a cubic graph with a 5-matching-cutset M and with a 4-total coloring. Let X be a connected component defined by $G - M$. The number of vertices of X with the same color c is even if and only if the number of edges in M with color c is odd.* □

2.3 The star product

The *star product* is another operation that generates a snark G from a snark G_1 and a cubic graph G_2 (or vice-versa), denoted by $G_1 \star G_2$ [6]. This product operates as follows: remove one vertex v_1 of G_1 and remove one vertex v_2 of G_2. Now, letting z_0, z_1 and z_2 be the three vertices adjacent to v_1 in G_1 and u_0, u_1 and u_2 be the three vertices adjacent to v_2 in G_2, include edges z_0u_0, z_1u_1 and z_2u_2, see Figure 5. As happened for previous products, the star product $G_1 \star G_2$ may produce different snarks.

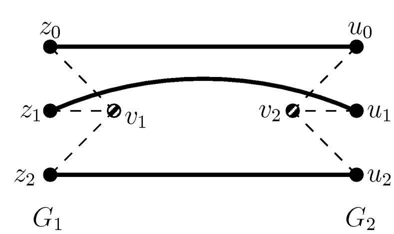


Figure 5: $G_1 \star G_2$, we depict only the relevant elements: vertex v_1 of G_1 and vertex v_2 of G_2.

We prove that it is possible to obtain a 4-total coloring of $G_1 \star G_2$ from specific 4-total colorings of G_1 and G_2. See Figure 6 for one of the 4-total colorings of $G_1 \star G_2$ obtained by Lemma 2.4.

Lemma 2.4. *Let π_i be a 4-total coloring of G_i, $i = 1, 2$ (a snark and a cubic graph). If pairs of vertices z_0, u_0 and z_1, u_1, and z_2, u_2 have different colors in addition to vertices v_1 of G_1 and v_2 of G_2 having the same color, then there exists a 4-total coloring of $G_1 \star G_2$ such that corresponding elements (vertices and edges) preserve the colors used in π_i, $i = 1, 2$.* □

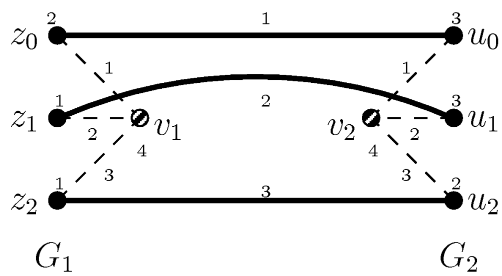


Figure 6: A specific 4-total coloring of $G_1 \star G_2$.

Once more, we define a parity property about 4-total coloring for general cubic graphs obtained by the star product. This proof is analogous to the proof of Lemma 2.1.

Lemma 2.5. *Let G be a cubic graph with a 3-matching-cutset M and with a 4-total coloring. Let X be a connected component defined by $G - M$. The number of vertices of X with the same color c is even if and only if the number of edges in M with color c is odd.* □

2.3.1 The $F_3 \star ... \star F_3$, $G_3 \star ... \star G_3$ and $LO_3 \star ... \star LO_3$ subfamilies

We determine a 4-total coloring of subfamilies of snarks obtained by the recursive star product of the first member of the following infinite families: the Flower Snark family (F_i), the Goldberg family (G_i) and the Loupekhine family (LO_i). We recall that the star products $F_3 \star ... \star F_3$, $G_3 \star ... \star G_3$ and $LO_3 \star ... \star LO_3$ are not unique. Therefore, note that there exist several families obtained by the recursive star product of F_3 (resp. G_3, LO_3). This specific subfamilies were obtained by removing one specific vertex of each F_3 (resp. G_3, LO_3). See an example in Figure 7.

Theorem 2. *Each of the subfamilies of* $F_3 \star ... \star F_3$*,* $G_3 \star ... \star G_3$ *and* $LO_3 \star ... \star LO_3$*, is 4-total colorable.*

Sketch of proof. We construct the 4-total colorings of these families by using Lemma 2.4 and the 4-total colorings given by [2, 10]. □

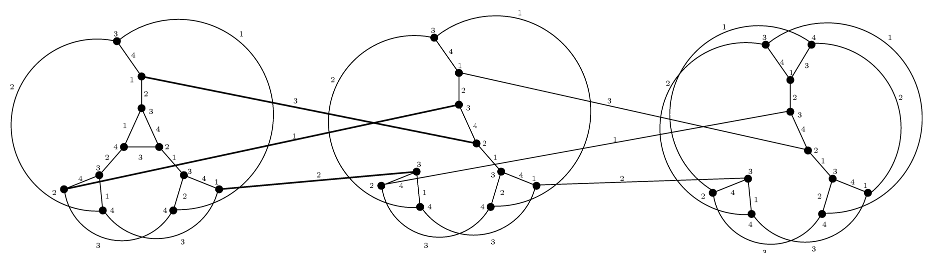


Figure 7: A 4-total coloring of a snark $F_3 \star F_3 \star F_3$.

3 Conclusion

We are seeking an answer to the problem proposed by Cavicchioli et al. [4] of finding, if one exists, the smallest snark which is not 4-total colorable. Paper [2] points out that the answer to this problem may be that there are no snarks that are not 4-total colorable. In this direction, we studied several different operations between snarks with respect to 4-total coloring; we proved properties about the dot product, the square product and the star product with respect to 4-total coloring. All results of this paper contribute to the evidence that all snarks are 4-total colorable.

Table 1 presents the state of the art about 4-total colorings of well-known snarks in the literature.

Snarks	**Reference for $\chi_T = 4$**
Snarks $\leq$ 30 (including the Petersen graph, the Double-star snark, and the Celmins-Swart snarks 1 and 2)	[4]
The Flower Snark family and the Goldberg family	[2]
The Loupekhine family	[10]
Blanusa subfamilies (including the Blanusa snarks, the Descartes snarks, and the Szekeres snarks)	[10]
Subfamilies $F_i \cdot P^j$, $LO_i \cdot P^j$ and $G_i \cdot P^j$	[10]
Subfamilies $F_3 \star ... \star F_3$, $G_3 \star ... \star G_3$ and $LO_3 \star ... \star LO_3$	this paper

Table 1: State of the art of 4-total coloring of well-known snarks.

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