Matemática Contemporânea, Vol 39, 31-38
https://doi.org/10.21711/231766362010/rmc394

Convex covers of graphs *

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Abstract

The closed interval $I[S]$ of a subset of vertices S of a simple graph G is the set of all vertices lying on a shortest path between any pair of vertices of S. If $I[S] = S$, we say that set S is convex. We consider the concept of convex covers of graphs. If there exists a cover of $V(G)$ into p convex sets we say that G has a convex p-cover. We prove that is NP-complete to decide if a general graph G has a convex p-cover for a fixed integer $p \geq 3$. We show that all connected chordal graphs have a convex p-cover, for any $1 \leq p \leq n$. We also establish conditions on n and k to decide if a power of cycle has a convex p-cover. Finally, we develop an algorithm for disconnected graphs.

1 Introduction

In the last three decades many authors have developed concepts of continuous mathematics to discrete mathematics. In particular, one of the first articles that extended convexity to graph theory was [6]. An extensive survey about convexity can be found in [8].

We denote by G a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *convexity* $\mathcal{C}$ in a graph G is a family of subsets S of $V(G)$ such that $\mathcal{C}$ is closed under intersection and contains both $V(G)$ and $\emptyset$. The elements of $\mathcal{C}$ are called *convex sets*.

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*2000 *AMS Subject Classification.* 68R10, 05C75, 05C85.
Key Words and Phrases: convex cover, convex partition, convexity.
*This research was supported by CNPq and FAPERJ.

If each set S in $\mathcal{C}$ contains the vertices of all shortest paths connecting any pair of vertices in S, then $\mathcal{C}$ is called a *geodesic convexity* or, as we call, just convexity. Other classes of convexities have been studied just by considering different path types such as chordless paths [6].

Many structures were created using geodesic convexity and other aspects like geodetic sets, geodetic number, hull and convexity numbers [4, 5].

A *geodesic* of v and w in G is a minimum path between v and w in the graph. The *distance* in G between two vertices v and w, $d_G(v, w)$, is the number of edges in a geodesic between v and w in G. The *closed interval* $I[v, w]$ is the set of all vertices lying on a geodesic between v and w. For a set S, $I[S] = \bigcup_{u,v \in S} I[u, v]$. If $I[S] = S$, we say that S is a *convex set*.

Next, we define the concept of convex p-cover in a graph, as a cover of the vertex set of a graph into p convex sets. So, a graph G has a *convex p-cover*, if $V(G)$ could be covered by p convex sets, i.e., there exists $\mathcal{V} = (V_1, \ldots, V_p)$, $p \in \mathbb{N}$, such that $V(G) = \bigcup_{1 \leq i \leq p} V_i$; set V_i is convex and $V_i \not\subseteq \bigcup_{\substack{1 \leq j \leq p \\ i \neq j}} V_j$, for $1 \leq i \leq p$. In particular, if all sets of $\mathcal{V}$ are disjoint, $\mathcal{V}$ is a *convex p-partition* of $V(G)$. The concept of convex p-partition was defined in [1]. A set $K \subseteq V(G)$ is a *clique* if every pair of vertices of K is adjacent in G. If $\mathcal{V}$ is a convex p-partition and all the sets of $\mathcal{V}$ are cliques, $\mathcal{V}$ is a clique p-partition of $V(G)$.

The *convex cover number* $\varphi_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-cover. The *convex partition number* $\Theta_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-partition.

It is clear that, for any graph G, we have $\varphi_c(G) \leq \Theta_c(G)$. An example where equality holds is the complete bipartite graph $G = K_{q,q}$, that is, $\varphi_c(G) = \Theta_c(G) = q$. On the other hand, we present a graph G, in Figure 1(a), such that G has a convex 2-cover and G does not have a convex 2-partition. Hence $\varphi_c(G) < \Theta_c(G)$. Figure 1($b$) shows an infinite family of graphs G such that G has a convex 2-cover and G does not have a convex 2-partition, consequently, $\varphi_c(G) < \Theta_c(G)$.

In this work, we prove that it is NP-complete to decide if a general graph

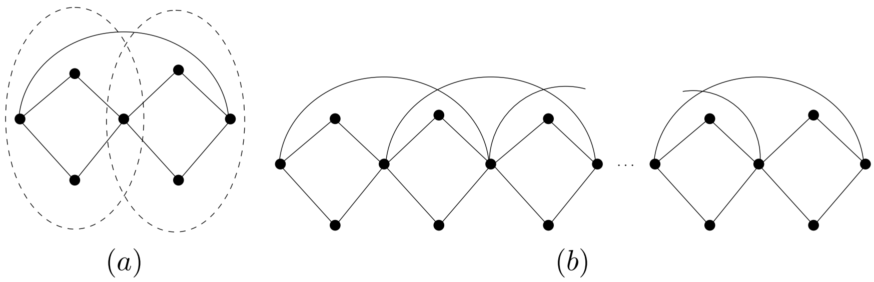

Figure 1: (a) The graph G has a convex 2-cover and does not have a convex 2-partition; (b) Generalization of graph G.

has a convex p-cover, for a fixed $p \geq 3$. This result motivates the study of the complexity of this problem for different classes of graphs. In particular we consider the chordal graphs and powers of cycles. Recent topics on these classes can be found in [4].

We show that all connected chordal graphs have a convex p-cover, for any integer p. For the class of powers of cycles, we determine the cases where C_n^k has a convex p-cover. Finally, we examine convex p-covers of disconnected graphs.

2 NP-completeness

In this section, we discuss the complexity of deciding whether a graph has a convex p-cover for a fixed integer $1 \leq p \leq n$. This problem is called CONVEX p-COVER. Let CONVEX p-PARTITION be the problem of deciding whether a graph G has a convex p-partition; and let CLIQUE p-PARTITION be the problem of deciding whether a graph G has a clique p-partition.

If a graph G has a clique p-partition, then G has a clique $(p+1)$-partition. We observe that this property does not always occur with convex p-cover, i.e., the fact that G has a convex p-cover does not imply that G has a convex $(p+1)$-cover, for $p < |V(G)|$.

We prove that CONVEX p-COVER is NP-complete, for a fixed $p \geq 3$, from a reduction of the NP-complete CLIQUE p-PARTITION problem, for a fixed $p \geq 3$ (Karp [7]).

Theorem 1. *Let G be a graph and $p \geq 3$ be a fixed integer. It is NP-complete to decide whether G has a convex p-cover.*

Proof. The problem is in NP because it is simple to verify in polynomial-time if a subset of $V(G)$ is convex [5].

Let G be a general instance of CLIQUE p-PARTITION. Without loss of generality, let G be a graph with $|V(G)| \geq 2$, such that $V(G)$ is not a clique. We construct a particular instance G' from G as follows. Let u, w be two auxiliary vertices such that $N(u) = N(w) = V(G)$. Then graph G' has $V(G') = V(G) \cup \{u, w\}$ and $E(G') = E(G) \cup \{\{u, v\}, \{w, v\} | v \in V(G)\}$.

We claim that every proper convex set of G' is a clique. Suppose that S is a proper convex set of G' which is not a clique. Hence, S contains two non-adjacent vertices x, y. This implies that $\{u, w\} \subseteq I[x, y] \subseteq S$, and consequently $S = I[S] = V(G)$, a contradiction.

If G' has a convex p-cover $\mathcal{V} = (V_1, \ldots, V_p)$, since $V_i \not\subseteq \bigcup_{\substack{1 \leq j \leq p \\ i \neq j}} V_j$, for $1 \leq i \leq p$, we can obtain a clique p-partition of $V(G')$ eliminating repeated vertices of $\mathcal{V}$. The converse is trivial. ■

The unique case which is not covered by the above theorem is $p = 2$. By [1], we know that it is NP-complete to decide if a graph has a convex p-partition, for $p \geq 2$, but the problem of deciding if a graph has a convex 2-cover remains open.

3 Graph classes

In this section, we examine convex covers of chordal graphs and powers of cycles. A graph is *chordal* if every cycle of length at least 4 has a chord.

Theorem 2. *[3] If G is a connected chordal graph, then G has a convex p-partition, for all $1 \leq p \leq n$.* ■

Corollary 3. *If G is a connected chordal graph, then G has a convex p-cover, for all $1 \leq p \leq n$.* ■

A graph C_n is a *cycle*, with length n, if it is a finite sequence $v_0, v_1, \ldots, v_n$ of vertices, $n \geq 3$, such that $\{v_{i-1}, v_i\} \in E(C_n)$, for $1 \leq i \leq n$, and $v_0 = v_n$.

A *power of cycle* C_n^k, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, is a graph such that $V(C_n^k) = V(C_n)$ and $E(C_n^k) = \{\{v_i, v_j\}|\ v_i, v_j \in V(C_n^k)\ \text{ and }\ d_{C_n}(v_i, v_j) \leq k\}$. We denote the vertices of C_n^k by $v_0, \ldots, v_n$, where v_{i-1} and v_i, for $1 \leq i \leq n$, are consecutive in C_n and $v_n = v_0$.

Lemma 4. *Let S be a proper convex set of a power of cycle C_n^k, such that $n > 2k + 2$ and $n \not\equiv 0, 1, 2 \pmod{2k}$. Then $|S| < \lceil \frac{n}{2} \rceil$.*

Sketch of the proof. Suppose that $|S| \geq \lceil \frac{n}{2} \rceil$. Clearly, S has a pair of vertices v and w such that $d_{C_n}(v, w) = \lceil \frac{n}{2} \rceil - 1$. We show that $I[v, w] = V(G)$, a contradiction. ■

Finally, Theorem 5 establishes conditions for a convex p-cover in a power of cycle C_n^k.

Theorem 5. *A power of cycle C_n^k always has a convex p-cover, except when $p = 2$, $n > 2k + 2$ and $n \not\equiv 0, 1, 2$ (mod $2k$).*

Proof. If one of the conditions is false, then, by [2], C_n^k has a convex p-partition. If all conditions are true then, by Lemma 4, C_n^k does not have a convex p-cover. ■

4 Disconnected graphs

Now, we describe a method for reducing the problem of deciding whether a disconnected graph admits a convex p-cover into a similar problem for a connected graph.

Theorem 6. *Let G be a graph with connected components $G_1, \ldots, G_\omega$. Then G has a convex p-cover if and only if there are values p_i, $1 \leq i \leq \omega$, such that:*

1) *G_i has a convex p_i-cover;*

2) $\sum_{1 \leq i \leq \omega} p_i \geq p$, *and each* $p_i \leq p$. ∎

Theorem 6 leads to Algorithm 1, for reducing the problem of deciding whether a disconnected graph G has a convex p-cover to the problem of deciding whether its connected components G_i have a convex p_i-cover, for $1 \leq p \leq n$.

Algorithm 1. CONVEX p-COVER OF A DISCONNECTED GRAPH.

Input: *Disconnected graph G with connected components $G_1, \ldots, G_\omega$;*

Output: *YES or NO.*

1) *For each i, $1 \leq i \leq \omega$, define $p_i = p$. If G_i does not have a convex p_i-cover, then decrease p_i by one and repeat this test;*

2) *If $\sum_{1 \leq i \leq \omega} p_i \geq p$, then return YES: G has a convex p-cover; otherwise return NO: G has not a convex p-cover.*

5 Conclusion

We have considered the problem of deciding whether a graph G can be covered by p convex sets. We have proved that the problem is NP-complete for fixed integers $p \geq 3$ and remains open for $p = 2$.

We have shown that chordal graphs have convex p-covers, for $1 \leq p \leq n$. We also have established conditions on n and k that determine whether a power of cycle has a convex p-cover. There exists a linear-time algorithm to decide if a cograph has a convex p-partition [1], but the problem of determining if a cograph has a convex p-cover is open.

Finally, we developed an algorithm that reduces the problem of deciding if a disconnected graph has convex p-cover to a similar problem on its connected components.

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