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On $L(2,1)$-coloring split permutation graphs

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Abstract

A k-$L(2,1)$-*coloring* of a graph is an assignment of colors in $\{0,\dots,k\}$ to its vertices such that adjacent vertices get colors at least two apart, and vertices at distance two get distinct colors. Let $\lambda(G)$ be the minimum value of k such that there exists a k-$L(2,1)$-coloring of G. We show a linear-time algorithm to find $\lambda(G)$ when G belongs to the class of split permutation graphs, a subclass of clique-Helly graphs. Furthermore, we show a polynomial-time algorithm that obtains an $L(2,1)$-coloring of these graphs using at most the color $\lambda(G)$.

1 Introduction

Let $G=(V,E)$ be a simple graph, where $n=|V|$ and $m=|E|$. For a vertex v of V, $N(v)$ is the set of vertices of G that are adjacent to v in G. The *degree* of a vertex v of V, denoted by $d(v)$, is given by $d(v)=|N(v)|$, and $\Delta=\max\limits_{v\in V}\{d(v)\}$. The *distance* between two vertices u and v of V, denoted by $dist(u,v)$, is the number of edges in a shortest path between these vertices. Let $G[S]$ be the subgraph of G induced by the vertices in S. A set $S\subseteq V$ is a *stable set* if there is no edge between any pair of vertices of S in G, whereas S is a *clique* if the subgraph induced by S is a complete graph. The minimum number of vertex-disjoint paths that cover V is denoted by $pv(G)$.

A k-$L(2,1)$-*coloring* of a graph G is a function $f:V(G)\to\{0,\dots,k\}$ such that if $uv\in E$, then $|f(u)-f(v)|\geq 2$ and if $dist(x,y)=2$, then $f(x)\neq$

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$f(y)$. The $L(2,1)$-coloring problem, introduced by Griggs and Yeh [9], aims to determine $\lambda(G)$, the minimum k for which there exists a k-$L(2,1)$-coloring of G. The $L(2,1)$-coloring problem is $\mathcal{NP}$-hard even when restricted to split graphs [3], however there are linear-time algorithms for computing $\lambda(G)$ for bipartite chain graphs [1] and for P_4-tidy graphs [11]. Furthermore, its complexity is still open even for proper interval graphs and bipartite permutation graphs.

A graph $G = (V, E)$ is: **(i)** *path* if $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} \mid 1 \leq i < n\}$; **(ii)** *complete* if there is an edge between every possible pair of vertices; **(iii)** *complement* of a graph $G^c = (V', E')$ if $V = V'$ and $E' = \{(V \times V) \setminus (E \cup vv) \mid v \in V\}$; **(iv)** *diameter two* if the distance between every pair of vertices is equal or less than two; **(v)** *comparability* if there is a transitive orientation of its edges; **(vi)** *cocomparability* if its complement graph is a comparability graph; **(vii)** *permutation* if it is the intersection graph of line segments between two parallel lines; **(viii)** *interval* if it is the intersection graph of intervals in the real line; **(ix)** *cointerval* if its complement graph is an interval graph; **(x)** *q-partite* if its vertices can be partitioned into q stable sets; **(xi)** *bipartite* if its vertices can be partitioned into two stable sets; **(xii)** *split* if its vertices can be partitioned into one stable set S and a clique K; **(xiii)** *clique-Helly* if its family of maximal cliques satisfies the Helly property.

2 Preliminaries and related works

We need the following theorems to our proofs.

Theorem 1 ([7]). *If G is a diameter two graph, then $\lambda(G) = n + pv(G^c) - 2$.*

Theorem 2 ([2]). *If G is an interval graph, then $pv(G)$ can be computed in linear-time.*

Theorem 3 (cf. [4]). *G is a permutation graph if and only if both G and G^c are comparability graphs.*

Theorem 4 (cf. [4])**.** *G is a split permutation graph if and only if both G and G^c are interval graphs.*

Table 1 summarizes the known results on the complexity of the $L(2,1)$-coloring problem. For more information on this problem, we refer to [11].

Class	Complexity
bipartite chain	$O(n+m)$ [Araki [1] - 2009]
cograph	$O(n+m)$ [Chang and Kuo [7] - 1996]
t-almost tree	$O(n2^t)$ [Fiala et al. [8] - 2001]
P_4-tidy	$O(n+m)$ [C. and P. [11] - 2010]
regular grid	$O(n)$ [Calamoneri e Petreschi [6] - 2001]
tree	$O(n)$ [Hasunuma *et al.* [10] - 2009]

Table 1: Complexity of determining $\lambda(G)$

3 Split permutation graphs

3.1 Properties

To $L(2,1)$-coloring a split permutation graph a better understanding of the structure of these graphs is needed. The following lemmas, whose proofs are omitted due to lack of space, assure that the vertices of a split permutation graph can be partitioned and arranged in a very special way displaying all the suffice structural properties.

Lemma 1. *For a split permutation graph $G=(V,E)$, the sets S and K can be partitioned into three sets each: S_L, S_M, S_R and K_L, K_M, K_R in such way that: **(i)** there is a vertex $u \in K_L$ and a vertex $v \in K_R$ such that $N(u) \cup N(v) = V$; **(ii)** if $w \in S_R$, then $N(w) \subseteq K_R$; **(iii)** if $w \in S_L$, then $N(w) \subseteq K_L$.*

Lemma 2. *For a split permutation graph G, if S_M, K_L and K_R are as described in Lemma 1, then for each $w \in S_M$, $K_R \subseteq N(w)$ or $K_L \subseteq N(w)$.*

Lemma 3. *For a split permutation graph G, if S_L, S_M, S_R, K_L, K_R are as described in Lemma 1, then there is an ordering $(a_1, a_2, \ldots, a_x)$ of the vertices of $S_L \cup S_M$ in such way that in $H = G[S_L \cup S_M \cup K_L]$, the neighborhood of a vertex in the ordering is contained in the neighborhood of the vertices that succeed it, i.e., $N_H(a_i) \subseteq N_H(a_{i+1})$. The same happens to the vertices of $S_R \cup S_M$, in $G[S_R \cup S_M \cup K_R]$.*

3.2 $L(2,1)$-coloring split permutation graphs

Let $G = (V, E)$ be a split permutation graph, S_L and S_R obtained as described in Lemma 1, $G_L = G[V \setminus S_R]$ and $G_R = G[V \setminus S_L]$.

Theorem 5. *For a split permutation graph G, $\lambda(G) = \max\{\lambda(G_L), \lambda(G_R)\}$.*

Sketch of the proof. As G_L and G_R are induced subgraphs of G it is straightforward that $\lambda(G) \geq \max\{\lambda(G_L), \lambda(G_R)\}$.

The proof that $\lambda(G) \leq \max\{\lambda(G_L), \lambda(G_R)\}$ is given by an analysis of the linear-time algorithm to find $pv(G)$ of interval graphs described in Theorem 2, with a specific interval model as input.

By Theorem 4, G^c is an interval graph. Given an $L(2,1)$-coloring of G_L, let $F(S_R)$ be the minimum number of forbidden colors to the vertices of S_R. We show how to assign colors to the vertices of G_L in such way that an $L(2,1)$-coloring of G_L using at most the color $\lambda(G_L)$ can be obtained. Moreover, $F(S_R) \leq \lambda(G_R)$.

By Theorem 1, we know that for a diameter two graph G, $\lambda(G) = n + pv(G^c) - 2$. As G_L and G_R are diameter two graphs, we can determine $\lambda(G_L)$ and $\lambda(G_R)$ using $pv(G_L^c)$ and $pv(G_R^c)$, respectively.

This can be done using the algorithm of Theorem 2 in the subgraph G_L^c, including vertices with lowest degrees of S_R in the interval model of G_L^c until there is no change, after removing the vertices of S_R, in the vertex-disjoint paths that cover the vertices of the new G_L^c, and prioritizing the vertices of K_R. We show that these vertices of S_R will not add any new vertex-disjoint path, after removing the vertices of S_R, in the covering of the new G_L^c, it will only reorder the paths. Furthermore, if one prioritize the vertices of K_R, it will allow us to show the upper bound $F(S_R) \leq \lambda(G_R)$.

In Figure 1 it is shown how the vertices of K_R can be prioritized. There are three cases in the interval model of G_L^c that we have to deal with: (i) if there is a vertex v of K_M that will receive a color before a vertex w of K_R and $N(v) = N(w)$, we can move the interval that represents v beyond the interval that represents w; (ii) the same of case (i), but now $N(w) \subset N(v)$, and if one consider the subgraph induced by the vertices that do not have received colors, their neighborhood is the same, in that case we show that the interval of the vertex w can also be moved beyond the interval that represents w; (iii) otherwise, we have that the vertex z of S_M that will appear in the path cover of G_L^c after the vertex w is adjacent to v in G_L, when this occurs, the model is not modified.

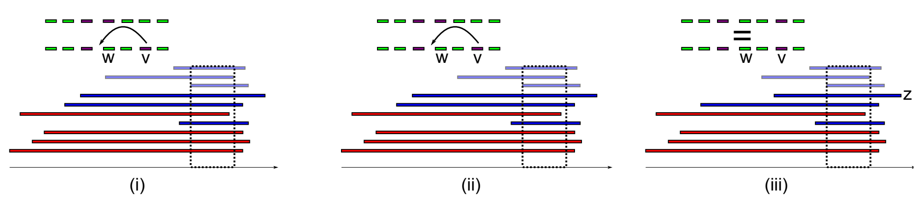

Figure 1: Modification of the interval model of G_L^c

By Theorem 1, $\lambda(G_R) = |K_R| + |S_M| + |K_L| + |K_M| + |S_R| + pv(G_R^c) - 2$. And applying the new interval model of G_L in the algorithm of Theorem 2, one can verify that $F(S_R) = 2|K_R| + j + |S_M| - j + |K_L| + |K_M| + |S_R| - 1 - C_R - C_{MR}$, where j is the number of consecutive colors used in S_M, C_{MR} is the number of intermediate colors used in K_R that are also used in K_M, and C_R is the number of intermediate colors used in K_R that are also in S_R. Thus, $\lambda(G_R) = F(S_R)$ will hold when $pv(G_R) = |K_R| - C_R - C_{MR} + 2$, and using our interval model of G_L^c as input of the algorithm of Theorem 2, and the properties of Lemmas 1, 2 and 3, we show that this will always be true.

Corollary 1. *If G is a split permutation graph, then $\lambda(G)$ can be computed in linear time.*

Proof. One can determine G_L and G_R as described in Lemma 1 in linear time: (i) construct the interval model of G; (ii) take u and v the vertices represented by the leftmost interval and the rightmost interval; (iii) let w be a vertex of K with degree greater than the degree of u (or the degree of v) if it exists; (iv) if $N(u) \subseteq N(w)$ (or $N(v) \subseteq N(w)$) the vertex w become the new u (or the new v); (v) if there was such vertex w go back to (iii). This can be done in $O(n+m)$ time using $O(n)$ space, as each vertex w of K is considered only once.

By Theorem 5, $\lambda(G) = \max\{\lambda(G_L), \lambda(G_R)\}$. As G_L and G_R are diameter two graphs, by Theorem 1, we only need to determine $pv(G_L^c)$ and $pv(G_R^c)$ in order to determine $\lambda(G_L)$ and $\lambda(G_R)$. By Theorem 4, G_L and G_R are both cointerval, and as described in Theorem 2, there is a linear-time algorithm that can determine the minimum number of vertex-disjoint paths that cover the vertices of their complement graphs.

Corollary 2. *If G is a split permutation graph, then an $L(2,1)$-coloring can be obtained in $O(n^2)$.*

Proof. The proof of Theorem 5 also gives an assignment of colors to the vertices of the split permutation graph that is an $L(2,1)$-coloring using at most the color $\lambda(G)$.

4 Conclusion

Although the $L(2,1)$-coloring problem has been widely studied in the last two decades, there are only few classes of graphs for which there are efficient algorithms. It is known that the $L(2,1)$-coloring is $\mathcal{NP}$-complete even when restricted to split graphs and the complexity of this problem is still open even for bipartite permutation graphs.

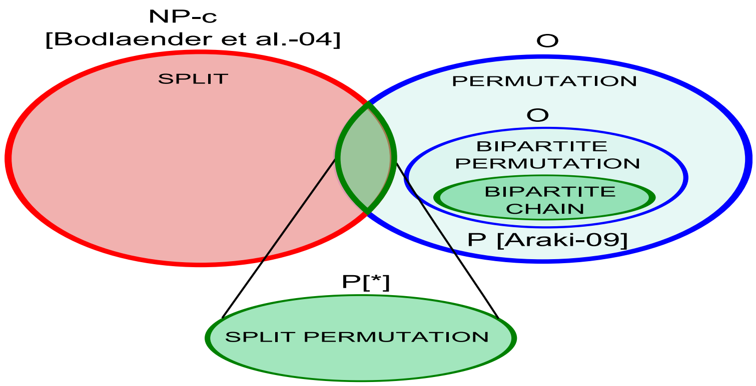

Figure 2: $L(2,1)$-coloring of split permutation graphs

In this work we give not only a linear-time algorithm to determine $\lambda(G)$ of split permutation graphs, but also a polynomial-time algorithm to assign the colors to the vertices of graphs in this class using at most the color $\lambda(G)$. Therefore, the class of split permutation graphs becomes one of the classes the $L(2,1)$-coloring problem is efficiently solved.

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