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On class 2 split graphs *

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Abstract

The *Classification Problem* is the problem of deciding whether a simple graph has chromatic index equal to Δ or $\Delta + 1$. In the first case, the graphs are called *Classe 1*, otherwise, they are *Class 2*. A *split graph* is a graph whose vertex set admits a partition into a stable set and a clique. Split graphs are a subclass of chordal graphs. Figueiredo at al. state that a chordal graph is Class 2 if and only if it is neighborhood-overfull. In this paper, we give a characterization of neighborhood-overfull split graphs.

1 Introduction

An *edge-coloring* of G is an assignment of one color to each edge of G such that no adjacent edges have the same color. The *chromatic index*, $\chi'(G)$, is the minimum number of colors for which G has an edge-coloring.

An easy lower bound for the chromatic index is the maximum vertex degree Δ. A celebrated theorem by Vizing [17] states that, for a simple graph, the chromatic index is at most $\Delta + 1$. It was the origin of the *Classification Problem*, that consists of deciding whether a given graph has chromatic index equals to Δ or $\Delta + 1$. Graphs whose chromatic index is equal to Δ are said to be *Class* 1; graphs whose chromatic index is equal

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to $\Delta + 1$ are said to be *Class* 2. Despite the restriction imposed by Vizing, it is NP-complete to determine, in general, if a graph is Class 1 [10]. In 1991, Cai and Ellis [1] proved that this holds also when the problem is restricted to some classes of graphs such as perfect graphs. However, the classification problem is entirely solved for a few known sets of graphs that include the complete graphs, bipartite graphs [11], complete multipartite graphs [9], and graphs with universal vertices [13]. On the other hand, the complexity of the classification problem is unknown for several well-studied strong structured graph classes such as cographs [6], join graphs [12, 16], planar graphs [15], chordal graphs, and several subclasses of chordal graphs such as split graphs [2], indifference graphs, interval graphs, and doubly chordal graphs [7].

By Vizing Theorem, to show that a graph G is Class 1 is enough to construct an edge-coloring for G with $\Delta(G)$ colors, however to show that G is Class 2 we must prove that G does not have an edge-coloring with $\Delta(G)$ colors. Considering a simple graph G, the inequality $|E(G)| > \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ is a useful sufficient condition to classify G as a Class 2 graph. In such a way, this condition implies that G has "many edges" and it is called *overfull graph*. Note that if a graph G is overfull, then G has an odd number of vertices and, since at most $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$ edges of G can be colored with a same color, it is Class 2. Moreover, if a graph G has an overfull subgraph H with $\Delta(H) = \Delta(G)$, it is a *subgraph-overfull graph* [8]. When the overfull subgraph H is induced by a $\Delta(G)$-vertex v and all its neighbors, we say that G is a *neighborhood-overfull graph* [5]. Overfull, subgraph-overfull, and neighborhood-overfull graphs are Class 2. Although very rare, there are examples of Class 2 graphs that are neither subgraph-overfull nor neighborhood-overfull. The smallest one is P^*, the graph obtained from the Petersen graph by removing an arbitrary vertex.

Hilton and Chetwynd conjectured that being Class 2 is equivalent to being subgraph-overfull, when the graph has a maximum degree greater than $\frac{|V(G)|}{3}$ [3]. This conjecture is known as the *Overfull Conjecture*. Every Class 2 graph with maximum degree at least $|V(G)| - 3$ is subgraph-overfull [4, 13, 14]; every Class 2 complete multipartite graph is overfull [9]. These classes provide evidence for the Overfull Conjecture. Note that if the Overfull Conjecture is true, the resulting theorem can not be improved, since $\frac{|V(P^*)|}{3} = \Delta(P^*)$.

A *split graph* is a graph whose vertex set admits a partition into a stable set and a clique. It has been proved, in [5], that every overfull split graph contains a universal vertex and therefore is neighborhood-overfull. Moreover, every subgraph-overfull split graph is in fact neighborhood-overfull. In the same article, the authors have posed the following conjecture for chordal graphs (graphs without induced cycles C_n with $n \geq 4$), a superclass of split graphs.

Conjecture 1. *Every Class* 2 *chordal graph is neighborhood-overfull.*

Note that the validity of this conjecture for chordal graphs and, therefore, for split graphs implies that the Classification Problem for the corresponding class is in P since being neighborhood-overfull can be easily verified.

In this work, we present a structural characterization of the neighborhood-overfull split graphs. If the Conjecture 1 is true for split graphs, we are presenting a structural characterization of the unique Class 2 split graphs.

In section 2, we recall some known results that we use in the successive sections. In section 3, we give a characterization of neighborhood-overfull split graphs.

2 Theoretical framework

In this paper, G denotes a simple, finite, undirected and connected graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, denote by $G[X]$ the subgraph induced by X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X. For any v in $V(G)$, the set of vertices adjacent to v is denoted by $N(v)$ and $N[v] = \{v\} \cup N(v)$. The subgraph induced by $N(v)$ and $N[v]$ are called *neighborhood* of v and *closed neighborhood* of v, respectively. Two vertices, u and v, of a graph G are *twin vertices* if $N[u]{=}N[v]$ in G. For $X \subseteq V(G)$, $N(X) = \bigcup_{v \in X} N(v)$. The *degree* of a vertex v is $d_G(v) = |N(v)|$. The *maximum degree* of G is, then, $\Delta(G) = \max_{v \in V(G)} \{d_G(v)\}$. A $\Delta(G)$-vertex is a vertex v with $d_G(v) = \Delta(G)$. When there is no ambiguity, we remove the symbol G from the notations. A *clique* is a set of pairwise adjacent vertices of a graph. A *maximal clique* is a clique that is not properly contained in any other clique. A *stable set* is a set of pairwise non adjacent vertices.

In the following we shall use some known results that we recall for reader's convenience.

Theorem 2. *[13] Let G be a graph with $\Delta(G) = |V(G)| - 1$. Then G is Class 1 if and only if $|E(\overline{G})| \geq \frac{\Delta(G)}{2}$.*

Theorem 3. *[2] Let $G = \{Q, S\}$ be a split graph. If $\Delta(G)$ is odd, then G is Class 1.*

Theorem 4. *[5] Let $G = \{Q, S\}$ be a split graph. If G is overfull, then G has a universal vertex. Moreover, G is subgraph-overfull if and only if G is neighborhood-overfull.*

3 A Class 2 split graph

By Theorem 4, a subgraph-overfull split graph is a neighborhood-overfull split graph. Hence, in this section, we give a structural characterization of split graphs that are neighborhood-overfull. To the best of our knowledge, these graphs are the unique known Class 2 split graphs.

From now on, we consider a split graph G with a partition $\{Q, S\}$, where Q is a maximal clique and S is a stable set. Note that since Q is a maximal clique, all $\Delta(G)$-vertices belong to Q. To every split graph G we shall associate the bipartite graph B obtained from G by removing all edges of the subgraph of G induced by Q. Let $d(Q)$ be the maximum degree of a vertex of Q in the bipartite graph B, i.e., $d(Q) = \max_{v \in Q}\{d_B(v)\}$. Then $\Delta(G) = |Q| - 1 + d(Q)$.

Lemma 5. *Let $G = \{Q, S\}$ be a split graph. If G is a neighborhood-overfull graph, then Q and $d(Q)$ must have different parity and $|Q| = (d(Q))^2 + i$ with i odd, $i \geq 3$.*

Proof. Let $G = \{Q, S\}$ be a split graph. If $\Delta(G)$ is odd, by Theorem 3, G is Class 1, and, therefore, G is not neighborhood-overfull. Hence $\Delta(G) = |Q| + d(Q) - 1$ must be even. This implies that $|Q|$ and $d(Q)$ have different parity.

Let us assume that G is neighborhood-overfull. If G is a complete graph, it is known that $|Q|$ must be odd with $|Q| \geq 3$ and the lemma follows. Therefore, we consider $S \neq \emptyset$. By definition of neighborhood-overfull graph, there exists a $\Delta(G)$-vertex $v \in Q$ such that $|E(\overline{G[N[v]]})| \leq \frac{\Delta(G)}{2} - 1$. Since Q is a maximal clique, for each $u \in N[v] \cap S$ there exists at least a $w \in Q$ such that $\{u, w\} \notin E(G)$. Then $\binom{d(Q)}{2} + d(Q) \leq |E(\overline{G[N[v]]})| \leq \frac{\Delta(G)}{2} - 1$. The parities of Q and $d(Q)$ imply that $|Q| = (d(Q))^2 + i$ with i odd, $i \geq 3$. ∎

Now we give a characterization of neighborhood-overfull split graphs. It is relevant to note that the next theorem guarantees that every neighborhood-overfull split graph G contains a minimum number of $\Delta(G)$-vertices that have the same closed neighborhood.

Theorem 6. *Let $G = \{Q, S\}$ be a split graph. The graph G is neighborhood-overfull if and only if the following conditions hold:*

1. *$\Delta(G)$ is even; and*
2. *there exist a set $X \subseteq Q$ with at least $k = |Q| - \frac{\Delta(G)}{2} + \binom{d(Q)}{2} + 1$ $\Delta(G)$-vertices that are twins and, for a $v \in X$, the number of edges of $\overline{G[N[v]]}$ incident to vertices of $Q \setminus X$ is at most $|Q| - k$.*

Proof. Let $G = \{Q, S\}$ be a split graph. Suppose that G is neighborhood-overfull. Then, by Lemma 5, condition (1) is true.

Let us assume that G is neighborhood-overfull. If G is a complete graph, every vertex is a $\Delta(G)$-vertex and all the conditions are trivially true. Therefore, we consider $S \neq \emptyset$. Since G is neighborhood-overfull, G contains a $\Delta(G)$-vertex v such that $G[N[v]]$ is overfull. Hence, by Theorem 2, $|E(\overline{G[N[v]]})| < \frac{\Delta(G)}{2}$. So, there are at most $\frac{\Delta(G)}{2} - 1 - \binom{d(Q)}{2}$ vertices in Q which are not adjacent to at least one vertex in $N[v] \cap S$. Therefore, $G[N[v]]$ contains at least $k = |Q| - \frac{\Delta(G)}{2} + \binom{d(Q)}{2} + 1$ vertices of maximum degree. Let X be the set of the vertices of maximum degree in $G[N[v]]$ ($|X| \geq k$). Since $|N[v] \cap S| = d(Q)$, all vertices in X are $\Delta(G)$-vertices and they are twins. Furthermore, $\sum_{w \in Q \setminus X} d_{\overline{G[N[v]]}}(w) \leq \frac{\Delta(G)}{2} - 1 - \binom{d(Q)}{2} = |Q| - k$.

Now suppose that the conditions (1) and (2) are true. Let v be one of the k $\Delta(G)$-vertices that are twins and consider $G[N[v]]$. Then, by condition (2), we have $|E(\overline{G[N[v]]})| \leq \binom{d(Q)}{2} + |Q| - k = \frac{\Delta(G)}{2} - 1$. Since, by condition (1), G has even maximum degree, then G is a neighborhood-overfull graph. ■

Corollary 7. *Let $G = \{Q, S\}$ be a neighborhood-overfull split graph. Then $\Delta(G) > \frac{|V(G)|}{3}$.*

Proof. Let $G = \{Q, S\}$ be a neighborhood-overfull split graph. By Theorem 6, $|S| \leq d(Q) + (\frac{\Delta(G)}{2} - 1) - \binom{d(Q)}{2}$. This implies $|V(G)| = |Q| + |S| \leq |Q| + d(Q) - 1 + \frac{\Delta(G)}{2} - \binom{d(Q)}{2}$. Recall that $\Delta(G) = |Q| + d(Q) - 1$. So, $|V(G)| \leq \Delta(G) + \frac{\Delta(G)}{2} - \binom{d(Q)}{2}$. Therefore, $\Delta(G) \geq 2\left(\frac{|V(G)|}{3}\right) + \frac{2}{3}\binom{d(Q)}{2} > \frac{|V(G)|}{3}$. ■

The split graphs described in Theorem 6 are Class 2. Therefore, if the Conjecture 1 were true, these graphs would be the unique Class 2 split graphs and every split graph $G = \{Q, S\}$ with $\Delta(G)$ even and $|Q| < (d(Q))^2 + 3$ would be Class 1.

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