Matemática Contemporânea, Vol 39,121-128
https://doi.org/10.21711/231766362010/rmc3914

An evidence for Lovász conjecture about Hamiltonian paths and cycles *

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Abstract

It was conjectured by Lovász in 1970 that every connected vertex-transitive graph has a Hamiltonian path [5]. So far, only four connected vertex-transitive graphs with more than two vertices but without Hamiltonian cycles are known [4]. These four graphs have Hamiltonian paths. However, since none of these four graphs is a Cayley graph, we can look at the Lovász conjecture as stating that every connected Cayley graph with more than two vertices has a Hamiltonian cycle.

In this work, we show some properties of the gadget graph $H_{l,p}$ which was used by Holyer to prove the $\mathcal{NP}-$completeness of the problem of the edge-partition into cliques [3]. We show that the graph $H_{l,p}$ is a Cayley graph and present two constructions of a Hamiltonian cycle, which corroborates the Lovász conjecture.

1 Introduction

This paper considers a family of graphs, denoted by graph $H_{l,p}$. The graph $H_{l,p}$ was used by Holyer to prove the $\mathcal{NP}$-completeness of the problem of edge-partition into clique with l vertices K_l [3]. We follow the usual notation of K_l for the complete graph on l vertices and refer to a complete subgraph as a clique.

*2000 *AMS Subject Classification.* 68R10, 05C75, 05C85.
Key Words and Phrases. Cayley graphs, Abelian group, vertex-transitive graphs, Hamiltonian cycle, Lovász conjecture, graph $H_{l,p}$.
*This research was supported by CAPES, FAPERJ and CNPq.

We show that the graph $H_{l,p}$ is a Cayley graph of an Abelian group, that $H_{l,p}$ has a Hamiltonian cycle and we have an evidence for Lovász conjecture.

In the next section, we define the graph $H_{l,p}$. In the third section, we show that the graph $H_{l,p}$ is a Cayley graph of an Abelian group. In the fourth section, we show that the graph $H_{l,p}$ has Hamiltonian cycle and present a construction of the cycle. In the fifth section, we present another construction of the Hamiltonian cycle. Finally, we have the conclusion of this work.

2 Preliminaries

The graph $H_{l,p}$ was defined as a gadget graph in the study of the problem of edge partition. The problem of edge partition in K_l's, denoted EP_l, considers the partition of the edge-set of the graph into subsets of edges such that each subset induces a complete graph of l vertices.

For each $l \geq 3$ and $p \geq 3$, define a graph $H_{l,p} = (V_{l,p}, E_{l,p})$ having

$$V_{l,p} = \{x = (x_1, ..., x_l) \in Z_p^l \text{ with } \sum_{i=1}^{l} x_i \equiv 0 \bmod p\}$$

$$E_{l,p} = \{xy : \quad \text{there are distinct} \quad i, j \quad \text{such that} \quad y_k \equiv_p x_k$$

$$\text{for} \quad k \neq \{i, j\} \quad \text{and} \quad y_i \equiv_p x_i + 1, \;\; y_j \equiv_p x_j - 1\}$$

The graph $H_{l,p}$ can be edge-partitioned into subgraphs isomorphic to K_l in two different ways and the graph $H_{l,p}$ is a gadget of the graph used in the proof of $\mathcal{NP}-$ completeness of the problem EP_l.

Theorem 2.1 ([3]). *The edge-partition problem EP_l is $\mathcal{NP}-$complete for each $l \geq 3$.*

Figure 1 presents a representation with repeated vertices of the graph $H_{3,4}$. The graph $H_{3,4}$ is not planar because it is regular graph of degree 6, but it admits a representation on the torus without crossing edges. We can

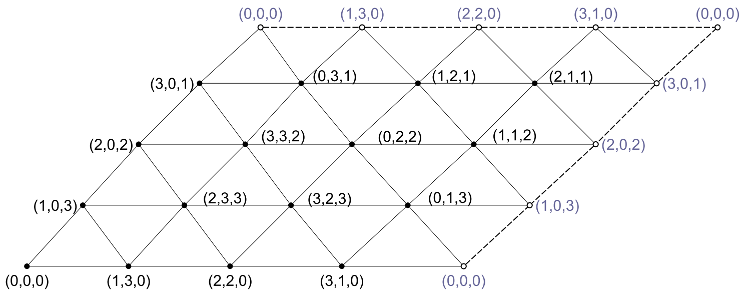

Figure 1: Graph $H_{3,4}$

observe that for $l = 3$, we have a torus "grid" of size $p \times p$. Some properties of graph $H_{l,p}$ are given by Lemma 2.2.

Lemma 2.2 ([3]). *The graph $H_{l,p}$ has the following properties:*

1. *the degree of each vertex is $2\binom{l}{2}$;*
2. *the number of vertices is p^{l-1};*
3. *the number of edges is $\binom{l}{2}p^{l-1}$.*

The graph $H_{l,p}$ was used by Holyer in the proof of $\mathcal{NP}$-completeness of the problem of edge partition [3]. Seen as a decision problem, this problem is to decide whether there is a partitioning set of edges E in K_l's.

3 Cayley graph

In this section, we demonstrate that the graph $H_{l,p}$ is a Cayley graph of an Abelian group.

A group $\mathcal{G}$ is finite when it contains a finite number of elements. For an example we can cite the group $(Z_n, +)$. This group is finite because $Z_n = \{\overline{0}, \overline{1}, ..., \overline{n-1}\}$ contains n equivalence class distinct, wherever $\overline{x}$ is the number natural that it has remainder x when divided for n.

The vertices of the graph $H_{l,p}$ are the elements of a finite group $\mathcal{G}(V_{l,p}, +)$, wherever the operation $+$ is $(f_{a,1}, ..., f_{a,l}) + (f_{b,1}, ..., f_{b_l}) = (f_{a,1} + f_{b,1}, ..., f_{a,l} + f_{b,l})$, wherever $f_{a,i}$ and $f_{b_i} \in (Z_p, +)$.

Lemma 3.1. *$\mathcal{G} = (V_{l,p}, +)$ is a finite group.*

A group $\mathcal{G} = (L, +)$ is Abelian (or commutative) group if $a+b = b+a$ for all $a, b \in L$. We consider $(Z, +)$ an Abelian group, so we have the following result.

Lemma 3.2. *The group $\mathcal{G} = (V_{l,p}, +)$ is Abelian.*

Let $\mathcal{G}$ be a finite group. The subset S of a group $\mathcal{G}$ is called generators of $\mathcal{G}$, and S is said to be a generating set, if every element of G can be expressed as a finite product of elements in S. We also say that G is generated by S. A subset S of G is identity free if $e \notin S$ and it is symmetric if $s \in S$ implies $s^{-1} \in S$.

We present a generating set of the group $\mathcal{G} = (V_{l,p}, +)$. We define the set $S_l = \{(f_1, ..., f_l) \in V_{l,p} : \exists i, j \in \{1, ..., l\}, i \neq j$, such that $f_i = 1, f_j = -1$ and, $\forall k \in \{1, ..., l\} - \{i, j\}$, we have that $f_k = 0\}$.

Lemma 3.3. *S_l is generating set of the $\mathcal{G} = (V_{l,p}, +)$.*

Let $S \subset \mathcal{G}$ be an identity free and symmetric generating set of a finite group $\mathcal{G}$. In the Cayley graph $\Gamma(\mathcal{G}, S_l) = (V, E)$ vertices associated to the elements of the group, i.e., $V(\Gamma) = \mathcal{G}$, and the edge $<v, w> \in E(\Gamma)$, if there is $s \in S_l$ and $v \in \mathcal{G}$ such that $w = v + s$. The identity free condition means that there are no loops in Γ, and the symmetry condition means that when there is an edge from v to $v + s$, there is also an edge from $v + s$ to $(v + s) - s = v$.

Now we have all definitions required to demonstrate that the graph $H_{l,p}$ is a Cayley graph of the group $\mathcal{G} = (V_{l,p}, +)$ and with the generating set S_l.

Theorem 3.4. *The graph $H_{l,p}$ is Cayley graph of the group $\mathcal{G} = (V_{l,p}, +)$ with the generating set S_l.*

In the next section, we utilize the result that the graph $H_{l,p}$ is a Cayley graph of an Abelian group to find a Hamiltonian cycle. So we will corroborate the conjecture of Lovász.

4 Hamiltonian cycle

In this section, we present the result of Marušič about Hamiltonian cycle in Cayley graph of an Abelian group [6]. Another results about Hamiltonian cycle in Cayley graph can be found in [7]. We show that the graph $H_{l,p}$ has a Hamiltonian cycle. Firstly, we present the definitions uses for Marušič.

Let $\mathcal{G}$ a group and e identity element. If $g \in G$, then $|g| = j$ denotes the order of g, such that, $g^j = e$.

Let M a subset of $\mathcal{G}$, we have following definitions: $M^{-1} = \{x^{-1} : x \in M\}$; $M_0 = M - e$; $M^* = M_0 \cup M_0^{-1}$; $\langle M \rangle$ is subset of $\mathcal{G}$ generated for M; If $\langle M \rangle = \mathcal{G}$, then M is calling generating set of $\mathcal{G}$.

A sequence on $\mathcal{G}$ is a sequence all of whose terms are elements of $\mathcal{G}$. The sequence without terms is denoted by $\emptyset$. Let the sequences $S = [s_1, s_2, ..., s_r]$ and $T = [t_1, t_2, ..., t_q]$, where $s_i \in G$ for $1 \leq i \leq r$ and $t_k \in G$ for $1 \leq k \leq q$, the product ST is denoted by $[s_1, s_2, ..., s_r, t_1, t_2, ...t_q]$. The partial product $\Pi_i(S)$ of S is formed by the group operation $s_1 s_2 ... s_i$.

Let $S = [s_1, s_2, ..., s_r]$ a sequence of $\mathcal{G}$, then we say that S is a Hamiltonian sequence if: $r = |\mathcal{G}|$; $\Pi_r(S) = e$; $\Pi_i(S) \neq \Pi_j(S)$ if $i \neq j$ for $1 \leq i, j \leq r$.

If $s_i \in M$ for $i = 1, 2, ..., r$ then a sequence S is denoted by $M-$sequence, and if S is Hamiltonian, then S is calling for M^*-sequence. The set $\mathcal{H}(M, G)$ is formed for all M^*-sequences of $\mathcal{G}$.

Lemma 4.1. *[6] The Cayley graph $\Gamma(\mathcal{G}, M)$ has Hamiltonian cycle if and only if $\mathcal{H}(M, \mathcal{G}) \neq \emptyset$.*

Lemma 4.2. *[6] Let M be a generating set of Abelian group $\mathcal{G}$ and M' be a non-empty subset of M_0. If S, $T \in \mathcal{H}(M', \left\langle M' \right\rangle)$ and $l_s = l_r$, then there is a sequence Q on $\mathcal{G}$, such that $\bar{S}Q$, $\bar{T}Q \in \mathcal{H}(M, \mathcal{G})$. Where $\bar{S}$ is the sequence S without the last element.*

Corollary 4.3. *[6] Every connected Cayley graph of an Abelian group of order at least three is Hamiltonian.*

By Theorem 3.4, we have that graph $H_{l,p}$ is a Cayley graph of an Abelian group. So, we obtain the following result.

Corollary 4.4. *The graph $H_{l,p}$ is Hamiltonian.*

We can use the results of Marušič to construct an algorithm that returns a Hamiltonian cycle in the graph $H_{l,p}$. By Lemma 4.2 there are two sequences $\bar{S}Q$ and $\bar{T}Q$. The algorithm only needs to construct a sequence $\bar{S}Q$. Let $\bar{S}R$ a Hamiltonian sequence of $\mathcal{H}(M', \left\langle M' \right\rangle)$. So, if $W = \bar{S}R$, let Q be the sequence $\bar{R}(W, [g]^j[l_w][g^{-1}]^{j-1}$ if j is odd and the sequence $\bar{R}(W, [g]^j[g](\bar{W})[g^{-1}]^{j-1}$ if j is even. We can apply the sequence $\bar{S}Q$ in any vertex to get an Hamiltonian cycle. This operation can be implemented with the complexity of $O(p^{l-1})$, because $|\bar{S}Q| = p^{l-1}$.

For example, in the graph $H_{3,4}$ we have a Hamiltonian sequence $\bar{S}Q = \{(1,0,3), (1,0,3), (1,0,3), (0,1,3), (3,0,1), (3,0,1), (0,1,3), (1,0,3), (1,0,3), (0,1,3), (3,0,1), (3,0,1), (3,0,1), (0,3,1), (0,3,1), (0,3,1)\ \}$. If we apply this sequence in the vertex $(0,0,0)$, we obtain the Hamiltonian cycle of graph $H_{3,4}$ $\{(1,0,3), (2,0,2), (3,0,1), (3,1,0), (2,1,1), (1,1,2), (1,2,1), (2,2,0), (3,2,3), (3,3,2), (2,3,3), (1,3,0), (0,3,1), (0,2,2), (0,1,3), (0,0,0)\ \}$.

5 Hamiltonian cycle with Gray code

The reflected binary codes are also known as Gray codes. These codes were invented by Frank Gray [1]. This term is used to refer any code of distance only one, i.e., adjacent code words differ only by one digit and one position. Some Gray codes are cyclic, i.e., the last element is at distance one from the first element.

In this section, we present another construction of a Hamiltonian cycle in the graph $H_{l,p}$. We use the $(p,l)-$Gray code. The idea is to create a graph G of the $(p,l-1)-$Gray code with $l-1$ elements in the base p and we show that the graph G is homomorphic to a cycle of the graph $H_{l,p}$.

Algorithm 1: $(p,l)-$Gray code

```
input  : valueBase10, l, p
output: valueGray
begin
    temp = valueBase10;
    for i = l - 1 until 0 do
        baseP[i] = temp mod p;
        temp = temp/p;
    temp = 0;
    for i = 0 until l - 1 do
        valueGray[i] = (baseP[i] - temp) mod p;
        temp = temp + (valueGray[i] - p);
    return valueGray;
```

Algorithm 1 transforms any value in a $(p,l)-$Gray code format with l digits in the base p. For example the sequence of elements in $(4,2)$-Gray code is $G = \{(0,0), (0,1), (0,2), (0,3), (1,3), (1,0), (1,1), (1,2), (2,2), (2,3), (2,0), (2,1), (3,1), (3,2), (3,3), (3,0)\}$. The Gray code G is homomorphic to the cycle $C = \{(0,0,0), (0,1,3), (0,2,2), (0,3,1), (1,3,0), (1,0,3), (1,1,2), (1,2,1), (2,2,0), (2,3,3), (2,0,2), (2,1,1), (3,1,0), (3,2,3), (3,3,2), (3,0,1)\}$. The cycle C is in the graph $H_{3,4}$.

We show that the last element of $(p,l)-$Gray code stays distance one digit to the first element, so we have a Gray code cycle.

Theorem 5.1. *The $(p,l)-$Gray code of Algorithm 1 is a cycle.*

We can implement an algorithm in time $O(lp^{l-1})$ to construct the vertices of the Hamiltonian cycle. For this, we utilize values between 0 and p^{l-1} and by Algorithm 1 we have values in the Gray code. But Algorithm 1 consumes time in the order constant of l digits for each code.

6 Conclusion

In this work, we study the properties of the graph $H_{l,p}$. This graph is important because it was used in the proof of $\mathcal{NP}$-completeness of the problem of the partition of the set of edges into cliques of size l. We show that the graph $H_{l,p}$ is a Cayley graph of an Abelian group. We corroborate with the conjecture of Lovász because the graph $H_{l,p}$ has a Hamiltonian cycle.

Finally, we present two different constructions for finding the Hamiltonian cycle. The first construction uses the result of Marušič. The second uses the Gray code. With the second construction, we have a more natural order of the vertices between 0 to p^{l-1} (the values in decimal base there are near in the Hamiltonian cycle), and it is possible to obtain the position in the cycle for a particular element by a formula, without visiting other vertices. We suggest as future work the use of Gray code to determine the position of each vertex in the Hamiltonian cycle in the Cayley graphs of Abelian groups.

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