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Adjacent-vertex-distinguishing total coloring of indifference graphs *

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Abstract

In this work, we study a variant of the classical total coloring problem of graphs, called adjacent-vertex-distinguishing total coloring. Given a graph with a total coloring, let $C(v)$ be the set of colors containing the color of vertex v and the colors of all edges incident to v. This problem asks for the minimum number of colors, χ''_a, of a total coloring such that any two adjacent vertices u and v, have $C(u) \neq C(v)$. It was conjectured that for a simple graph G, $\chi''_a(G) \leq \Delta(G)+3$, where $\Delta(G)$ denotes the maximum degree of G.

We restricted ourselves to indifference graphs, a subclass of interval graphs, which contains graphs that can be represented by fixed-size intervals of the real line. We prove the conjecture cited above for all indifference graphs. To do this, we design an algorithm that produces an adjacent-vertex-distinguishing total coloring for indifference graphs using at most $\Delta(G) + 3$ colors. Moreover, this algorithm gives an optimal solution depending on the input graph.

1 Introduction

A *total coloring* of a graph is a coloring of its vertices and edges such that no adjacent vertices, no adjacent edges, and no incident vertices and edges get the same color. *Edge coloring* and *vertex coloring* are partial cases of total coloring when only edges or only vertices are colored, respectively. A

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total coloring of a graph is an *adjacent-vertex-distinguishing total coloring* if, for any pair of adjacent vertices u and v, the set of colors, $C(u)$, assigned to the edges incident to u and the color assigned to u, differs from $C(v)$. We call the adjacent-vertex-distinguishing total coloring as *strong-total coloring*. The minimum number of colors needed in each of these problems is called the *total-chromatic number*, the *chromatic index*, the *chromatic number*, and the *strong-total-chromatic number*, respectively, and they are denoted by $\chi''(G)$, $\chi'(G)$, $\chi(G)$, and $\chi''_a(G)$, respectively.

In this paper, we consider the problem of determining the strong-total-chromatic number of a simple graph. This problem was introduced by Zhang et al. [20]. In the same paper, they have determined the strong-total-chromatic number for some families of graphs such as complete graphs, cycles, complete bipartite graphs, fan, wheel, and trees. Moreover, the authors have posed the following conjecture:

Conjecture 1. *Let G be a connected simple graph with at least two vertices. Then $\chi''_a(G) \leq \Delta(G) + 3$, where $\Delta(G)$ denotes the maximum degree of G.*

It is well known the upper bound for the chromatic index of a simple graph G, $\chi'(G) \leq \Delta(G) + 1$, given by Vizing [16], as well the *Total-coloring Conjecture*, $\chi''(G) \leq \Delta(G) + 2$, given, independently, by Behzad [3] and Vizing [16, 17]. The upper bound in Conjecture 1 is tight, since the strong-total-chromatic number for a complete graph $\chi''(K_p) = p + 2 = \Delta(K_p) + 3$, when $p \geq 3$ is odd. Moreover, Conjecture 1 has been confirmed for graphs with $\Delta(G) = 3$, independently by Wang [18] and Chen [6], hypercubes [5], outerplanar graphs [19], and Halin graphs [7]. Also, concise proofs for adjacent-vertex-distinguishing total coloring of some graphs are given by Hulgan [13].

The adjacent-vertex-distinguishing total coloring of a simple graph is related to vertex-distinguishing edge coloring, first considered by Burris and Schelp [4] and after discussed by many others authors [1, 2, 5, 21]. This problem is a restriction to the classical problem of edge coloring where the edges incident to two adjacent vertices must not be colored using the same set of colors. It is well known that it is NP-hard to determine the chromatic index and the total-chromatic number of a graph [15, 16]. The status of the complexity of the problem of determining the strong-total-chromatic number is not known.

In this work, we focus on indifference graphs. They are a well-known subclass of interval graphs (see, for instance, [12]) in which all intervals have the same (unitary) length. They can also be characterized as interval graphs without induced $K_{1,3}$ (or $K_{1,3}$-free interval graphs) [11] and can be recognized in linear time [8, 9]. Indifference graphs too can be characterized by a linear order: their vertices can be linearly ordered so that vertices contained in the same maximal cliques are consecutive [14]. This order is called *indifference order*.

The coloring problems so mentioned have been considered for indifference graphs, but both problems of determining the chromatic index and the total-chromatic number of a graph in the class remain open. Some partial results are known. Let G be an indifference graph. If $\Delta(G)$ is odd, $\chi'(G) = \Delta(G)$, otherwise $\chi'(G) \leq \Delta(G) + 1$ (this inequality is given by Vizing's theorem). If $\Delta(G)$ is even, $\chi''(G) = \Delta(G) + 1$, otherwise $\chi''(G) \leq \Delta(G) + 2$, i.e., the total-coloring conjecture holds [10]. These results were obtained using a *pullback technique*, which is a way to color a graph G transferring colors from a colored graph H to G. In this article, we use this technique to calculate an upper bound for the strong-total-chromatic number of indifference graphs and point out some cases where it is the optimum value.

In Section 2 we give notations used in this paper and bring some known results to the reader. The new results are described in Section 3 and a summary of conclusions of this work is given in Section 4.

2 Preliminaries and definitions

In this paper, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex and edge sets of G, respectively. Moreover, given a vertex v of $V(G)$ the set of vertices incident to v (called *open neighborhood*) is denoted as $N(v)$, while $N[v] = N(v) \cup \{v\}$ (the *closed neighborhood*). The degree of a vertex v is $|N(v)|$ and the maximum degree of all vertices in a graph is denoted as $\Delta(G)$.

In this work we use the pullback technique to give a strong-total coloring for any indifference graph. This technique was used in the context of the edge and total coloring of dually chordal graphs, a super class of indifference graphs.

For convenience of the reader, we recall the results from [10].

A *pullback* from G to G' is a function $f : V(G) \to V(G')$, such that if $uv \in E(G)$, then $f(u)f(v) \in E(G')$ and f is injective when restricted to $N[v]$, for all $x \in V(G)$.

We denote by G^2 the graph having $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if u and v are distinct and their distance in G is at most 2.

Lemma 2 ([10])**.** *If G is a complete graph K_p, a pullback from a graph G to K_p exists if and only if $\chi(G^2) \leq p$.*

Lemma 3 ([10])**.** *Let G be a dually chordal graph. Then $\chi(G^2) \leq \Delta(G)+1$.*

Since an indifference graph G is a dually chordal graph, there exists a pullback from G to $K_{\Delta(G)+1}$.

We say that α *appears* in v if $\alpha \in C(v)$, otherwise α is *missing* in v.

3 Strong-total coloring of indifference graphs

In this section, we consider a pullback of a complete graph to give a strong-total coloring for indifference graphs. This coloring is optimal if the graph satisfies some additional properties, discussed further. In any cases, the obtained coloring has at most $\Delta(G)+3$ colors, which is enough to prove the Conjecture 1 for indifference graphs.

For this purpose, we consider a known total coloring c of a complete graph K_p with p an odd number. It is known that $\chi''(K_p) = p$. Label the vertices of K_p with $\{v_0, \ldots, v_{p-1}\}$ and consider total coloring of K_p given by the following function $c : V(K_p) \cup E(K_p) \longrightarrow \{0, \ldots, p-1\}$.

$$c(v_i) = 2i \bmod p, \text{ and}$$
$$c(v_i, v_j) = (i + j) \bmod p.$$

Note that all colors appear in each vertex of K_p, because c is a total coloring of K_p with p colors.

Lemma 4. *Let G be an indifference graph and p be an odd integer such that $p \geq \Delta(G) + 2$. Then G has a strong-total coloring with p colors.*

Proof. Let $v_0, \ldots, v_{n-1}$ be an indifference order for an indifference graph G and let p be an odd integer such that $p \geq \Delta(G) + 2$. Consider the total coloring c of K_p as defined before and a function $\pi : V(G) \cup E(G) \longrightarrow \{0, \ldots, p-1\}$ defined bellow.

$$\begin{array}{ccccc} \pi(v_i) & = & c(v_{i \bmod p}) & = & 2i \bmod p, \text{ and} \\ \pi(v_i, v_j) & = & c(v_{i \bmod p}, v_{j \bmod p}) & = & (i+j) \bmod p \end{array}$$

By lemmas 2 and 3, there exists a pullback from G to a complete graph K_p. The function π is a pullback from G to a complete graph K_p and so π is a total coloring for G. In fact, the sequence $(0, 2, \ldots, p-1, 1, 3, \ldots, p-2)$ of colors is attributed repeatedly by the function π to vertices of G following the indifference order. Since for any vertex $v_i \in V(G)$, $|N[v_i]| \leq \Delta(G) + 1 < p$, the function π is injective when restricted to $N[v_i]$. Moreover, if $(v_i, v_j) \in E(G)$, the color $\pi(v_i, v_j)$ is the same color of the edge $(v_r, v_s) \in E(K_p)$ with $r = i \bmod p$ and $s = j \bmod p$.

Now we show that π is a strong-total coloring for G. For this purpose, we show that for each vertex v_i of G there exists a color α that is missing at v_i and α appears in each adjacent vertex v_j of v_i with $j > i$. In this way, for every two adjacent vertices of G, there is a color that appears in one of them and this same color misses at the other, proving the lemma.

Let v_i be a vertex of G and v_j be the rightmost vertex that is adjacent to v_i with respect to the indifference order. By construction, v_i is not adjacent to v_{j+1}. Moreover, v_i is not adjacent to v_{j+1-p}, otherwise v_i would have degree $p - 1 \geq \Delta(G) + 1$, a contradiction.

Since π is a pullback of a K_p, the color $\alpha = i + j + 1 \bmod p$ could appear only in edges (v_i, v_k) with $k = j+1+lp$, $l \in \mathbb{Z}$. This implies that any such v_k is not in $N[v_i]$ and v_i has no incident edge with color α. Moreover, $\pi(v_i) \neq \alpha$. In fact, if $\pi(v_i) = \alpha = \pi(v_i, v_{j+1})$, then $2i \bmod p = i + j + 1 \bmod p$. Thus $i = j + 1 + lp$, $l \in \mathbb{Z}$. On the other hand, by the choice of $j + 1$, this is impossible, since $i < j + 1$ and $j + 1 - p < i$. Therefore, $\alpha \notin C(v_i)$.

To finish the proof, let $\alpha = \pi(v_i, v_{j+1})$. Now, we show that color α appears in all vertices between v_{i+1} and v_j.

By definition of indifference order, $G[v_{i+1}, \ldots, v_j]$ is complete since $(v_i, v_j) \in E(G)$. Thus, by construction of π, $\alpha = \pi(v_i, v_{j+1}) = \pi(v_{i+k}, v_{j+1-k})$, for $1 \leq k < (j - i + 1)/2$. Moreover, if $j + 1 - i$ is even, the vertex $v_{(j+1-i)/2}$ also has color α. Therefore, color α appears in all vertices in $\{v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_j\}$ and the lemma follows. ■

Note that, for any graph G with at least two maximum-degree vertices that are adjacent, $\chi''_a(G) \geq \Delta(G) + 2$. The following corollaries follow immediately from Lemma 4.

Corollary 5. *Let G be an indifference graph with odd maximum degree. Then G has a strong-total coloring with $\Delta(G)+2$ colors. Moreover, if G has at least two adjacent vertices with maximum degree, then $\chi''_a(G) = \Delta(G)+2$.*

Corollary 6. *Let G be an indifference graph with even maximum degree. Then G has a strong-total coloring with $\Delta(G) + 3$ colors.*

These results give bounds on strong-total coloring of indifference graphs that are enough to state that indifference graphs satisfy the strong-total coloring conjecture.

Theorem 7. *If G is an indifference graph, then $\chi''_a(G) \leq \Delta(G) + 3$.*

Now, we consider a special case of indifference graphs with even maximum degree, and we give a total coloring which is also a strong-total coloring for these graphs.

Lemma 8. *Let G be an indifference graph with even maximum degree and no two adjacent maximum-degree vertices. Then $\chi''_a(G) = \Delta(G) + 1$.*

Proof. In this case, we are going to prove that $\chi''_a(G) = \chi''(G)$. By similar arguments of the Lemma 4, if $p = \Delta(G) + 1$ (which is an odd integer), the function π remains a pullback from G to K_p and, therefore, π is a total coloring of G.

Note that, since there are no adjacent maximum-degree vertices, $\chi''_a(G) \geq \Delta(G) + 1$. If v is a maximum-degree vertex of G, then all colors appear in v. A neighbor u of v has at least one color missing (because it is not a maximum-degree vertex), so $C(u) \neq C(v)$.

Like in Lemma 4, consider an indifference order of G and let v_i be a vertex that is not a maximum-degree vertex and v_j be the rightmost vertex in the order that is adjacent to v_i. Since, $i < j+1$ and $j+1-p = j-\Delta(G) < i$ (because $j < i + \Delta(G)$), the color $\alpha = \pi(v_i, v_{j+1})$ does not appear in v_i. By similar arguments of Lemma 4, color α appears in all vertices between v_{i+1} and v_j, which completes the proof. ■

4 Conclusions

We proved Conjecture 1 for all indifference graphs. The technique we used lead to linear-time algorithm for actually coloring the graph using at most $\Delta(G) + 3$ colors. Moreover, if the graph has even maximum degree and no adjacent maximum-degree vertices, or if the graph has odd maximum degree and at least two adjacent maximum-degree vertices, the algorithm finds an optimal coloring.

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