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LAWCliques'2010 problem session

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1 Open problems

In November 17th, 17h00 took place the LAWCliques'2010 problem session. Liliana Alcón, Aline Ribeiro de Almeida, Andreas Brandstädt, Luerbio Faria, Miguel Algel Pizanã, and Jayme Luiz Szwarcfiter posed problems such as, simplicial vertices on clique graphs, lions versus men at an arena, characterization of k-leaf powers, splitting number algorithms, fundamental group of clique-Helly graphs, and normal model of circular-arc graphs. In this section we catalog the contributions stated by these authors.

Open problems on hereditary clique graphs

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A *complete set* of a graph G is a subset of vertices inducing a complete subgraph. A *clique* is a maximal complete set. Let $\mathcal{C}(G)$ be the *clique family* of G.

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Key Words and Phrases: simplicial vertices on clique graphs, lions versus men at an arena, characterization of k-leaf powers, splitting number algorithms, fundamental group of clique-Helly graphs, normal model of circular-arc graphs.

The *clique graph* of G, denoted by $K(G)$, is the intersection graph of $\mathcal{C}(G)$. Say that G is *a clique graph* if there exists a graph H such that $G = K(H)$.

A vertex v is said to be simplicial when its open neighborhood $N(v)$ is a complete set.

We want to know if the following statement is true.

If v is a simplicial vertex of a clique graph G then $G - v$ is a clique graph.

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A lion and a man in an circular arena. Shall the man survive?

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There is a man and a lion in a circular closed arena. No one gets in, and no one gets out. The lion is hugely hungry and its goal is to catch the man. The man, in turn, wishes to keep himself alive and will try his best to do so. The arena's radius is R meters, where R is a finite real number. The lion and the man have equal maximum speed. Considering these facts, shall the man survive? In other words: is there a escape strategy such that, no matter how smart the lion is, the man can run away from the lion indefinitely?

I first read about this problem in a book by Bollobás[1], whose bibliography included [3] and [2]. The author explains that the problem was invented in the late 1930s by Richard Rado, and then popularized in [2].

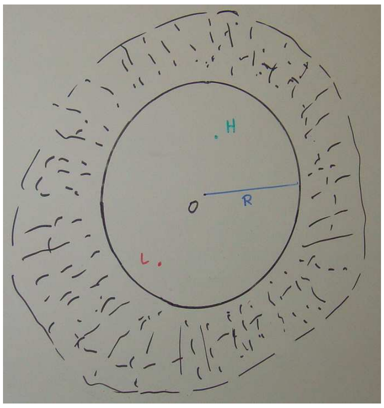

Figure 1: Lion L and man H in a circular closed arena of radius R.

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Open problems on Leaf Powers

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Nishimura, Ragde and Thilikos [4] defined the following notions: A tree T is a k*-leaf root* of a finite undirected graph $G = (V, E)$ if the set of leaves of T is V and for any two vertices $x, y \in V$, $xy \in E$ if and only if the distance of x and y in T is at most k. Graph G is a k*-leaf power* if it has a k-leaf root; it is a *leaf power* if it is a k-leaf power for some $k \geq 2$.

Obviously, a graph is a 2-leaf power if and only if it is the disjoint union of cliques. Characterizations of 3-leaf powers [1] and of 4-leaf powers [2, 5] as well as linear time recognition algorithms for 3-, 4- and 5-leaf powers are known; for 5-leaf powers, a linear time recognition algorithm was given in [3]. A nice characterization of this class, however, is missing. For $k \geq 6$, characterizing k-leaf powers as well as characterizing leaf powers is a challenging open problem.

The characterization of k-leaf powers for every $k \geq 6$ and the efficient recognition of these graphs remains a challenging open problem. A characterization and efficient recognition of leaf powers in general is also an open problem.

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A polynomial algorithm for checking whether the splitting number of a graph is 1

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Let $G = (V, E)$ be a graph and $v \in V$, the *open neighborhood* $N_G(v)$ of the vertex v consists of the vertices adjacent to v in G. The *splitting operation* of a vertex v of a graph $G = (V, E)$ obtains a graph G' from G by inserting a new vertex v' to G such that $(N_{G'}(v), N_{G'}(v'))$ is a partition for

$N_G(v)$. Notice that v and v' are nonadjacent in G'. The *splitting number* of a graph G is the minimum number of splitting operations which obtains a planar graph from G. Given a graph $G = (V, E)$ and a positive integer k it is known [2] that the combinatoric problem of checking whether the splitting number of G is less than or equal to k is NP-complete even for cubic graphs. However, it follows from the famous Robertson-Seymour minor Theorem [1] that checking whether a graph has splitting number less than or equal to k for a fixed k is a polynomial problem, since the class of the graphs with splitting number less than or equal to k is minor closed. Therefore, the problem is: devise a polynomial time algorithm to check whether the splitting number of a graph is less than or equal to 1.

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The fundamental group of clique-Helly graphs

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Problem: Is it true that if G is a non trivial, clique-Helly graph without dominated vertices, then its fundamental group $\pi_1(G)$ is non-trivial?

A number of definitions are involved in this problem:

A *clique* is a maximal complete subgraph. A graph G is *clique-Helly* if the set of all cliques satisfy the Helly property (every pairwise intersecting family of cliques has a non-empty total intersection). It is known [1, 6] that a graph G is clique-Helly if and only if every extended triangle (the subgraph induced by the vertices having at least two neighbors in a given triangle) is a cone (has a universal vertex). Hence, the clique-Helly property can be verified in polynomial time. We say that a vertex x is *dominated* if there is a (different) vertex y such that $N[x] \subseteq N[y]$ (i.e. the closed neighborhood of y contains that of x).

The *clique complex* $\Delta(G)$ of a graph G is the one whose simplices are exactly the complete subgraphs of G. Then the *geometric realization* $|\Delta(G)|$ of $\Delta(G)$ is a topological space [5]. Now, we can attach topological properties to a graph G by means of this geometric realization $|\Delta(G)|$ [2], in particular, we define the *fundamental group*[2, 5] $\pi_1(G)$ of a graph G as that of $|\Delta(G)|$, i.e. $\pi_1(G) := \pi_1(|\Delta(G)|)$.

The problem we present here concerns only with the non triviality of $\pi_1(G)$. Fortunately, there is a combinatorial characterization of such property (see [2, 3, 4, 5]), namely: $\pi_1(G)$ is trivial if and only if, every closed walk $x_0 \simeq x_1 \simeq x_2 \simeq \cdots \simeq x_n \simeq x_0$ (consecutive vertices are either adjacent or equal) in G can be transformed into a trivial closed walk $y \simeq y$, by means of the elementary operation that transforms $\cdots \simeq a \simeq b \simeq c \simeq \cdots$ into $\cdots \simeq a \simeq c \simeq \cdots$ and the inverse elementary operation. Note that $\{a, b, c\}$ must induce a complete subgraph in G in order to apply the said elementary operation to the closed walk. Also, note that a, b and c may not be different i.e. $1 \leq |\{a, b, c\}| \leq 3$. In the usual characterization [5, 4], it is required the base point (the initial and final vertex of the closed walk) x_0 to be always fixed, but it is not difficult to show that this condition is not required here.

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Open Problem on Circular-Arc Graphs

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A *(circular-arc) model* $(C, \mathcal{A})$ for a graph G is a circle C, together with a collection of arcs $\mathcal{A}$ of C, such that there is a one-to-one correspondence between vertices of G and arcs of $\mathcal{A}$, such that two vertices are adjacent if and only if the corresponding arcs intersect. A *circular-arc graph* is a graph admitting a model. A *normal model* is one in which no two arcs entirely cover the circle. Finally, a circular-arc graph is *normal* is it admits a normal model.

The problem consists of determining whether a circular-arc graph is normal. In addition, find a collection of minimal induced forbidden subgraphs for this class.

It has been proved by Hell and Huang [1] that interval bigraphs are precisely those bipartite graphs whose complements are normal. However, besides the characterization by forbidden subgraphs, some relevant questions also remain. What about if the graph is not co-bipartite ? What is the complexity of recognizing normal circular-arc graphs ? For co-bipartite graphs, there is a polynomial-time recognition algorithm by Müller [2], with complexity $O(n^5 m^6 \log n)$. Is it possible to formulate a faster algorithm ?

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