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Existence and qualitative properties of kinetic functions generated by diffusive-dispersive regularizations

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Abstract

We investigate the properties of traveling wave solutions to hyperbolic conservation laws augmented with diffusion and dispersion, and review the existence and qualitative properties of the associated kinetic functions, which characterize the class of admissible shock waves selected by such regularizations.

1 Introduction

In this paper, we investigate the properties of traveling wave solutions to hyperbolic conservation laws augmented with diffusion and dispersion, and reviews the existence and qualitative properties of the associated kinetic functions. Such a function characterizes the class of admissible shock waves, both compressive and undercompressive, selected by a given regularization. Building on the pioneering papers [24, 25, 1, 16], the mathematical research on undercompressive shocks generated by diffusive-dispersive

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limits developed intensively in the last fifteen years. For background on this topics and further material, we refer the reader to the reviews [17, 18, 19] and the extensive literature cited therein. The present review restrict attention to traveling waves and to a class of scalar equations.

The kinetic relation can be defined as follows. Recall that classical compressive shocks with a given left-hand state u_- (and wave family, when systems of equations are considered) form a one-parameter family of solutions, parametrized by their right-hand state u_+. By contrast, given any left-hand state u_- (and wave family), there typically exists a single undercompressive shock, and the kinetic function $\varphi^\flat$ precisely determines the right-hand state

$$u_+ = \varphi^\flat(u_-)$$

as a function of the left-hand side.

The fundamental questions of interest are the following ones: do there exist traveling wave solutions associated with classical and/or with non-classical shock waves ? Can one associate a kinetic function to the given model ? If so, is this kinetic function monotone ? What is the behavior of arbitrarily small shocks ? How does the kinetic function depend upon the parameters?

Answers to these questions were obtained first for the cubic flux function, by deriving *explicit* formulas for the kinetic function in Shearer et al. [15] and Hayes and LeFloch [12]. General flux-functions and general regularization were covered by Bedjaoui and LeFloch in the series of papers [3]–[7].

More generally, the existence and properties of traveling waves for the nonlinear elasticity and the Euler equations are known in both the hyperbolic [22, 4] and the hyperbolic-elliptic regimes [25, 23, 8, 5]. For all other models, only partial results on traveling waves are available.

The existence of nonclassical traveling wave solutions for the thin liquid film model is proven by Bertozzi and Shearer in [10]. For this model, no qualitative information on the properties of these traveling waves is known, and, in particular, the existence of the kinetic relation has not been rigorously established yet. The kinetic function was recently determined numerically in LeFloch and Mohamadian [20]. For the 3×3 Euler equations, we refer to [7].

Finally, we also recall that the Van de Waals model admits *two* inflection points and leads to *multiple* traveling wave solutions. Although the physical significance of the "second" inflection point is questionable, given that this model is extensively used in the applications it is important to investigate whether additional features arise. Indeed, it is established in [2] that *non-monotone nonclassical* traveling wave profiles exist, and that a single kinetic function is not sufficient to single out the physically relevant solutions.

An outline of this paper is as follows. In Section 2, we briefly discuss the case of the diffusion model, while the rest of the paper is concerned with the diffusion-dispersion model. We then begin with the case of a cubic flux-function for which explicit formulas can be derived. The main results are stated in Section 4 for general flux-functions having one inflection point. Sections 5 and 6 are concerned with the derivations of key properties of the traveling waves and kinetic function, corresponding to a fixed shock speed and to a fixed diffusion over dispersion ratio, respectively.

2 Traveling waves associated with the nonlinear diffusion model

Consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth mapping. We begin, in this section, with the *nonlinear diffusion model*

$$\partial_t u + \partial_x f(u) = \varepsilon \left(b(u)\, u_x\right)_x, \quad u = u^\varepsilon(x,t) \in \mathbb{R}, \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter. The *diffusion function* $b : \mathbb{R} \to \mathbb{R}_+$ is assumed to be smooth and bounded below:

$$b(u) \geq \bar{b} > 0, \tag{2.3}$$

so that the equation (2.2) is *uniformly parabolic.* We are going to establish that the shock set associated with the traveling wave solutions of (2.2) coincides with the one described by Oleinik entropy inequalities (see (2.9), below).

Recall that a *traveling wave* of (2.2) is a solution depending only upon the variable

$$y := \frac{x - \lambda\, t}{\varepsilon} \tag{2.4}$$

for some constant speed λ. Note that, after rescaling, the corresponding *trajectory* $y \mapsto u(y)$ is independent of the parameter ε. Fixing the left-hand state u_- we search for traveling waves of (2.2) *connecting* u_- *to some state* u_+, that is, solutions $y \mapsto u(y)$ of the ordinary differential equation

$$-\lambda\, u_y + f(u)_y = \big(b(u)\, u_y\big)_y \tag{2.5}$$

satisfying the boundary conditions

$$\lim_{y \to -\infty} u(y) = u_-, \qquad \lim_{y \to +\infty} u(y) = u_+, \qquad \lim_{|y| \to +\infty} u_y(y) = 0. \tag{2.6}$$

In view of (2.6) the equation (2.5) can be integrated once:

$$b(u(y))\, u_y(y) = -\lambda\, (u(y) - u_-) + f(u(y)) - f(u_-), \quad y \in \mathbb{R}. \tag{2.7}$$

The Rankine-Hugoniot condition

$$-\lambda\, (u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{2.8}$$

follows by letting $y \to +\infty$ in (2.7). The equation (2.7) is an ordinary differential equation (O.D.E) on the real line. The qualitative behavior of the solutions is easily determined, as follows.

Theorem 2.1 (Diffusive traveling waves)**.** *Consider the scalar conservation law* (2.1) *with general flux-function f together with the diffusive model* (2.2)*. Fix a left-hand state u_- and a right-hand state $u_+ \neq u_-$. Then, there exists a traveling wave of* (2.7) *associated with the nonlinear diffusion model* (2.2) *if and only if u_- and u_+ satisfy* Oleinik entropy *inequalities in the strict* sense, *that is:*

$$\frac{f(v)-f(u_-)}{v-u_-} > \frac{f(u_+)-f(u_-)}{u_+-u_-} \quad \text{for all } v \text{ lying strictly between } u_- \text{ and } u_+. \tag{2.9}$$

Proof. *All the trajectories of interest are bounded, i.e., cannot escape to infinity. Namely, the shock profile satisfies the equation*

$$u' = \frac{u-u_-}{b(u)}\left(\frac{f(u)-f(u_-)}{u-u_-} - \frac{f(u_+)-f(u_-)}{u_+-u_-}\right). \tag{2.10}$$

It is not difficult to see that the solution exists and connects monotonically u_- to u_+ provided Oleinik entropy inequalities hold and the right-hand side of (2.10) *keeps (strictly) a constant sign (except at the end point $y = \pm\infty$ where it vanishes).*

□

We define the *shock set* associated with the nonlinear diffusion model as

$$S(u_-) := \big\{ u_+ \,/\, \text{ there exists a solution of (2.6)} - -\text{(2.8)} \big\}.$$

From Theorem 2.1 one can deduce the following.

Theorem 2.2 (Shock set based on diffusive limits)**.** *Consider the scalar conservation law* (2.1) *when the flux f is convex, concave-convex, or convex-concave (see* (4.2)*, below). Then, for any u_-, the shock set $S(u_-)$ associated with the nonlinear diffusion model* (2.2) *and* (2.3) *is independent of the diffusion function b, and the closure of $S(u_-)$ coincides with the shock set characterized by Oleinik entropy inequalities (or, equivalently, Lax shock inequalities).*

Remark 2.3. *The conclusions of Theorem 2.2 do not hold for more general flux-functions. This is due to the fact that a strict inequality is required in* (2.9) *for the existence of the traveling waves. The set based on traveling waves may be strictly smaller than the one based on Oleinik entropy inequalities.*

3 Kinetic functions associated with cubic flux-functions

Investigating traveling wave solutions of diffusive-dispersive regularizations of (2.1) is considerably more involved than what was done in Section 2. Besides proving the existence of associated (classical and nonclassical) traveling waves our main objective will be to derive the corresponding kinetic functions for nonclassical shocks.

To explain the main difficulty and ideas it will be useful to treat first, in the present section, the specific *diffusive-dispersive model with cubic flux*

$$\partial_t u + \partial_x u^3 = \varepsilon\, u_{xx} + \delta\, u_{xxx}, \tag{3.1}$$

which, formally as $\varepsilon, \delta \to 0$, converges to the *conservation law with cubic flux*

$$\partial_t u + \partial_x u^3 = 0. \tag{3.2}$$

We are interested in the singular limit $\varepsilon \to 0$ in (3.1) when the ratio

$$\alpha = \frac{\varepsilon}{\sqrt{\delta}} \tag{3.3}$$

is kept constant. We assume also that the dispersion coefficient δ is positive. Later, in Theorem 4.5 below, we will see that all traveling waves are classical when $\delta < 0$ which motivates us to restrict attention to $\delta > 0$.

We search for traveling wave solutions of (3.1) depending on the rescaled variable

$$y := \alpha\, \frac{x - \lambda\, t}{\varepsilon} = \frac{x - \lambda\, t}{\sqrt{\delta}}. \tag{3.4}$$

Proceeding along the same lines as those in Section 2 we find that a traveling wave $y \mapsto u(y)$ should satisfy

$$-\lambda\, u_y + (u^3)_y = \alpha\, u_{yy} + u_{yyy}, \tag{3.5}$$

together with the boundary conditions

$$\begin{aligned} &\lim_{y\to\pm\infty} u(y) = u_\pm, \\ &\lim_{y\to\pm\infty} u_y(y) = \lim_{y\to\pm\infty} u_{yy}(y) = 0, \end{aligned} \tag{3.6}$$

where $u_- \neq u_+$ and λ are constants. Integrating (3.5) once we obtain

$$\alpha\, u_y(y) + u_{yy}(y) = -\lambda\,(u(y) - u_-) + u(y)^3 - u_-^3, \quad y \in \mathbb{R}, \tag{3.7}$$

which also implies

$$\lambda = \frac{u_+^3 - u_-^3}{u_+ - u_-} = u_-^2 + u_-\, u_+ + u_+^2. \tag{3.8}$$

To describe the family of traveling waves it is convenient to fix the left-hand state (with for definiteness $u_- > 0$) and to use the speed λ as a parameter. Given u_-, there is a range of speeds,

$$\lambda \in (3\, u_-^2/4, 3\, u_-^2),$$

for which the line passing through the point with coordinates (u_-, u_-^3) and with slope λ intersects the graph of the flux $f(u) := u^3$ at three distinct points. For the discussion in this section we restrict attention to this situation, which is most interesting. There exist *three equilibria* at which the right-hand side of (3.7) vanishes. The notation

$$u_2 < u_1 < u_0 := u_-$$

will be used, where u_2 and u_1 are the two distinct roots of the polynomial

$$u^2 + u_0\, u + u_0^2 = \lambda. \tag{3.9}$$

Observe in passing that $u_2 + u_1 + u_0 = 0$.

Consider a trajectory $y \mapsto u(y)$ leaving from u_- at $-\infty$. We want to determine which point, among u_1 or u_2, the trajectory will reach at $+\infty$. Clearly, the trajectory is associated with a so-called *classical shock* if it reaches u_1 and with a so-called *nonclassical shock* if it reaches u_2. Accordingly, we will refer to it as a *classical trajectory* or as a *nonclassical trajectory*, respectively.

We reformulate (3.7) as a differential system of two equations,

$$\frac{d}{dy}\begin{pmatrix} u \\ v \end{pmatrix} = K(u,v), \tag{3.10}$$

where

$$K(u,v) = \begin{pmatrix} v \\ -\alpha\, v + g(u,\lambda) - g(u_-,\lambda) \end{pmatrix}, \quad g(u,\lambda) = u^3 - \lambda\, u. \tag{3.11}$$

The function K vanishes precisely at the three equilibria $(u_0, 0)$, $(u_1, 0)$, and $(u_2, 0)$ of (3.10). The eigenvalues of the Jacobian matrix of $K(u,v)$ at any point $(u,0)$ are $-\alpha/2 \pm \sqrt{\alpha^2/4 + g'_u(u,\lambda)}$. So we set

$$\begin{aligned} \underline{\mu}(u) &= \frac{1}{2}\Big(-\alpha - \sqrt{\alpha^2 + 4\,(3\,u^2 - \lambda)}\Big), \\ \overline{\mu}(u) &= \frac{1}{2}\Big(-\alpha + \sqrt{\alpha^2 + 4\,(3\,u^2 - \lambda)}\Big). \end{aligned} \tag{3.12}$$

At this juncture, we recall the following standard definition and result. (See the bibliographical notes for references.)

Definition 3.1 (Nature of equilibrium points)**.** *Consider a differential system of the form* (3.10) *where K is a smooth mapping. Let $(u_*, v_*) \in \mathbb{R}^2$ be an equilibrium point, that is, a root of $K(u_*, v_*) = 0$. Denote by $\underline{\mu} = \underline{\mu}(u_*, v_*)$ and $\overline{\mu} = \overline{\mu}(u_*, v_*)$ the two (real or complex) eigenvalues of the Jacobian matrix of K at (u_*, v_*), and suppose that a basis of corresponding eigenvectors $\underline{r}(u_*, v_*)$ and $\overline{r}(u_*, v_*)$ exists. Then, the equilibrium (u_*, v_*) is called*

1. *a stable point if $Re(\underline{\mu})$ and $Re(\overline{\mu})$ are both negative,*

2. *a saddle point if* $Re(\underline{\mu})$ *and* $Re(\overline{\mu})$ *have opposite sign,*

3. *or an unstable point if* $Re(\underline{\mu})$ *and* $Re(\overline{\mu})$ *are both positive.*

Moreover, a stable or unstable point is called a node *if the eigenvalues are real and a* spiral *if they are complex conjugate.*

Theorem 3.2 (Local behavior of trajectories)**.** *Consider the differential system* (3.10) *under the same assumptions as in Definition 2.1. If* (u_*, v_*) *is a saddle point, there are two trajectories defined on some interval* $(-\infty, y_*)$ *and two trajectories defined on some interval* $(y_*, +\infty)$ *and converging to* (u_*, v_*) *at* $-\infty$ *and* $+\infty$*, respectively. The trajectories are tangent to the eigenvectors* $\underline{r}(u_*, v_*)$ *and* $\overline{r}(u_*, v_*)$*, respectively.*

Returning to (3.11) and (3.12) we conclude that, since $g'_u(u, \lambda) = 3u^2 - \lambda$ is positive at both $u = u_2$ and $u = u_0$, we have

$$\underline{\mu}(u_0) < 0 < \overline{\mu}(u_0), \quad \underline{\mu}(u_2) < 0 < \overline{\mu}(u_2).$$

Thus both points u_2 and u_0 are *saddle points.* On the other hand, since we have $g'_u(u_1, \lambda) < 0$, the point u_1 is stable: it is a *node* if $\alpha^2 + 4\,(3\,u_1^2 - \lambda) \geq 0$ or a *spiral* if $\alpha^2 + 4\,(3\,u_1^2 - \lambda) < 0$. In summary, for the system (3.10)-(3.11)

$$\begin{aligned} &u_2 \text{ and } u_0 \text{ are saddle points and} \\ &u_1 \text{ is a stable point (either a node or a spiral).} \end{aligned} \tag{3.13}$$

In the present section we check solely that, in some range of the parameters u_0, λ, and α, there exists a *nonclassical trajectory* connecting the two saddle points u_0 and u_2. *Saddle-saddle connections* are not "generic" and, as we will show, arise only when a special relation (the kinetic relation) holds between u_0, λ, and α or, equivalently, between u_0, u_2, and α; see (3.15) below.

For the cubic model (3.1) an *explicit formula* is now derived for the nonclassical trajectory. Motivated by the fact that the function g in (3.11) is a cubic, we a priori assume that $v = u_y$ is a *parabola in the variable u*.

Since v must vanish at the two equilibria we write

$$v(y) = a\,(u(y) - u_2)\,(u(y) - u_0), \quad y \in \mathbb{R}, \tag{3.14}$$

where a is a constant to be determined. Substituting (3.14) into (3.10)-(3.11), we obtain an expression of v_y:

$$\begin{aligned} v_y &= -\alpha\, v + u^3 - u_0^3 - \lambda\,(u - u_0) \\ &= -\alpha\, v + (u - u_2)\,(u - u_0)\,(u + u_0 + u_2) \\ &= v\,\big(-\alpha + \frac{1}{a}\,(u + u_0 + u_2)\big). \end{aligned}$$

But, differentiating (3.14) directly we have also

$$\begin{aligned} v_y &= a\, u_y\,(2\,u - u_0 - u_2) \\ &= a\, v\,(2\,u - u_0 - u_2). \end{aligned}$$

The two expressions of v_y above coincide if we choose

$$\frac{1}{a} = 2\,a, \quad -\alpha + \frac{1}{a}\,(u_0 + u_2) = -a(u_0 + u_2).$$

So, $a = 1/\sqrt{2}$ (since clearly we need $v < 0$) and the three parameters u_0, u_2, and α satisfy the *explicit relation*

$$u_2 = -u_0 + \frac{\sqrt{2}}{3}\,\alpha. \tag{3.15}$$

Since $u_1 = -u_0 - u_2$ we see that the trajectory (3.14) is the saddle-saddle connection we are looking for, only if $u_2 < u_1$ as expected, that is, only if

$$u_0 > \frac{2\sqrt{2}}{3}\,\alpha. \tag{3.16}$$

Now, by integrating (3.14), it is not difficult to arrive at the following *explicit formula for the nonclassical trajectory*:

$$\begin{aligned} u(y) &= \frac{u_0 + u_2}{2} - \frac{u_0 - u_2}{2}\,\tanh\Big(\frac{u_0 - u_2}{2\sqrt{2}}\,y\Big) \\ &= \frac{\alpha}{3\sqrt{2}} - \Big(u_- - \frac{\alpha}{3\sqrt{2}}\Big)\,\tanh\Big(\Big(u_- - \frac{\alpha}{3\sqrt{2}}\Big)\,\frac{y}{\sqrt{2}}\Big). \end{aligned} \tag{3.17}$$

We conclude that, given any left-hand state $u_0 > 2\sqrt{2}\,\alpha/3$, there exists a saddle-saddle connection connecting u_0 to $-u_0 + \sqrt{2}\,\alpha/3$ which is given by (3.17). Later, in Section 4 and followings, we will prove that the trajectory just found is actually the *only* saddle-saddle trajectory leaving from $u_0 > 2\sqrt{2}\,\alpha/3$ and that no such trajectory exists when u_0 is below that threshold.

Now, denote by $\mathcal{S}_\alpha(u_-)$ the set of all right-hand states u_+ attainable through a diffusive-dispersive traveling wave of (3.1) with $\delta > 0$ and $\varepsilon/\sqrt{\delta} = \alpha$ fixed. In the case of the equation (3.1) the results to be established in the following sections can be summarized as follows.

Theorem 3.3 (Kinetic function and shock set for the cubic flux)**.** *The kinetic function associated with the diffusive-dispersive model* (3.1) *is*

$$\varphi^\flat_\alpha(u_-) = \begin{cases} -u_- - \tilde{\alpha}/2, & u_- \le -\tilde{\alpha}, \\ -u_-/2, & |u_-| \le \tilde{\alpha}, \\ -u_- + \tilde{\alpha}/2, & u_- \ge \tilde{\alpha}, \end{cases} \tag{3.18}$$

with $\tilde{\alpha} := 2\,\alpha\,\sqrt{2}/3$, *while the corresponding shock set is*

$$\mathcal{S}_\alpha(u_-) = \begin{cases} (u_-, \tilde{\alpha}/2] \cup \big\{ -u_- - \tilde{\alpha}/2 \big\}, & u_- \le -\tilde{\alpha}, \\ [-u_-/2, u_-), & -\tilde{\alpha} \le u_- \le \tilde{\alpha}, \\ \big\{ -u_- + \tilde{\alpha}/2 \big\} \cup [-\tilde{\alpha}/2, u_-), & u_- \ge \tilde{\alpha}. \end{cases} \tag{3.19}$$

In agreement with the general theory of the kinetic function, (3.18) is monotone decreasing and lies between the limiting functions $\varphi^\natural(u) := -u/2$ and $\varphi^\flat_0(u) := -u$. Depending on u_- the shock set can be either an interval or the union of a point and an interval.

Consider next the *entropy dissipation* associated with the nonclassical shock:

$$\begin{aligned} E(u_-;\alpha,U) := &- \big(\varphi^\flat_\alpha(u_-)^2 + \varphi^\flat_\alpha(u_-)\,u_- + u_-^2\big)\,\big(U(\varphi^\flat_\alpha(u_-)) - U(u_-)\big) \\ &+ F(\varphi^\flat_\alpha(u_-)) - F(u_-), \end{aligned} \tag{3.20}$$

where (U, F) is any convex entropy pair of the equation (3.2). By multiplying (3.5) by $U'(u(y))$ and integrating over $y \in \mathbb{R}$ we find the equivalent expression

$$\begin{aligned} E(u_-; \alpha, U) &= \int_{\mathbb{R}} U'(u(y)) \big(\alpha\, u_{yy}(y) + u_{yyy}(y)\big)\, dy \\ &= \int_{\mathbb{R}} \big(-\alpha\, U''(u)\, u_y^2 + U'''(u)\, u_y^3/2\big)\, dy. \end{aligned} \tag{3.21}$$

So, the sign of the entropy dissipation can also be determined from the explicit form (3.17) of the traveling wave.

Theorem 3.4 (Entropy inequalities)**.** *1. For the quadratic entropy*

$$U(z) = z^2/2, \quad z \in \mathbb{R},$$

the entropy dissipation $E(u_-; \alpha, U)$ is non-positive for all real u_- and all $\alpha \geq 0$.

2. *For all convex entropy U the entropy dissipation $E(u_-; \alpha, U)$ is non-positive for all $\alpha > 0$ and all $|u_-| \leq 2\sqrt{2}\,\alpha/3$.*
3. *Consider $|u_-| > 2\sqrt{2}\,\alpha/3$ and any (convex) entropy U whose third derivative is sufficiently small, specifically*

$$\big(|u_-| - \alpha/(3\sqrt{2})\big)^2\, |U'''(z)| \leq 2\alpha\,\sqrt{2}\, U''(z), \quad z \in \mathbb{R}. \tag{3.22}$$

Then, the entropy dissipation $E(u_-; \alpha, U)$ is also non-positive.

4. *Finally given any $|u_-| > 2\sqrt{2}\,\alpha/3$ there exists infinitely many strictly convex entropies for which $E(u_-; \alpha, U)$ is positive.*

Proof. *When U is quadratic (with $U'' \geq 0$ and $U''' \equiv 0$) we already observed that Item 1 follows immediately from* (3.21)*. The statement Item 2 is also obvious since the function $\varphi^\flat$ reduces to a classical value in the range under consideration. Under the condition* (3.22) *the integrand of* (3.21) *is non-positive, as follows from the inequality (see* (3.14)*)*

$$|u_y| \leq \frac{1}{4\sqrt{2}}\,(u_0 - u_2)^2 = \frac{1}{\sqrt{2}}\,\big(u_- - \alpha/(3\sqrt{2})\big)^2.$$

This implies the statement Item 3. Finally, to derive Item 4 we use the (Lipschitz continuous) Kruzkov entropy pairs

$$U_k(z) := |z-k|, \quad F_k(z) := \operatorname{sgn}(z-k)(z^3-k^3), \quad z \in \mathbb{R}, \tag{3.23}$$

with the choice $k = -u_-/2$. *We obtain*

$$E(u_-;\alpha, U_k) = \frac{3}{4}\,|u_-|\,\big(|u_-| - 2\alpha\sqrt{2}/3\big)^2 > 0.$$

By continuity, $E(u_-;\alpha,U_k)$ *is also strictly positive for all* k *in a small neighborhood of* $-u_-/2$. *The desired conclusion follows by observing that any smooth convex function can be represented by a weighted sum of Kruzkov entropies.*

□

Remark 3.5. *We collect here the explicit expressions of some functions associated with the model* (3.1)*. From now on we restrict attention to the entropy pair*

$$U(u) = u^2/2, \quad F(u) = 3\,u^4/4.$$

First of all, recall that for the equation (3.2) *the following two functions*

$$\varphi^\natural(u) = -\frac{u}{2}, \quad \varphi_0^\flat(u) = -u, \quad u \in \mathbb{R}. \tag{3.24}$$

determine the admissible range of the kinetic functions.

We define the critical diffusion-dispersion ratio

$$A(u_0,u_2) = \frac{3}{\sqrt{2}}\,(u_0+u_2) \tag{3.25}$$

for $u_0 \geq 0$ *and* $u_2 \in (-u_0, -u_0/2)$ *and for* $u_0 \leq 0$ *and* $u_2 \in (-u_0/2, -u_0)$*. In view of Theorem 3.3 (see also* (3.15)*), a nonclassical trajectory connecting* u_0 *to* u_2 *exists if and only if the parameter* $\alpha = \varepsilon/\sqrt{\delta}$ *equals* $A(u_0,u_2)$*. The function* A *increases monotonically in* u_2 *from the value* 0 *to the threshold diffusion-dispersion ratio* ($u_0 > 0$)

$$A^\natural(u_0) = \frac{3\,u_0}{2\sqrt{2}}. \tag{3.26}$$

For each fixed state $u_0 > 0$ there exists a nonclassical trajectory leaving from u_0 if and only if α is less than $A^\natural(u_0)$. On the other hand, for each fixed α there exists a nonclassical trajectory leaving from u_0 if and only if the left-hand state u_0 is greater than ${A^\natural}^{-1}(\alpha)$. The function $A^\natural$ is a linear function (for $u_0 > 0$) with range extending therefore from $\underline{A^\natural} = 0$ to $\overline{A^\natural} = +\infty$.

Remark 3.6. *It is straightforward to check that if* (3.1) *is replaced with the more general equation*

$$\partial_t u + \partial_x \big(K\, u^3\big) = \varepsilon\, u_{xx} + \delta\, C\, u_{xxx}, \tag{3.27}$$

where C and K are positive constants, then (3.26) *becomes*

$$A^\natural(u_0) = \frac{3\,u_0}{2\,\sqrt{2}} \sqrt{K\,C}. \tag{3.28}$$

Remark 3.7. *Clearly, there is a one-parameter family of traveling waves connecting the same end states: If $u = u(y)$ is a solution of* (3.5) *and* (3.6)*, then the translated function $u = u(y + b)$ ($b \in \mathbb{R}$) satisfies the same conditions. However, one could show that the trajectory in the phase plane connecting two given end states is unique.*

4 Kinetic functions associated with general flux-functions

Consider now the general *diffusive-dispersive conservation law*

$$\partial_t u + \partial_x f(u) = \varepsilon\, \big(b(u)\, u_x\big)_x + \delta\, \big(c_1(u)\, (c_2(u)\, u_x)_x\big)_x, \quad u = u^{\varepsilon,\delta}(x,t), \tag{4.1}$$

where the diffusion coefficient $b(u) > 0$ and dispersion coefficients $c_1(u), c_2(u) > 0$ are given smooth functions. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a *concave-*

convex function satisfying, by definition,

$$
\begin{aligned}
& u\, f''(u) > 0 \quad \text{ for all } u \neq 0, \\
& f'''(0) \neq 0, \quad \lim_{|u| \to +\infty} f'(u) = +\infty.
\end{aligned}
\tag{4.2}
$$

We are interested in the singular limit $\varepsilon \to 0$ when $\delta > 0$ and the ratio $\alpha = \varepsilon / \sqrt{\delta}$ is kept constant. The limiting equation associated with (4.1), formally, is the scalar conservation law

$$
\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in \mathbb{R}.
$$

It can be checked that the entropy inequality

$$
\partial_t U(u) + \partial_x F(u) \leq 0
$$

holds, provided the entropy pair (U, F) is chosen such that

$$
U''(u) := \frac{c_2(u)}{c_1(u)}, \quad F'(u) := U'(u)\, f'(u), \quad u \in \mathbb{R},
\tag{4.3}
$$

which we assume in the rest of this paper. Since $c_1, c_2 > 0$ the function U is strictly convex.

Given two states $u_\pm$ and the corresponding propagation speed

$$
\lambda = \overline{a}(u_-, u_+) :=
\begin{cases}
\frac{f(u_+) - f(u_-)}{u_+ - u_-}, & u_+ \neq u_-, \\
f'(u_-), & u_+ = u_-,
\end{cases}
$$

we search for traveling wave solutions $u = u(y)$ of (4.1) depending on the rescaled variable $y := (x - \lambda\, t)\, \alpha / \varepsilon$. Following the same lines as those in Sections 1 and 2 we find that the trajectory satisfies

$$
c_1(u)\, (c_2(u)\, u_y)_y + \alpha\, b(u) u_y = -\lambda\, (u - u_-) + f(u) - f(u_-), \quad u = u(y),
\tag{4.4}
$$

and the boundary conditions

$$
\lim_{y \to \pm\infty} u(y) = u_\pm, \quad \lim_{y \to \pm\infty} u_y(y) = 0.
$$

Setting now

$$v = c_2(u)\, u_y,$$

we rewrite (4.4) in the general form (3.10) for the unknowns $u = u(y)$ and $v = v(y)$ ($y \in \mathbb{R}$), i.e.,

$$\frac{d}{dy}\begin{pmatrix} u \\ v \end{pmatrix} = K(u, v) \tag{4.5}$$

with

$$K(u, v) = \begin{pmatrix} \frac{v}{c_2(u)} \\ -\alpha\, \frac{b(u)}{c_1(u) c_2(u)}\, v + \frac{g(u,\lambda) - g(u_-,\lambda)}{c_1(u)} \end{pmatrix}, \quad g(u, \lambda) := f(u) - \lambda\, u, \tag{4.6}$$

while the boundary conditions take the form

$$\lim_{y \to \pm\infty} u(y) = u_\pm, \quad \lim_{y \to \pm\infty} v(y) = 0. \tag{4.7}$$

The function K in (4.6) vanishes at the *equilibrium points* $(u, v) \in \mathbb{R}^2$ satisfying

$$g(u, \lambda) = g(u_-, \lambda), \quad v = 0. \tag{4.8}$$

In view of the assumption (4.2), given a left-hand state u_- and a speed λ there exist at most three equilibria u satisfying (4.8) (including u_- itself). Considering a trajectory leaving from u_- at $-\infty$, we will determine whether this trajectory diverges to infinity or else which equilibria (if there is more than one equilibria) it actually connects to at $+\infty$. Before stating our main result (cf. Theorem 4.3, below) let us derive some fundamental inequalities satisfied by states u_- and u_+ connected by a traveling wave.

Consider the *entropy dissipation*

$$E(u_-, u_+) := -\overline{a}(u_-, u_+)\,\big(U(u_+) - U(u_-)\big) + F(u_+) - F(u_-) \tag{4.9}$$

or, equivalently, using (4.3) and (4.7)

$$
\begin{aligned}
E(u_-, u_+) &= \int_{-\infty}^{+\infty} U'(u(y)) \big(-\lambda\, u_y(y) + f(u(y))_y\big)\, dy \\
&= -\int_{-\infty}^{+\infty} U''(u(y)) \big(-\lambda\, (u(y) - u_-) + f(u) - f(u_-)\big)\, u_y(y)\, dy \\
&= -\int_{u_-}^{u_+} \big(g(z, \overline{a}(u_-, u_+)) - g(u_-, \overline{a}(u_-, u_+))\big)\, \frac{c_2(z)}{c_1(z)}\, dz.
\end{aligned}
\tag{4.10}
$$

In view of

$$
\begin{aligned}
E(u_-, u_+) &= \int_{-\infty}^{+\infty} U'(u) \Big(\alpha \big(b(u)\, u_y\big)_y + \big(c_1(u)\, (c_2(u)\, u_y)_y\big)_y\Big)\, dy \\
&= -\int_{-\infty}^{+\infty} \alpha\, U''(u)\, b(u)\, u_y^2\, dy,
\end{aligned}
$$

we have immediately the following.

Lemma 4.1 (Entropy inequality)**.** *If there exists a traveling wave of* (4.4) *connecting* u_- *to* u_+*, then the corresponding entropy dissipation is non-positive,*

$$
E(u_-, u_+) \le E(u_-, u_-) = 0.
$$

From the graph of the function f we define the functions $\varphi^\natural$ and $\lambda^\natural$ by

$$
\lambda^\natural(u) := f'\big(\varphi^\natural(u)\big) = \frac{f(u) - f\big(\varphi^\natural(u)\big)}{u - \varphi^\natural(u)}, \quad u \neq 0.
$$

We have $u\, \varphi^\natural(u) < 0$ and by continuity $\varphi^\natural(0) = 0$ and, thanks to (4.2), the map $\varphi^\natural : \mathbb{R} \to \mathbb{R}$ is decreasing and onto. It is invertible and its inverse function is denoted by $\varphi^{-\natural}$. Observe in passing that, u_- being kept fixed, $\lambda^\natural(u_-)$ is a *lower bound* for all shock speeds λ satisfying the Rankine-Hugoniot relation

$$
-\lambda\, (u_+ - u_-) + f(u_+) - f(u_-) = 0
$$

for some u_+.

The properties of the entropy dissipation (4.9) are determined from the *zero-entropy dissipation* function $\varphi_0^\flat$ was introduced.

Lemma 4.2 (Entropy dissipation function)**.** *There exists a decreasing function* $\varphi_0^\flat : \mathbb{R} \to \mathbb{R}$ *such that for all* $u_- > 0$ *(for instance)*

$$E(u_-, u_+) = 0 \text{ and } u_+ \neq u_- \quad \text{if and only if} \quad u_+ = \varphi_0^\flat(u_-),$$

$$E(u_-, u_+) < 0 \quad \text{if and only if} \quad \varphi_0^\flat(u_-) < u_+ < u_-,$$

and

$$\varphi^{-\natural}(u_-) < \varphi_0^\flat(u_-) < \varphi^\natural(u_-).$$

In passing, define also the function $\varphi_0^\sharp = \varphi_0^\sharp(u_-)$ and the speed $\lambda_0 = \lambda_0(u_-)$ by

$$\lambda_0(u_-) = \frac{f(u_-) - f(\varphi_0^\flat(u_-))}{u_- - \varphi_0^\flat(u_-)} = \frac{f(u_-) - f(\varphi_0^\sharp(u_-))}{u_- - \varphi_0^\sharp(u_-)}, \quad u_- \neq 0. \quad (4.11)$$

Combining Lemmas 4.1 and 4.2 together we conclude that, if there exists a traveling wave connecting u_- to u_+, necessarily

$$u_+ \text{ belongs to the interval } [\varphi_0^\flat(u_-), u_-]. \quad (4.12)$$

In particular, the states $u_+ > u_-$ and $u_+ < \varphi^{-\natural}(u_-)$ *cannot* be reached by a traveling wave and, therefore, it is not restrictive to focus on the case that three equilibria exist.

Next, for each $u_- > 0$ we define the *shock set* generated by the diffusive-dispersive model (4.1) by

$$\mathcal{S}_\alpha(u_-) := \Big\{ u_+ \,/\, \text{ there exists a traveling wave of (4.4) connecting } u_- \text{ to } u_+ \Big\}.$$

Theorem 4.3 (Kinetic function and shock set for general flux)**.** *Given a concave-convex flux-function* f *(see* (4.2)*), consider the diffusive-dispersive model* (4.1) *in which the ratio* $\alpha = \varepsilon/\sqrt{\delta} > 0$ *is fixed. Then, there exists a locally Lipschitz continuous and decreasing kinetic function* $\varphi_\alpha^\flat : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\begin{aligned} \varphi^\natural(u) \leq \varphi_\alpha^\flat(u) < \varphi_0^\flat(u), \quad u < 0, \\ \varphi_0^\flat(u) < \varphi_\alpha^\flat(u) \leq \varphi^\natural(u), \quad u > 0, \end{aligned} \quad (4.13)$$

and such that

$$\mathcal{S}_\alpha(u_-) = \begin{cases} \big[u_-, \varphi^\sharp_\alpha(u_-)\big) \cup \big\{\varphi^\flat_\alpha(u_-)\big\}, & u_- < 0, \\ \big\{\varphi^\flat_\alpha(u_-)\big\} \cup \big(\varphi^\sharp_\alpha(u_-), u_-\big], & u_- > 0. \end{cases} \tag{4.14}$$

Here, the function $\varphi^\sharp_\alpha$ is defined from the kinetic function $\varphi^\flat_\alpha$ by

$$\frac{f(u) - f\big(\varphi^\sharp_\alpha(u)\big)}{u - \varphi^\sharp_\alpha(u)} = \frac{f(u) - f\big(\varphi^\flat_\alpha(u)\big)}{u - \varphi^\flat_\alpha(u)}, \quad u \neq 0,$$

with the constraint

$$\begin{aligned} &\varphi^\sharp_0(u) < \varphi^\sharp_\alpha(u) \le \varphi^\natural(u), \quad u < 0, \\ &\varphi^\natural(u) \le \varphi^\sharp_\alpha(u) < \varphi^\sharp_0(u), \quad u > 0. \end{aligned} \tag{4.15}$$

Moreover, there exists a function

$$A^\natural : \mathbb{R} \to [0, +\infty),$$

called the threshold diffusion-dispersion ratio, which is smooth away from $u = 0$, Lipschitz continuous at $u = 0$, increasing in $u > 0$, and decreasing in $u < 0$ with

$$A^\natural(u) \sim C\,|u| \quad \text{as } u \to 0, \tag{4.16}$$

(where $C > 0$ depends upon f, b, c_1, and c_2 only) and such that

$$\varphi^\flat_\alpha(u) = \varphi^\natural(u) \quad \text{when } \alpha \ge A^\natural(u). \tag{4.17}$$

Additionally we have

$$\varphi^\flat_\alpha(u) \to \varphi^\flat_0(u) \quad \text{as } \alpha \to 0 \quad \text{for each } u \in \mathbb{R}. \tag{4.18}$$

The 4.3 will be the subject of Sections 4 and 5 below. The kinetic function $\varphi^\flat_\alpha : \mathbb{R} \to \mathbb{R}$ completely characterizes the dynamics of the nonclassical shock waves associated with (4.1). In view of Theorem 4.3 one can solve the Riemann problem. The kinetic function $\varphi^\flat_\alpha$ is decreasing and its range is limited by the functions $\varphi^\natural$ and $\varphi^\flat_0$. Therefore we can solve the Riemann problem, uniquely in the class of nonclassical entropy solutions selected by the kinetic function $\varphi^\flat_\alpha$.

The statements (4.17) and (4.18) provide us with important qualitative properties of the nonclassical shocks:

1. The shocks leaving from u_- are always classical if the ratio α is chosen to be sufficiently large or if u_- is sufficiently small.

2. The shocks leaving from u_- are always nonclassical if the ratio α is chosen to be sufficiently small.

Furthermore, under a mild assumption on the growth of f at infinity, one could also establish that the shock leaving from u_- are always nonclassical if the state u_- is sufficiently large. (See the bibliographical notes.)

In this rest of this section we introduce some important notation and investigate the limiting case when the diffusion is identically zero ($\alpha = 0$). We always suppose that $u_- > 0$ (for definiteness) and we set

$$u_0 = u_-.$$

The shock speed λ is regarded as a parameter allowing us to describe the set of attainable right-hand states. Precisely, given a speed in the interval

$$\lambda \in \big(\lambda^\natural(u_0), f'(u_0)\big),$$

there exist exactly three distinct solutions denoted by u_0, u_1, and u_2 of the equation (4.8) with

$$u_2 < \varphi^\natural(u_0) < u_1 < u_0. \tag{4.19}$$

Recall that no trajectory exists when λ is chosen outside the interval limited by $\lambda^\natural(u_0)$ and $f'(u_0)$.

From Lemmas 4.1 and 4.2 (see (4.12)) it follows that a trajectory either is *classical* if u_0 is connected to

$$u_1 \in [\varphi^\natural(u_0), u_0] \text{ with } \lambda \in \big[\lambda^\natural(u_0), f'(u_0)\big] \tag{4.20}$$

or else is *nonclassical* if u_0 is connected to

$$u_2 \in [\varphi_0^\flat(u_0), \varphi^\natural(u_0)) \text{ with } \lambda \in \big(\lambda^\natural(u_0), \lambda_0(u_0)\big]. \tag{4.21}$$

For the sake of completeness we cover here both cases of positive and negative dispersions. For the statements in Lemma 4.4 and Theorem 4.5

below *only* we will set $\alpha := \varepsilon/\sqrt{|\delta|}$ and $\eta = \operatorname{sgn}(\delta) = \pm 1$. If (u, v) is an equilibrium point, the eigenvalues of the Jacobian matrix of the function $K(u, v)$ in (4.6) are found to be

$$\mu = \frac{1}{2}\left(-\eta\,\alpha\,\frac{b(u)}{c_1(u)c_2(u)} \pm \sqrt{\alpha^2\,\frac{b(u)^2}{c_1(u)^2 c_2(u)^2} + 4\eta\,\frac{f'(u)-\lambda}{c_1(u)\,c_2(u)}}\right).$$

So, we set

$$\begin{aligned}
\underline{\mu}(u;\lambda,\alpha) &= \frac{\eta\,\alpha}{2}\,\frac{b(u)}{c_1(u)c_2(u)}\left(-1 - \eta\sqrt{1 + \frac{4\eta}{\alpha^2}\,\frac{c_1(u)c_2(u)}{b(u)^2}\,(f'(u)-\lambda)}\right),\\
\overline{\mu}(u;\lambda,\alpha) &= \frac{\eta\,\alpha}{2}\,\frac{b(u)}{c_1(u)c_2(u)}\left(-1 + \eta\sqrt{1 + \frac{4\eta}{\alpha^2}\,\frac{c_1(u)c_2(u)}{b(u)^2}\,(f'(u)-\lambda)}\right).
\end{aligned} \tag{4.22}$$

Lemma 4.4 (Nature of equilibrium points)**.** *Fix some values u_- and λ and denote by $(u_*, 0)$ any one of the three equilibrium points satisfying* (4.8)*.*

1. *If $\eta = +1$ and $f'(u_*) - \lambda < 0$, then $(u_*, 0)$ is a* stable point.
2. *If $\eta\,(f'(u_*) - \lambda) > 0$, then $(u_*, 0)$ is a* saddle point.
3. *If $\eta = -1$ and $f'(u_*) - \lambda > 0$, then $(u_*, 0)$ is an* unstable point.

Furthermore, in the two cases that $\eta\,(f'(u_*)-\lambda) < 0$ we have the additional result: When $\alpha^2\, b(u_*)^2 + 4\,\eta\, c_1(u_*)\, c_2(u_*)\,(f'(u_*) - \lambda) \geq 0$ the equilibrium is a node, *and is a* spiral *otherwise.*

For negative dispersion coefficient δ, that is, when $\eta = -1$, we see that both u_1 and u_2 are *unstable points* which no trajectory can attain at $+\infty$, while u_1 is a stable point. So, in this case, we obtain immediately:

Theorem 4.5 (Traveling waves for negative dispersion)**.** *Consider the diffusive-dispersive model* (4.1) *where the flux satisfies* (4.2)*. If $\varepsilon > 0$ and $\delta < 0$, then only classical trajectories exist.*

Some additional analysis (along similar lines) would be necessary to establish the existence of these classical trajectories and conclude that

$$\mathcal{S}_\alpha(u_-) = \mathcal{S}(u_-) := \begin{cases} \big[\varphi^\natural(u_-), u_-\big], & u_- \geq 0 \\ & \\ \big[u_-, \varphi^\natural(u_-)\big], & u_- \leq 0 \end{cases} \quad \text{when } \delta < 0,$$

which is the shock set already found in Section 2 when $\delta = 0$.

We return to the case of a positive dispersion which is of main interest here. (From now on $\eta = +1$.) Since $g'_u(u, \lambda)$ is positive at both $u = u_2$ and $u = u_0$, we have

$$\underline{\mu}(u_0) < 0 < \overline{\mu}(u_0), \quad \underline{\mu}(u_2) < 0 < \overline{\mu}(u_2),$$

and both points u_2 and u_0 are *saddle*. On the other hand, since $g'_u(u_1, \lambda) < 0$, the equilibrium u_1 is a stable point which may be a *node* or a *spiral*. These properties are the same as the ones already established for the equation with cubic flux. The following result is easily checked from the expressions (4.22).

Lemma 4.6 (Monotonicity properties of eigenvalues)**.** *In the range of parameters where* $\underline{\mu}(u, \lambda, \alpha)$ *and* $\overline{\mu}(u; \lambda, \alpha)$ *remain real-valued, we have*

$$\frac{\partial \underline{\mu}}{\partial \lambda}(u; \lambda, \alpha) > 0, \quad \frac{\partial \underline{\mu}}{\partial \alpha}(u; \lambda, \alpha) < 0 \quad \frac{\partial \overline{\mu}}{\partial \lambda}(u; \lambda, \alpha) < 0,$$

and, under the assumption $f'(u) - \lambda > 0$,

$$\frac{\partial \overline{\mu}}{\partial \alpha}(u; \lambda, \alpha) < 0.$$

To the state u_0 and the speed $\lambda \in \big(\lambda^\natural(u_0), \lambda_0(u_0)\big)$ we associate the following function of the variable u, which will play an important role throughout,

$$G(u; u_0, \lambda) := \int_{u_0}^{u} \big(g(z, \lambda) - g(u_0, \lambda)\big) \, \frac{c_2(z)}{c_1(z)} \, dz.$$

Observe, using (4.10), that the functions G and E are closely related:

$$G(u; u_0, \lambda) = -E(u_0, u) \quad \text{when} \quad \lambda = \overline{a}(u_0, u). \tag{4.23}$$

Note also that the derivative $\partial_u G(u; u_0, \lambda)$ vanishes exactly at the equilibria u_0, u_1, and u_2 satisfying (4.8). Using the function G we rewrite now the main equations (4.5)-(4.6) in the form

$$c_2(u)\, u_y = v, \tag{4.24}$$

$$c_2(u)\, v_y = -\alpha\, \frac{b(u)}{c_1(u)}\, v + G'_u(u; u_0, \lambda), \tag{4.25}$$

which we will often use in the rest of the discussion.

We collect now some fundamental properties of the function G.

Theorem 4.7 (Monotonicity properties of the function G)**.** *Fix some $u_0 > 0$ and $\lambda \in \big(\lambda^\natural(u_0), f'(u_0)\big)$ and consider the associated states u_1 and u_2. Then, the function $u \mapsto \tilde{G}(u) := G(u; u_0, \lambda)$ satisfies the monotonicity properties*

$$\begin{aligned} &\tilde{G}'(u) < 0, \quad u < u_2 \text{ or } u \in (u_1, u_0), \\ &\tilde{G}'(u) > 0, \quad u \in (u_2, u_1) \text{ or } u > u_0. \end{aligned}$$

Moreover, if $\lambda \in \big(\lambda^\natural(u_0), \lambda_0(u_0)\big)$ we have

$$\tilde{G}(u_0) = 0 < \tilde{G}(u_2) < \tilde{G}(u_1), \tag{4.26}$$

while, if $\lambda = \lambda_0(u_0)$,

$$\tilde{G}(u_0) = \tilde{G}(u_2) = 0 < \tilde{G}(u_1) \tag{4.27}$$

and finally, if $\lambda \in \big(\lambda_0(u_0), f'(u_0)\big)$,

$$\tilde{G}(u_2) < 0 = \tilde{G}(u_0) < \tilde{G}(u_1). \tag{4.28}$$

Proof. *The sign of $\tilde{G}'$ is the same as the sign of the function*

$$g(u, \lambda) - g(u_0, \lambda) = (u - u_0) \Big(\frac{f(u) - f(u_0)}{u - u_0} - \lambda \Big).$$

So, the sign of $\tilde{G}'$ is easy determined geometrically from the graph of the function f. To derive (4.26)–(4.28) *note that $\tilde{G}(u_0) = 0$ and (by the monotonicity properties above) $\tilde{G}(u_1) > \tilde{G}(u_0)$. To complete the argument we only need the sign of $\tilde{G}(u_2)$. But by* (4.23) *we have $\tilde{G}(u_2) = -E(u_0, u_2)$ whose sign is given by Lemma 4.2.*

□

We conclude this section with the special case that the diffusion is zero. Note that the shock set below *is not* the obvious limit from (4.14).

Theorem 4.8 (Dispersive traveling waves)**.** *Consider the traveling wave equation* (4.4) *in the limiting case $\alpha = 0$ (*not included in Theorem 4.3*) under the assumption that the flux f satisfies* (4.2)*. Then, the corresponding shock set reduces to*

$$\mathcal{S}_0(u_-) = \big\{\varphi_0^\flat(u_-), u_-\big\}, \quad u_- \in \mathbb{R}.$$

Proof. *Suppose that there exists a trajectory connecting a state $u_- > 0$ to a state $u_+ \neq u_-$ for the speed $\lambda = \overline{a}(u_-, u_+)$ and satisfying (see* (4.24)-(4.25)*)*

$$\begin{aligned} c_2(u)\, u_y &= v, \\ c_1(u)\, v_y &= g(u, \lambda) - g(u_-, \lambda). \end{aligned} \tag{4.29}$$

Multiplying the second equation in (4.29) *by $v/c_1(u) = c_2(u)\, u_y / c_1(u)$, we find*

$$\frac{1}{2}\left(v^2\right)_y = \big(g(u, \lambda) - g(u_-, \lambda)\big)\, \frac{c_2(u)}{c_1(u)}\, u_y$$

and, after integration over some interval $(-\infty, y]$,

$$\frac{1}{2}\, v^2(y) = G(u(y); u_-, \lambda), \quad y \in \mathbb{R}. \tag{4.30}$$

Letting $y \to +\infty$ in (4.30) *and using that $v(y) \to 0$ we obtain*

$$G(u_+; u_-, \lambda) = 0$$

which, by (4.23)*, is equivalent to*

$$E(u_+, u_-) = 0.$$

Using Lemma 4.2 we conclude that the right-hand state u_+ is uniquely determined, by the zero-entropy dissipation function:

$$u_+ = \varphi_0^\flat(u_-), \quad \lambda = \lambda_0(u_-). \tag{4.31}$$

Then, by assuming (4.31) *and $u_- > 0$, Theorem 4.7 implies that the function $u \mapsto G(u; u_-, \lambda)$ remains strictly positive for all u (strictly) between u_+ and u_-. Since $v < 0$ we get from* (4.30)

$$v(y) = -\sqrt{2\, G(u(y); u_-, \lambda)}. \tag{4.32}$$

In other words, we obtain the trajectory in the (u, v) plane:

$$v = \bar{v}(u) = -\sqrt{2\, G(u; u_-, \lambda)}, \quad u \in [u_+, u_-],$$

supplemented with the boundary conditions

$$\bar{v}(u_-) = \bar{v}(u_+) = 0.$$

Clearly, the function $\bar{v}$ is well-defined and satisfies $\bar{v}(u) < 0$ for all $u \in (u_+, u_-)$. Finally, based on the change of variable $y \in [-\infty, +\infty] \mapsto u = u(y) \in [u_+, u_-]$ given by

$$dy = \frac{c_2(u)}{\bar{v}(u)}\, du,$$

we immediately recover from the curve $v = \bar{v}(u)$ the (unique) trajectory

$$y \mapsto \big(u(y), v(y)\big).$$

This completes the 4.8.

□

5 Traveling waves corresponding to a given speed

We prove in this section that, given u_0, u_2, and $\lambda = \overline{a}(u_0, u_2)$ in the range (see (4.21))

$$u_2 \in \big[\varphi_0^\flat(u_0), \varphi^\natural(u_0)\big), \quad \lambda \in \big(\lambda^\natural(u_0), \lambda_0(u_0)\big], \tag{5.1}$$

a nonclassical connection always exists if the ratio α *is chosen appropriately.* As we will show in the next section this result is the key step in the 4.3. The main existence result proven in the present section is stated as follows.

Theorem 5.1 (Nonclassical trajectories for a fixed speed). *Consider two states $u_0 > 0$ and $u_2 < 0$ associated with a speed*

$$\lambda = \overline{a}(u_0, u_2) \in \big(\lambda^\natural(u_0), \lambda_0(u_0)\big].$$

Then, there exists a unique value $\alpha \geq 0$ such that u_0 is connected to u_2 by a diffusive-dispersive traveling wave solution.

By Lemma 4.4, u_0 is a saddle point and we have $\overline{\mu}(u_0) > 0$ and from Theorem 3.2 it follows that there are two trajectories leaving from u_0 at $y = -\infty$, both of them satisfying

$$\lim_{y \to -\infty} \frac{v(y)}{u(y) - u_0} = \overline{\mu}(u_0; \lambda, \alpha)\, c_2(u_0). \tag{5.2}$$

One trajectory approaches $(u_0, 0)$ in the quadrant $Q_1 = \big\{u > u_0,\, v > 0\big\}$, the other in the quadrant $Q_2 = \big\{u < u_0,\, v < 0\big\}$. On the other hand, u_2 is also a saddle point and there exist two trajectories reaching u_2 at $y = +\infty$, both of them satisfying

$$\lim_{y \to +\infty} \frac{v(y)}{u(y) - u_2} = \underline{\mu}(u_2; \lambda, \alpha)\, c_2(u_2). \tag{5.3}$$

One trajectory approaches $(u_2, 0)$ in the quadrant $Q_3 = \big\{u > u_2,\, v < 0\big\}$, the other in the quadrant $Q_4 = \big\{u < u_2,\, v > 0\big\}$.

Lemma 5.2. *A traveling wave solution connecting u_0 to u_2 must leave the equilibrium $(u_0, 0)$ at $y = -\infty$ in the quadrant Q_2, and reach $(u_2, 0)$ in the quadrant Q_3 at $y = +\infty$.*

Proof. *Consider the trajectory leaving from the quadrant Q_1, that is, satisfying $u > u_0$ and $v > 0$ in a neighborhood of the point $(u_0, 0)$. By* *contradiction, suppose it would reach the state* u_2 *at* $+\infty$*. Since* $u_2 < u_0$ *by continuity there would exist* y_0 *such that*

$$u(y_0) = u_0.$$

Multiplying (4.25) *by* $u_y = v/c_2(u)$ *we find*

$$\left(v^2/2\right)_y + \alpha \, \frac{b(u)}{c_1(u)\, c_2(u)} \, v^2 = G'_u(u; u_0, \lambda)\, u_y.$$

Integrating over $(-\infty, y_0]$ *we arrive at*

$$\frac{v^2(y_0)}{2} + \alpha \int_{-\infty}^{y_0} v^2 \, \frac{b(u)}{c_1(u)\, c_2(u)} \, dy = G(u(y_0); u_0, \lambda) = 0. \tag{5.4}$$

Therefore $v(y_0) = 0$ *and, since* $u(y_0) = u_0$*, a standard uniqueness theorem for the Cauchy problem associated with* (4.24)-(4.25) *implies that* $u \equiv u_0$ *and* $v \equiv 0$ *on* $\mathbb{R}$*. This contradicts the assumption that the trajectory would connect to* u_2 *at* $+\infty$*.*

The argument around the equilibrium $(u_2, 0)$ *is somewhat different. Suppose that the trajectory satisfies* $u < u_2$ *and* $v > 0$ *in a neighborhood of the point* $(u_2, 0)$*. There would exist some value* y_1 *achieving a local minimum, that is, such that*

$$u(y_1) < u_2, \quad u_y(y_1) = 0, \quad u_{yy}(y_1) \geq 0.$$

From (4.24) *we would obtain* $v(y_1) = 0$ *and, by differentiation of* (4.24),

$$v_y(y_1) = u_{yy}(y_1)\, c_2(u(y_1)) \geq 0.$$

Combining the last two relations with (4.25) *we would obtain*

$$G'_u(u(y_1); u_0, \lambda) \geq 0$$

which is in contradiction with Theorem 4.7 since $u(y_1) < u_2$ *and* $G'_u(u(y_1); u_0, \lambda) < 0.$

□

Next, we determine some intervals in which the traveling waves are always monotone.

Lemma 5.3. *Consider a trajectory $u = u(y)$ leaving from u_0 at $-\infty$ and denote by $\underline{\xi}$ the largest value such that $u_1 < u(y) \le u_0$ for all $y \in (-\infty, \underline{\xi})$ and $u(\underline{\xi}) = u_1$. Then, we have*

$$u_y < 0 \qquad \text{on the interval } (-\infty, \underline{\xi}).$$

Similarly, if $u = u(y)$ is a trajectory connecting to u_2 at $+\infty$, denote by $\overline{\xi}$ the smallest value such that $u_2 \le u(y) < u_1$ for all $y \in (\overline{\xi}, +\infty)$ and $u(\overline{\xi}) = u_1$. Then, we have

$$u_y < 0 \qquad \text{on the interval } (\overline{\xi}, +\infty).$$

In other words, a trajectory cannot change its monotonicity before reaching the value u_1.

Proof. *We only check the first statement, the proof of the second one being similar. By contradiction, there would exist $y_1 \in (-\infty, \underline{\xi})$ such that*

$$u_y(y_1) = 0, \quad u_{yy}(y_1) \ge 0, \quad u_1 < u(y_1) \le u_0.$$

Then, using the equation (4.25) would yield $G'_u(u(y_1); u_0, \lambda) \ge 0$, which is in contradiction with the monotonicity properties in Theorem 4.7.

□

Proof of Theorem 5.1 *For each $\alpha \ge 0$ we consider the orbit leaving from u_0 and satisfying $u < u_0$ and $v < 0$ in a neighborhood of $(u_0, 0)$. This trajectory reaches the line $\{u = u_1\}$ for the "first time" at some point denoted by $(u_1, V_-(\alpha))$. In view of Lemma 5.3 this part of trajectory is the graph of a function*

$$[u_1, u_0] \ni u \mapsto v_-(u; \lambda, \alpha)$$

with of course $v_-(u_1; \lambda, \alpha) = V_-(\alpha)$. Moreover, by standard theorems on differential equations, v_- is a smooth function with respect to its argument $(u; \lambda, \alpha) \in [u_1, u_0] \times \big(\lambda^\natural(u_0), \lambda_0(u_0)\big] \times [0, +\infty)$.

Similarly, for each $\alpha \geq 0$ we consider the orbit arriving at u_2 and satisfying $u > u_2$ and $v < 0$ in a neighborhood of $(u_2, 0)$. This trajectory reaches the line $\{u = u_1\}$ for the "first time" as y decreases from $+\infty$ at some point $(u_1, V_+(\alpha))$. By Lemma 5.3 this trajectory is the graph of a function

$$[u_2, u_1] \ni u \mapsto v_+(u; \lambda, \alpha).$$

The mapping v_+ depends smoothly upon $(u, \lambda, \alpha) \in [u_2, u_1] \times \big(\lambda^\natural(u_0), \lambda_0(u_0)\big] \times [0, +\infty)$.

For each of these curves $u \mapsto v_-(u)$ and $u \mapsto v_+(u)$ we derive easily from (4.24)-(4.25) a differential equation in the (u, v) plane:

$$v(u)\,\frac{dv}{du}(u) + \alpha\,\frac{b(u)}{c_1(u)}\,v(u) = G'_u(u, u_0, \lambda). \tag{5.5}$$

Clearly, the function

$$\begin{aligned} \alpha \in [0, +\infty) \mapsto W(\alpha) := & \ v_+(u_1; \lambda, \alpha) - v_-(u_1; \lambda, \alpha) \\ = & \ V_+(\alpha) - V_-(\alpha) \end{aligned}$$

measures the distance (in the phase plane) between the two trajectories at $u = u_1$. Therefore, the condition $W(\alpha) = 0$ characterizes the traveling wave solution of interest connecting u_0 to u_2. The existence of a root for the function W is obtained as follows.

□

Case 1: *Take first $\alpha = 0$.*

Integrating (5.5) with $v = v_-$ over the interval $[u_1, u_0]$ yields

$$\frac{1}{2}(V_-(0))^2 = G(u_1; u_0, \lambda) - G(u_0; u_0, \lambda) = G(u_1; u_0, \lambda),$$

while integrating (5.5) with $v = v_+$ over the interval $[u_2, u_1]$ gives

$$\frac{1}{2}(V_+(0))^2 = G(u_1; u_0, \lambda) - G(u_2; u_0, \lambda).$$

When $\lambda \neq \lambda_0(u_0)$, since $G(u_2; u_0, \lambda) > 0$ (Theorem 4.7) and $V_\pm(\alpha) < 0$ (Lemma 5.3) we conclude that $W(0) > 0$. When $\lambda = \lambda_0(u_0)$ we have $G(u_2; u_0, \lambda) = 0$ and $W(0) = 0$.

Case 2: *Consider next the limit $\alpha \to +\infty$.*

On one hand, since $v_- < 0$, for $\alpha > 0$ we get in the same way as in Case 1

$$\frac{1}{2}(V_-(\alpha))^2 < G(u_1; u_0, \lambda). \tag{5.6}$$

On the other hand, dividing (5.5) by $v = v_+$ and integrating over the interval $[u_2, u_1]$ we find

$$V_+(\alpha) = -\alpha \int_{u_2}^{u_1} \frac{b(u)}{c_1(u)}\, du + \int_{u_2}^{u_1} \frac{G'_u(u; u_0, \lambda)}{v_+(u)}\, du.$$

Since $v = c_2(u)\, u_y \le 0$ and $G'_u(u) \ge 0$ in the interval $[u_2, u_1]$ we obtain

$$V_+(\alpha) \le -\kappa\, \alpha\, (u_1 - u_2), \tag{5.7}$$

where $\kappa = \inf_{u \in [u_2, u_1]} b(u)/c_1(u) > 0$. Combining (5.6) and (5.7) and choosing α to be sufficiently large, we conclude that

$$W(\alpha) = V_+(\alpha) - V_-(\alpha) < 0.$$

Hence, by the intermediate value theorem there exists at least one value α such that

$$W(\alpha) = 0,$$

which establishes the existence of a trajectory connecting u_0 to u_2. Thanks to Lemma 5.3 it satisfies $u_y < 0$ globally.

The uniqueness of the solution is established as follows. Suppose that there would exist two orbits $v = v(u)$ and $v^* = v^*(u)$ associated with distinct values α and $\alpha^* > \alpha$, respectively. Then, Lemma 4.6 would imply that

$$\overline{\mu}(u_0; \lambda, \alpha^*) < \overline{\mu}(u_0; \lambda, \alpha), \quad \underline{\mu}(u_2; \lambda, \alpha^*) < \underline{\mu}(u_2; \lambda, \alpha).$$

So, there would exist $u_3 \in (u_2, u_0)$ satisfying

$$v(u_3) = v^*(u_3), \quad \frac{dv^*}{du}(u_3) \ge \frac{dv}{du}(u_3).$$

Comparing the equations (5.5) satisfied by both v and v^*, we get

$$v(u_3)\left(\frac{dv}{du}(u_3) - \frac{dv^*}{du}(u_3)\right) = (\alpha^* - \alpha)\, \frac{b(u_3)}{c_1(u_3)}\, v(u_3). \tag{5.8}$$

Now, since $v(u_3) \neq 0$ *(the connection with the third critical point* $(u_1, 0)$ *is impossible) we obtain a contradiction, as the two sides of* (5.8) *have opposite signs. This completes the proof of Theorem 5.1.*

□

Remark 5.4. *It is not difficult to see also that, in the proof of Theorem 5.1,*

$$\alpha \mapsto V_-(\alpha) \text{ is non-decreasing} \tag{5.9}$$

and

$$\alpha \mapsto V_+(\alpha) \text{ is decreasing.} \tag{5.10}$$

In particular, the function $W(\alpha) := V_+(\alpha) - V_-(\alpha)$ *is decreasing.*

Theorem 5.5 (Threshold function associated with nonclassical shocks)**.** *Consider the function* $A = A(u_0, u_2)$ *which is the unique value* α *for which there is a nonclassical traveling wave connecting* u_0 *to* u_2 *(Theorem 5.1). It is defined for* $u_0 > 0$ *and* $u_2 < 0$ *with* $u_2 \in \big[\varphi_0^\flat(u_0), \varphi^\natural(u_0)\big)$ *or, equivalently,* $u_0 \in \big[\varphi_0^\flat(u_2), \varphi^{-\natural}(u_2)\big)$*. Then we have the following two properties:*

1. *The function* $A\big(u_0, u_2\big)$ *is increasing in* u_2 *and maps* $[\varphi_0^\flat(u_0), \varphi^\natural(u_0))$ *onto some interval of the form* $\big[0, A^\natural(u_0)\big)$ *where* $A^\natural(u_0) \in (0, +\infty]$*.*
2. *The function* A *is also increasing in* u_0 *and maps the interval* $[\varphi_0^\flat(u_2), \varphi^{-\natural}(u_2))$ *onto the interval* $\big[0, A^\natural(\varphi^{-\natural}(u_2))\big)$*.*

Later (in Section 6) the function A will also determine the range in which *classical shocks* exist. From now on, we refer to the function A as the *critical diffusion-dispersion ratio.* On the other hand, the value $A^\natural(u_0)$ is called the *threshold diffusion-dispersion ratio* at u_0. Nonclassical trajectories leaving from u_0 exist if and only if $\alpha < A^\natural(u_0)$.

Observe that, in Theorem 5.5, we have $A(u_0, u_2) \to 0$ when $u_2 \to \varphi_0^\flat(u_0)$, which is exactly the desired property (4.18) in Theorem 4.3.

Proof. *We will only prove the first statement, the proof of the second one being completely similar. Fix* $u_0 > 0$ *and* $u_2^* < u_2 < u_0$ *so that*

$$\lambda^\natural(u_0) < \lambda = \frac{f(u_2) - f(u_0)}{u_2 - u_0} < \lambda^* = \frac{f(u_2^*) - f(u_0)}{u_2^* - u_0} \leq \lambda_0(u_0).$$

Proceeding by contradiction we assume that

$$\alpha^* := A(u_0, u_2^*) \geq \alpha := A(u_0, u_2).$$

Then, Lemma 4.6 implies

$$\overline{\mu}(u_0; \lambda, \alpha) \geq \overline{\mu}(u_0; \lambda, \alpha^*) > \overline{\mu}(u_0; \lambda^*, \alpha^*).$$

Let $v = v(u)$ *and* $v^* = v^*(u)$ *be the solutions of* (5.5) *associated with* α *and* α^**, respectively, and connecting* u_0 *to* u_2*, and* u_0 *to* u_2^**, respectively. Since* $u_2^* < u_2$*, by continuity there must exist some state* $u_3 \in (u_2, u_0)$ *such that*

$$v(u_3) = v^*(u_3), \quad \frac{dv^*}{du}(u_3) \geq \frac{dv}{du}(u_3).$$

On the other hand, (5.5) *which is satisfied by both* v *and* v^* *we obtain*

$$v(u_3)\left(\frac{dv^*}{du}(u_3) - \frac{dv}{du}(u_3)\right) + v(u_3)(\alpha^* - \alpha)\,\frac{b(u_3)}{c_1(u_3)} = (\lambda^* - \lambda)\,(u_0 - u_3)\,\frac{c_2(u_3)}{c_1(u_3)},$$

which leads to a contradiction since the left-hand side is non-positive and the right-hand side is positive. This completes the proof of Theorem 5.5.

□

We complete this section with some important asymptotic properties (which will establish (4.16)-(4.17) in Theorem 4.3).

Theorem 5.6. *The threshold diffusion-dispersion ratio satisfies the following two properties:*

1. $A^\natural(u_0) < +\infty$ *for all* u_0.
2. *There exists a traveling wave connecting* u_0 *to* $u_2 = \varphi^\natural(u_0)$ *for the value* $\alpha = A^\natural(u_0)$.

Proof. *Fix* $u_0 > 0$*. According to Theorem 5.1, given* $\lambda \in (\lambda^\natural(u_0), \lambda_0(u_0)]$ *there exists a nonclassical trajectory, denoted by* $u \mapsto v(u)$*, connecting* u_0 *to some* u_2 *with*

$$\lambda = \frac{f(u_2) - f(u_0)}{u_2 - u_0}, \quad u_2 < \varphi^\natural(u_0), \quad \alpha = A(u_0, u_2). \tag{5.11}$$

On the other hand, choosing any state $u_0^* > u_0$ *and setting*

$$\lambda^* = \frac{f(u_0^*) - f(u_1^*)}{u_0^* - u_1^*}, \quad u_1^* = \varphi^\natural(u_0),$$

it is easy to check from (4.22) *that, for all* α^* *sufficiently large,* $\underline{\mu}(u_1^*; \lambda^*, \alpha^*)$ *remains real with*

$$\underline{\mu}(u_1^*; \lambda^*, \alpha^*) < 0.$$

Then, consider the trajectory $u \mapsto v^*(u)$ *arriving at* u_1^* *and satisfying*

$$\lim_{\substack{u \to u_1 \\ u > u_1}} \frac{v^*(u)}{u - u_1^*} = \underline{\mu}(u_1^*; \lambda^*, \alpha^*)\, c_2(u_1^*) < 0.$$

Two different situations should be distinguished.

Case 1 : *The curve* $v^* = v^*(u)$ *crosses the curve* $v = v(u)$ *at some point* u_3 *where*

$$u_1^* < u_3 < u_0, \quad v(u_3) = v^*(u_3), \quad \frac{dv}{du}(u_3) \geq \frac{dv^*}{du}(u_3).$$

Using the equation (5.5) *satisfied by both* v *and* v^* *we get*

$$v(u_3)\Big(\frac{dv^*}{du}(u_3) - \frac{dv}{du}(u_3)\Big) + (\alpha^* - \alpha)\,\frac{b(u_3)}{c_1(u_3)}\,v(u_3) = G'_u(u_3; u_0^*, \lambda^*) - G'_u(u_3; u_0, \lambda)$$
$$< 0.$$

In view of our assumptions, since $v(u_3) < 0$ *we conclude that* $\alpha < \alpha^*$ *in this first case.*

Case 2 : $v^* = v^*(u)$ *does not cross the curve* $v = v(u)$ *on the interval* (u_1^*, u_0).

Then, the trajectory v^* *crosses the* u*-axis at some point* $u_4 \in (u_1^*, u_0]$. *Integrating the equation* (5.5) *for the function* v *on the interval* $[u_2, u_0]$ *we obtain*

$$\alpha \int_{u_0}^{u_2} \frac{b(u)}{c_1(u)}\, v(u)\, du = G(u_2; u_0, \lambda) - G(u_0; u_0, \lambda).$$

On the other hand, integrating (5.5) *for the solution* v^* *over* $[u_1^*, u_4]$ *we get*

$$\alpha^* \int_{u_4}^{u_1^*} \frac{b(u)}{c_1(u)}\, v^*(u)\, du = G(u_1^*; u_0^*, \lambda^*) - G(u_4; u_0^*, \lambda^*).$$

Since, by our assumption in this second case,

$$\int_{u_0}^{u_2} \frac{b(u)}{c_1(u)}\, v(u) du > \int_{u_4}^{u_1^*} \frac{b(u)}{c_1(u)}\, v^*(u) du,$$

we deduce from the former two equations that

$$\alpha \le \alpha^* \frac{G(u_2; u_0, \lambda) - G(u_0; u_0, \lambda)}{G(u_1^*; u_0^*, \lambda^*) - G(u_4; u_0^*, \lambda^*)} \le C \alpha^*,$$

where C is a constant independent of u_2. More precisely, u_2 describes a small neighborhood of $\varphi^\natural(u_0)$, while u_0^*, u_1^*, u_4, and λ^* remain fixed.

Finally, we conclude that in both cases

$$A(u_0, u_2) \le C' \, \alpha^*,$$

where α^* is sufficiently large (the condition depends on u_0 only) and C' is independent of the right-hand state u_2 under consideration. Hence, we have obtained an upper bound for the function $u_2 \mapsto A(u_0, u_2)$. This completes the proof of the first statement in the theorem.

The second statement is a consequence of the fact that $A(u_0, u_2)$ remains bounded as u_2 tends to $\varphi^\natural(u_0)$ and of the continuity of the traveling wave v with respect to the parameters λ and α, i.e., with obvious notation

$$v(.; \lambda^\natural(u_0), A^\natural(u_0)) = \lim_{u_2 \to \varphi^\natural(u_0)} v(.; \lambda(u_0, u_2), A(u_0, u_2)).$$

$\square$

The function $A^\natural = A^\natural(u_0)$ maps the interval $(0, +\infty)$ onto some interval $[\underline{A^\natural}, \overline{A^\natural}]$ where $0 \le \underline{A^\natural} \le \overline{A^\natural} \le +\infty$. The values $\underline{A^\natural}$ and $\overline{A^\natural}$ correspond to *lower* and *upper bounds* for the threshold ratio, respectively. The following theorem shows that the range of the function $A^\natural(u_0)$, in fact, has the form $\big[0, \overline{A^\natural}\big]$.

Theorem 5.7. *With the notation in Theorem 5.5 the asymptotic behavior of* $A^\natural(u_0)$ *as* $u_0 \to 0$ *is given by*

$$A^\natural(u_0) \sim \kappa\, u_0, \quad \kappa := \frac{c_1(0)c_2(0)}{4\,b(0)} \sqrt{3 f'''(0)} > 0. \tag{5.12}$$

Note that of course (4.2) implies that $f'''(0) > 0$. In particular, Theorem 5.7 shows that $\underline{A}^\natural(0) = A^\natural(0) = 0$. Theorem 5.7 is the only instance where the assumption $f'''(0) \neq 0$ (see (4.2)) is needed. In fact, if this assumption is dropped one still have $A^\natural(u_0) \to 0$ as $u_0 \to 0$. (See the bibliographical notes.)

Proof. *To estimate* $A^\natural$ *near the origin we compare it with the corresponding critical function* $A_*^\natural$ *determined explicitly from the third-order Taylor expansion* f^* *of* $f = f(u)$ *at* $u = 0$. *(See* (5.17) *below.) We rely on the results in Section 3, especially the formula* (3.26) *which provides the threshold ratio explicitly for the cubic flux.*

Fix some value $u_0 > 0$ *and the speed* $\lambda = \lambda^\natural(u_0)$ *so that, with the notation introduced earlier,* $u_2 = u_1 = \varphi^\natural(u_0)$. *Since* $f'''(0) \neq 0$ *it is not difficult to see that*

$$u_2 = \varphi^\natural(u_0) = -(1 + O(u_0))\, \frac{u_0}{2}$$

(as is the case for the cubic flux $f(u) = u^3$*). A straightforward Taylor expansion for the function*

$$G(u) := G(u; u_0, \lambda^\natural(u_0))$$

yields

$$\begin{aligned} G(u) - G(u_2) &= G(u) - G(\varphi^\natural(u_0)) \\ &= \frac{(u-u_2)^3}{24} \Big(f'''(0)\, \frac{c_2(0)}{c_1(0)}\, (3\,u_2 + u) + O(|u_2|^2 + |u|^2) \Big). \end{aligned}$$

Since, for all $u \in [u_2, u_0]$

$$4\,u_2 < u + 3\,u_2 < u_0 + 3\,u_2 = u_2\,(1 + O(u_0)),$$

we arrive at

$$\Big|G(u)-G(u_2)-f'''(0)\,\frac{c_2(0)}{c_1(0)}\,(u+3\,u_2)\,\frac{(u-u_2)^3}{24}\Big| \le C\,u_0\,|u+3\,u_2|\,(u-u_2)^3. \tag{5.13}$$

Now, given $\varepsilon > 0$, we can assume that u_0 is sufficiently small so that

$$\begin{aligned}
&(i) \quad -\frac{u_0}{2}\,(1+\varepsilon) \le u_2 \le -\frac{u_0}{2}\,(1-\varepsilon),\\
&(ii) \quad (1-\varepsilon)\,\frac{b(0)}{c_1(0)} \le \frac{b(u)}{c_1(u)} \le (1+\varepsilon)\,\frac{b(0)}{c_1(0)}, \quad u \in [u_2, u_0],\\
&(iii) \quad c_j(0)\,(1-\varepsilon) \le c_j(u) \le c_j(0)\,(1+\varepsilon), \quad u \in [u_2, u_0], \quad j = 1, 2.
\end{aligned} \tag{5.14}$$

Introduce next the flux-function

$$f_*(u) = k\,\frac{u^3}{6}, \quad k = (1+\varepsilon)\,f'''(0), \quad u \in \mathbb{R}. \tag{5.15}$$

Define the following (constant) functions

$$b^*(u) = b(0), \quad c_1^*(u) = c_1(0), \quad c_2^*(u) = c_2(0).$$

To these functions we can associate a function G_* by the general definition in Section 4. We are interested in traveling waves associated with the functions f_*, b^*, c_1^*, and c_2^*, and connecting the left-hand state u_0^* given by

$$u_0^* = -2\,u_2$$

to the right-hand state u_2 (which will also correspond to the traveling wave associated with f).

The corresponding function

$$G_*(u) := G_*(u; u_0^*, \lambda^\natural(u_0^*))$$

satisfies

$$G_*(u) - G_*(u_2) = f'''(0)\,\frac{c_2(0)}{c_1(0)}\,\frac{(1+\epsilon)}{24}\,(u+3\,u_2)\,(u-u_2)^3. \tag{5.16}$$

In view of Remark 2.6 the threshold function $A_*^\natural$ *associated with* f_*, b^*, c_1^*, *and* c_2^* *is*

$$A_*^\natural(u_0^*) = \frac{\sqrt{3k}\, c_1(0)c_2(0)}{4\, b(0)}\, u_0^*. \tag{5.17}$$

By Theorem 5.6, for the value $\alpha^* := A_*^\natural(u_0^*)$ *there exists also a traveling wave trajectory connecting* u_0^* *to* $u_2^* := u_2$, *which we denote by* $v^* = v^*(u)$. *By definition, in the phase plane it satisfies*

$$v^* \frac{dv^*}{du}(u) + \alpha^* \frac{b^*(u)}{c_1^*(u)}\, v^*(u) = G_*'(u), \tag{5.18}$$

with

$$G_*'(u) = \big(f_*(u) - f_*(u_0^*) - f_*'(u_2)\,(u - u_0^*)\big)\, \frac{c_2^*(u)}{c_1^*(u)}.$$

We consider also the traveling wave trajectory $u \mapsto v = v(u)$ *connecting* u_0 *to* u_2 *which is associated with the data* f, b, c_1, *and* c_2 *and the threshold value* $\alpha := A^\natural(u_0)$. *We will now establish lower and upper bounds on* $A^\natural(u_0)$; *see* (5.24) *and* (5.25) *below.*

Case 1 : *First of all, in the easy case that* $A^\natural(u_0)\,(1-\varepsilon) \le A_*^\natural(u_0^*)$, *we immediately obtain by* (5.17) *and then* (5.14)

$$\begin{aligned} A^\natural(u_0) &\le (1+2\,\varepsilon)\, A_*^\natural(u_0^*) = (1+2\,\varepsilon)\,\sqrt{3k}\,\frac{c_1(0)c_2(0)}{4\,b(0)}\, u_0^* \\ &\le (1+2\,\varepsilon)\,\sqrt{3k}\,\frac{c_1(0)c_2(0)}{4\,b(0)}\, u_0\,(1+\varepsilon) \\ &\le (1+C\,\varepsilon)\,\sqrt{3f'''(0)}\,\frac{c_1(0)c_2(0)}{4\,b(0)}\, u_0, \end{aligned}$$

which is the desired upper bound for the threshold function.

Case 2 : *Now, assume that* $A^\natural(u_0)\,(1-\varepsilon) > A_*^\natural(u_0^*)$ *and let us derive a similar inequality on* $A^\natural(u_0)$. *Since* $G'(u_2) = G_*'(u_2) = 0$, $G''(u_2) = G_*''(u_2) = 0$, *and*

$$v(u_2) = v(u_2^*) = 0, \quad \frac{dv}{du}(u_2) < 0, \quad \frac{dv^*}{du}(u_2) < 0,$$

it follows from the equation

$$\frac{dv}{du}(u) + \alpha \, \frac{b(u)}{c_1(u)} = \frac{G'_u(u; u_0, \lambda)}{v(u)}$$

by letting $u \to u_2$ *that*

$$\frac{dv}{du}(u_2) = -A_*^\natural(u_0) \, \frac{b(u_2)}{c_1(u_2)} < \frac{-1}{1-\varepsilon} \, A_*^\natural(u_0^*) \, \frac{b(0)}{c_1(0)} \, (1-\varepsilon) = \frac{dv^*}{du}(u_2).$$

This tells us that in a neighborhood of the point u_2 *the curve* v *is locally below the curve* v^**.*

Suppose that the two trajectories meet for the "first time" at some point $u_3 \in (u_2, u_0]$*, so*

$$v(u_3) = v^*(u_3) \qquad \text{with } \frac{dv}{du}(u_3) \geq \frac{dv^*}{du}(u_3).$$

From the equations (5.5) *satisfied by* $v = v(u)$ *and* $v^* = v^*(u)$*, we deduce*

$$\frac{1}{2} \, v(u_3)^2 + \alpha \int_{u_2}^{u_3} v(u) \, \frac{b(u)}{c_1(u)} \, du = G(u_3) - G(u_2),$$

and

$$\frac{1}{2} \, v^*(u_3)^2 + \alpha^* \int_{u_2}^{u_3} v^*(u) \, \frac{b(u)}{c_1(u)} \, du = G_*(u_3) - G_*(u_2),$$

respectively. Subtracting these two equations and using (5.13) *and* (5.16)*, we obtain*

$$\begin{aligned} \alpha \int_{u_2}^{u_3} v(u) \, \frac{b(u)}{c_1(u)} \, du - \alpha^* \int_{u_2}^{u_3} v^*(u) \, \frac{b^*(u)}{c_1^*(u)} \, du & \\ = G(u_3) - G(u_2) - \big(G_*(u_3) - G_*(u_2) \big) & \\ \geq (O(u_0) - C\,\varepsilon) \, (u_3 + 3\,u_2) \, (u_3 - u_2)^3. & \end{aligned} \tag{5.19}$$

But, by assumption the curve v *is locally below the curve* v^* *so that the left-hand side of* (5.19) *is negative, while its right-hand side of* (5.19) *is positive if one chooses* u_0 *sufficiently small. We conclude that the two trajectories intersect only at* u_2*, which implies that* $u_0^* \leq u_0$ *and thus*

$$\int_{u_2}^{u_0} |v(u)| \, du > \int_{u_2}^{u_0^*} |v^*(u)| \, du. \tag{5.20}$$

On the other hand we have by (5.14)

$$A^\natural(u_0)\,\frac{b(0)}{c_1(0)}\,(1-\varepsilon)\int_{u_2}^{u_0}|v(u)|\,du \le A^\natural(u_0)\int_{u_2}^{u_0}\frac{b(u)}{c_1(u)}\,|v(u)|\,du \\ = G(u_2)-G(u_0). \tag{5.21}$$

Now, in view of the property (i) in (5.14) *we have*

$$|3\,u_2+u_0| \le \frac{u_0}{2}\,(1+3\,\varepsilon) \le |u_2|\,\frac{1+3\varepsilon}{1-\varepsilon}, \quad |u_2-u_0| \le \frac{u_0}{2}\,(3+\varepsilon) \le |u_2|\,\frac{3+\varepsilon}{1-\varepsilon}.$$

Based on these inequalities we deduce from (5.13) *that*

$$G(u_2)-G(u_0) \le f'''(0)\,\frac{c_2(0)}{c_1(0)}\,\frac{9\,|u_2|^4}{8}\,(1+C\,\varepsilon). \tag{5.22}$$

Concerning the second curve, $v^* = v^*(u)$, we have

$$A_*^\natural(u_0^*)\,\frac{b(0)}{c_1(0)}\int_{u_2}^{u_0^*}|v^*(u)|\,du = G_*(u_2)-G_*(u_0^*) \\ = f'''(0)\,\frac{c_2(0)}{c_1(0)}\,\frac{9\,|u_2|^4}{8}\,(1+\varepsilon) \tag{5.23}$$

by using (5.16).

Finally, combining (5.20)–(5.23) we conclude that for every ε and for all sufficiently small u_0:

$$A^\natural(u_0) \le (1+C\,\varepsilon)\,A_*^\natural(u_0^*) \\ \le (1+C\,\varepsilon)\,\sqrt{3f'''(0)}\,\frac{c_1(0)c_2(0)}{4\,b(0)}\,u_0, \tag{5.24}$$

which is the desired upper bound. Exactly the same analysis as before but based on the cubic function $f_*(u) = k\,u^3$ with $k = (1-\varepsilon)\,f'''(0)$ (exchanging the role played by f_* and f, however) we can also derive the following inequality

$$A^\natural(u_0) \ge \sqrt{3f'''(0)}\,\frac{c_1(0)c_2(0)}{4\,b(0)}\,u_0\,(1-C\,\varepsilon). \tag{5.25}$$

The proof of Theorem 5.7 is thus completed since ε is arbitrary in (5.24) and (5.25).

□

6 Traveling waves corresponding to a given diffusion-dispersion ratio

Fixing the parameter α, we can now complete the proof of Theorem 4.3 by identifying the set of right-hand state attainable from u_0 by classical trajectories. We rely here mainly on Theorem 5.1 (existence of the nonclassical trajectories) and Theorem 5.5 (critical function).

Given $u_0 > 0$ and $\alpha > 0$, a classical traveling wave must connect $u_- = u_0$ to $u_+ = u_1$ for some shock speed $\lambda \in (\lambda^\natural(u_0), f'(u_0))$. Theorem 5.5, to each pair of states (u_0, u_2) we can associate the critical ratio $A(u_0, u_2)$. Equivalently, to each left-hand state u_0 and each speed λ, we can associate a critical value $B(\lambda, u_0) = A(u_0, u_2)$. The mapping

$$\lambda \mapsto B(\lambda, u_0)$$

is defined and decreasing from the interval $\big[\lambda^\natural(u_0), \lambda_0(u_0)\big]$ onto $\big[0, A^\natural(u_0)\big]$. It admits an inverse

$$\alpha \mapsto \Lambda_\alpha(u_0),$$

defined from the interval $\big[0, A^\natural(u_0)\big]$ onto $\big[\lambda^\natural(u_0), \lambda_0(u_0)\big]$. By construction, given any $\alpha \in \big(0, A^\natural(u_0)\big)$ there exists a nonclassical traveling trajectory (associated with the shock speed $\Lambda_\alpha(u_0)$) leaving from u_0 and solving the equation with the prescribed value α.

It is natural to *extend the definition of the function* $\Lambda_\alpha(u_0)$ to arbitrary values α by setting

$$\Lambda_\alpha(u_0) = \lambda^\natural(u_0), \quad \alpha \geq A^\natural(u_0).$$

The nonclassical traveling waves are considered here when α is a fixed parameter. So, we define the *kinetic function* for nonclassical shocks,

$$(u_0, \alpha) \mapsto \varphi^\flat_\alpha(u_0) = u_2,$$

where u_2 denotes the right-hand state of the nonclassical trajectory, so that

$$\frac{f(u_0) - f(u_2)}{u_0 - u_2} = \Lambda_\alpha(u_0). \tag{6.1}$$

Note that $\varphi^\flat_\alpha(u_0)$ makes sense for all $u_0 > 0$ but $\alpha < A^\natural(u_0)$.

Theorem 6.1. *For all $u_0 > 0$ and $\alpha > 0$ and for every speed satisfying*

$$\Lambda_\alpha(u_0) < \lambda \le f'(u_0),$$

there exists a unique traveling wave connecting $u_- = u_0$ to $u_+ = u_1$. Moreover, for $\alpha \ge A^\natural(u_0)$ there exists a traveling wave connecting $u_- = u_0$ to $u_+ = u_1$ for all

$$\lambda \in \big[\lambda^\natural(u_0), f'(u_0)\big].$$

Proof. *We first treat the case $\alpha \le A^\natural(u_0)$ and $\lambda \in \big(\Lambda_\alpha(u_0), f'(u_0)\big]$. Consider the curve $u \mapsto v_-(u;\lambda,\alpha)$ defined on $[u_1, u_0]$ that was introduced earlier in the proof of Theorem 5.1. We have either $v_-(u_1;\lambda,\alpha) = 0$ and the proof is completed, or else $v_-(u_1;\lambda,\alpha) < 0$. In the latter case, the function v_- is a solution of* (5.5) *that extends further on the left-hand side of u_- in the phase plane. On the other hand, this curve cannot cross the nonclassical trajectory $u \mapsto v(u)$ connecting $u_- = u_0$ to $u_+ = \varphi^\flat_\alpha(u_0)$. Indeed, by Lemma 4.6 we have*

$$\overline{\mu}(u_0;\lambda,\alpha) < \overline{\mu}(u_0;\Lambda_\alpha(u_0),\alpha).$$

If the two curves would cross, there would exist $u^* \in (\varphi^\flat_\alpha(u_0), u_1)$ such that

$$v(u^*) = v_-(u^*) \quad \text{and} \quad \frac{dv}{du}(u^*) \le \frac{dv_-}{du}(u^*).$$

By comparing the equations (5.5) *satisfied by these two trajectories we get*

$$v(u^*)\left(\frac{dv}{du}(u^*) - \frac{dv_-}{du}(u^*)\right) = \big(\lambda - \Lambda_\alpha(u_0)\big)\,(u^* - u_0)\,\frac{c_2(u^*)}{c_1(u^*)}. \qquad (6.2)$$

This leads to a contradiction since the right-hand side of (6.2) *is positive while the left-hand side is negative. We conclude that the function v_- must cross the u-axis at some point u_3 with $u_2 < \varphi^\flat_\alpha(u_0) < u_3 < u_1$. The curve $u \mapsto v_-(u,\lambda,\alpha)$ on the interval $[u_3, u_0]$ corresponds to a solution $y \mapsto u(y)$ in some interval $(-\infty, y_3]$ with $u_y(y_3) = 0$ and*

$$u_{yy}(y_3) = \frac{g\big(u(y_3),\lambda\big) - g\big(u_0,\lambda\big)}{c_1(u(y_3))\,c_2(u(y_3))} = \frac{G'_u(u_3;u_0,\lambda)}{c_2(u_3)^2}, \qquad (6.3)$$

which is positive by Theorem 4.7. Thus $u_{yy}(y_3) > 0$ *and necessarily* $u(y) > u_3$ *for* $y > y_3$*. Indeed, assume that there exists* $y_4 > y_3$*, such that* $u(y_4) = u(y_3) = u_3$*. Then, multiplying* (4.25) *by* v_-/c_2 *and integrating over* $[y_3, y_4]$*, we obtain*

$$\frac{1}{2}\, v_-^2(y_4) + \alpha \int_{y_3}^{y_4} \frac{b(u)}{c_1(u)\, c_2(u)}\, v_-^2\, dy = G(u_3; u_0, \lambda) - G(u_3; u_0, \lambda) = 0.$$

This would means that $u(y) = u_3$ *for all* y*, which is excluded since* $u_- = u_1$*.*

Now, since $u \leq u_0$ *we see that* u *is bounded. Finally, by integration over the interval* $(-\infty, y]$ *we obtain*

$$\frac{1}{2}\, v_-^2(y) + \alpha \int_{-\infty}^{y} \frac{b(u)}{c_1(u)\, c_2(u)}\, v_-^2\, dy = G(u(y)) - G(u_0),$$

which implies that v *is bounded and that the function* u *is defined on the whole real line* $\mathbb{R}$*. When* $y \to +\infty$ *the trajectory* (u, v) *converges to a critical point which can only be* $(u_1, 0)$*.*

Consider now the case $\alpha > A^\natural(u_0)$*. The proof is essentially same as the one given above. However, we replace the nonclassical trajectory with the curve* $u \mapsto v_+(u)$ *defined on the interval* $[u_2, u_1]$*. For each* λ *fixed in* $(\lambda^\natural(u_0), f'(u_0))$ *(since* $\alpha > A^\natural(u_0)$*) and thanks to Remark 4.4, the function,* $W = V_+ - V_-$ *(defined in the proof of Theorem 5.1, with* $v_-(u; \lambda, \alpha)$ *and* $v_+(u; \lambda, \alpha)$ *and extended to* $\lambda \in (f'(u_2), f'(u_0))$*) satisfies* $W(\alpha) < 0$*. On the left-hand side of* u_1*, with the same argument as in the first part above, we can prove that the extension of* v_- *does not intersect* v_+ *and must converge to* $(u_1, 0)$*. Finally, the case* $\lambda = \lambda^\natural(u_0)$ *is reached by continuity. This completes the proof of Theorem 6.1.*

□

Theorem 6.2. *If* $\lambda^\natural(u_0) < \lambda < \Lambda_\alpha(u_0)$ *there is no traveling wave connecting* $u_- = u_0$ *to* $u_+ = u_1$*.*

Proof. *Assume that there exists a traveling wave connecting* u_0 *to* u_1*. As in Lemma 5.2, we prove easily that such a curve must approach* $(u_0, 0)$

from the quadrant Q_1 *and coincide with the function* v_- *on the interval* $[u_1, u_0]$. *On the other hand, as in the proof of Theorem 6.1, we see that this curve does not cross the nonclassical trajectories. On the other hand, we have*

$$\overline{\mu}(u_0; \lambda, \alpha) \geq \overline{\mu}\big(u_0; \Lambda_\alpha(u_0), \alpha\big),$$

thus, the classical curve remains "under" the nonclassical one. So we have

$$v_-\big(\varphi_\alpha^\flat(u_0)\big) < v\big(\varphi_\alpha^\flat(u_0)\big),$$

where $u \mapsto (u, v(u))$ *denotes the nonclassical trajectory. Assume now that the curve* $(u, v_-(u))$ *meets the u-axis for the first time at some point* $(u_3, 0)$ with $u_3 < \varphi_\alpha^\flat(u_0) < u_2$. *The previous curve defined on* $[u_3, u_0]$ *corresponds to a solution* $y \mapsto u(y)$ *defined on some interval* $(-\infty, y_3]$ *with* $u_y(y_3) = 0$ *and* $u_{yy}(y_3) \geq 0$. *Thus* $v_y(y_3)$ *satisfies* (6.3) *and is negative (Lemma 5.3). This implies that* $u_{yy}(y_3) < 0$ *which is a contradiction. Finally, the trajectory remains under the u-axis for* $u < u_2$, *and cannot converge to any critical point.*

□

According to Theorem 6.1 the kinetic function can now be extended to all values of α by setting

$$\varphi_\alpha^\flat(u_0) = \varphi^\natural(u_0), \quad \alpha \geq A^\natural(u_0). \tag{6.4}$$

Finally we have:

Theorem 6.3. (Monotonicity of the kinetic function.) *For each* $\alpha > 0$ *the mapping* $u_0 \mapsto \varphi_\alpha^\flat(u_0)$ *is decreasing.*

Proof. *Fix* $u_0 > 0$, $\alpha > 0$, $\lambda = \Lambda_\alpha(u_0)$ *and* $u_2 = \varphi_\alpha^\flat(u_0)$. *First suppose that* $\alpha \geq A^\natural(u_0)$. *Then, for all* $u_0^* > u_0$, *since* $\varphi^\natural$ *is known to be strictly monotone, it is clear that*

$$\varphi_\alpha^\flat(u_0^*) \leq \varphi^\natural(u_0^*) < \varphi^\natural(u_0) = \varphi_\alpha^\flat(u_0).$$

Suppose now that $\alpha < A^\natural(u_0)$*. Then, for* $u_0^* > u_0$ *in a neighborhood of* u_0*, the speed* $\lambda^* = \frac{f(u_0^*)-f(u_2)}{u_0^*-u_2}$ *satisfies* $\lambda^* \in \big(\lambda^\natural(u_0^*), \lambda_0(u_0^*)\big)$*. Then, there exists a nonclassical traveling wave connecting* u_0^* *to* u_2 *for some* $\alpha^* = A(u_0^*, u_2)$*. The second statement in Theorem 5.5 gives* $\alpha^* > \alpha$*. Since the function* Λ_α *is decreasing (by the first statement in Theorem 5.5) we have* $\Lambda_{\alpha_*}(u_0^*) < \Lambda_\alpha(u_0^*)$ *and thus* $\varphi^\flat_\alpha(u_0^*) < u_2 = \varphi^\flat_\alpha(u_0)$ *and the proof of Theorem 6.3 is completed.*

□

Proof of Theorem 4.3 *Section 5 provides us with the existence of nonclassical trajectories, while Theorems 6.1 and 6.2 are concerned with classical trajectories. These results prove that the shock set is given by* (4.14)*. By standard theorems on solutions of ordinary differential equations the kinetic function is smooth in the region* $\big\{\alpha \le A^\natural(u_0)\big\}$ *while it coincides with the (smooth) function* $\varphi^\natural$ *in the region* $\big\{\alpha \ge A^\natural(u_0)\big\}$*. Additionally, by construction the kinetic function is continuous along* $\alpha = A^\natural(u_0)$*. This proves that* $\varphi^\flat$ *is Lipschitz continuous on each compact interval. On the other hand, the monotonicity of the kinetic function is provided by Theorem 6.3. The asymptotic behavior was the subject of Theorem 5.7.*

□

Remark 6.4. *To a large extend the techniques presented in this paper extend to* systems of equations*, in particular to a classical model of elastodynamics and phase transitions. The corresponding traveling wave solutions* $(v, w) = (v(y), w(y))$ *must solve*

$$\begin{aligned} &-s\, v_y - \Sigma\big(w, w_y, w_{yy}\big)_y = \big(\mu(w)\, v_y\big)_y, \\ &s\, w_y + v_y = 0, \end{aligned}$$

where s *denotes the speed of the traveling wave,* Σ *is the total stress function, and* $\mu(w)$ *is the viscosity coefficient. When* Σ *is given, some inte-*

gration with respect to y we arrive at

$$
\begin{aligned}
&-s\,(v-v_-)-\sigma(w)+\sigma(w_-)-\mu(w)\,v_y=\frac{\lambda'(w)}{2}\,w_y^2-\big(\lambda(w)\,w_y\big)_y,\\
&\quad s\,(w-w_-)+v-v_-=0,
\end{aligned}
$$

where (v_-, w_-) denotes the upper left-hand limit and $\lambda(w)$ the capillarity coefficient. Using the second equation above we can eliminate the unknown $v(y)$, namely

$$
\lambda(w)^{1/2}\,\big(\lambda(w)^{1/2}\,w_y\big)_y+\mu(w)\,v_y=s^2\,(w-w_-)-\sigma(w)+\sigma(w_-), \quad (6.5)
$$

which has precisely the structure of the equation (4.4) *studied in the present paper, so that most of our results extend to the equation* (6.5).

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