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# Elliptic Singular Problems with a Quadratic Gradient Term

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**Abstract**

We deal with existence and nonexistence of positive classical solutions to the Dirichlet problem for the quasilinear singular elliptic equation $-\Delta u = \lambda\, \beta(u)\, |\nabla u|^2 + \Psi(x)$ in $\Omega$, where $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, $\beta : (0,\infty) \to (0,\infty)$ is a $C^1$-function, possibly singular at zero in the sense that $\beta(s) \overset{s\to 0}{\to} \infty$, and $\Psi : \overline{\Omega} \to [0,\infty)$ is continuous. No monotonicity condition whatsoever is imposed upon $\beta$.

## 1 Introduction

The aim of the present paper is to study existence and non-existence of classical solutions for the quasilinear problem

$$\begin{cases} -\Delta u = \lambda\, \beta(u)\, |\nabla u|^2 + \Psi(x) \ \text{ in } \ \Omega, \\ \\ u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

*Key Words and Phrases:* elliptic equations, gradient term, fixed points, lower and upper solutions, singular problems.

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where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $\beta : (0, \infty) \to (0, \infty)$ is $C^1$ and $\Psi : \Omega \to [0, \infty)$ are suitable functions.

Our main interest is in the case $\beta$ is singular at zero, in the sense that $\beta(s) \overset{s \to 0}{\to} \infty$ and no monotonicity assumption whatsoever is required from $\beta$.

Problems like (1.1) with $\beta$ non-singular have been intensively investigated. In the recent, inspiring paper [2], Abdellaoui, Dall'Aglio & Peral, motivated in part by some features of the parabolic equation

$$u_t - \epsilon \Delta u = |\nabla u|^2,$$

where $\epsilon > 0$, investigated existence, non-existence, multiplicity and regularity of solutions of the non-singular problem

$$\begin{cases} -\Delta u = \beta(u)\ |\nabla u|^2 + \lambda \Psi(x) \ \text{ in } \ \Omega, \\ \\ u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\beta : [0, \infty) \to [0, \infty)$ is some continuous function and $\Psi$ belongs to a suitable Lebesgue space. Connections of (1.2) with elliptic problems involving measure data are also dealt with.

There is by now a broad literature on problems like (1.1) in the case $\beta$ is non-singular. We refer the reader to Kazdan & Kramer [4], Porreta & Segura de Leon [1], Boccardo, Segura de Leon & Trombetti [5], where even operators more genreral than $\Delta$ are treated, as well as their references.

Our research is in part motivated by the study of the parabolic equation

$$u_t = u\Delta u - \lambda |\nabla u|^2,$$

where $\lambda > 0$ is a parameter. This equation appears in the investigation of physical phenomena such as the filtration of a fluid through a porous medium, see e.g. Zeng'an Yao & Wenshu Zhou [9], Bertsch, Dal Passo & Ughi [6] and their references.

Notice that the singular equation

$$-\Delta u = \frac{\lambda}{u}\,|\nabla u|^2,$$

which represents the stationary part of the parabolic equation above is a special case of the equation in (1.1).

It is interesting to point out that the non-singular problem

$$-\Delta u = \lambda|\nabla u|^2 \ \text{ in } \ B, \quad u > 0 \ \text{ in } \ B, \quad u = 0 \ \text{ on } \ \partial B$$

where $B \subset \mathbf{R}^3$ is the unit ball and $\lambda > 0$, has a weak solution in $H_0^1(B)$, namely

$$u(x) := \lambda \log(|x|^{-1})$$

which, in fact blows-up to $\infty$ at the origin, while the singular problem

$$-\Delta u = \frac{\lambda}{u}\,|\nabla u|^2 \ \text{ in } \ B, \quad u > 0 \ \text{ in } \ B, \quad u = 0 \ \text{ on } \ \partial B$$

admits no weak solution in $H_0^1(B)$ if $\lambda \in (0,1)$.

The reader is additionally referred to Cui [8], García-Melián [3], Zhang [10] and references therein for further results on singular problems in the presence of a gradient term.

We shall assume that $N \geq 3$ and $\Psi \in C^{0,\alpha}_{loc}(\Omega) \cap C(\overline{\Omega})$ is a non-negative function. Our main result is.

**Theorem 1.1.** *Assume $\beta : (0,\infty) \to (0,\infty)$ is $C^1$-function and satisfies*

$$\limsup_{s\to\infty} \frac{\beta(s)}{s^\theta} < \infty, \quad \textit{for some} \ \ \theta > 0. \tag{1.3}$$

*Then* (1.1) *admits:*

(*i*) *a solution in* $C^2(\Omega)\cap C(\overline{\Omega})$ *if* $\Psi \not\equiv 0$ *and* $0 < \lambda \le \lambda^*$ *for some* $\lambda^* > 0$,

(*ii*) *no solution in* $C^2(\Omega)\cap C(\overline{\Omega})$ *if* $\Psi \equiv 0$ *and* $\lambda > 0$.

Examples to which theorem 1.1 applies are given below. Let $a, \beta > 0$ and $b, c, \kappa \ge 0$ be constants. Our result applies to problems where $\beta(u)$ is not necessarily monotone and eventually oscillating such as

$$\begin{cases} -\Delta u = \lambda \ \Big[\dfrac{a}{u^\beta} + b(1+\dfrac{\sin(1/u)}{2})u + cu^\kappa\Big] \ |\nabla u|^2 + \Psi \ \text{ in } \ \Omega \\ \\ \qquad\qquad u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Taking into account that $\beta$ is not necessarily monotone we introduce the auxiliary monotone non-increasing function $h : (0,\infty) \to (0,\infty)$,

$$h(s) := h_\theta(s) = \sup_{t \ge s} \frac{\beta(t)}{t^\theta}, \quad s > 0. \tag{1.4}$$

By (1.3) this new function $h$ is well defined and satisfies the conditions,

$$\text{(i)} \ \ h > 0 \ \text{ is non-increasing in } \ (0,\infty), \quad \text{(ii)} \quad h \in Lip_{loc}(0,\infty). \tag{1.5}$$

The auxiliary problem associated to the possibly singular monotone function $h$,

$$\begin{cases} -\Delta u = \lambda \ h(u) \ |\nabla u|^2 + \Psi(x) \ \text{ in } \ \Omega \\ \\ \quad u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{1.6}$$

will play a key role. Arguments on lower and upper solutions will be applied in a crucial form in this paper and the result below, whose proof is given in Section 3, represents a main step in the sense that it will provide us with an upper solution for (1.1).

**Theorem 1.2.** *If the function* $\beta$ *satisfies* (1.3) *then there is* $\Lambda_\Omega > 0$ *such that* (1.6) *has a solution in* $C^2(\Omega)\cap C(\overline{\Omega})$ *provided* $0 < \lambda \le \Lambda_\Omega$.

After constructing a lower solution of (1.1) we will apply the following special form of a result by Cui [8].

**Lemma 1.3.** (Cui [8, lemma 3]) *Let* $\beta : (0,\infty) \to (0,\infty)$ *be* $C^1$. *If* $\underline{u}, \overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ *are respectively lower and upper solutions of* (1.1) *in the sense that*

$$\text{(i)} \; -\Delta \underline{u} \leq \lambda\, \beta(\underline{u})\, |\nabla \underline{u}|^2 + \Psi \;\text{ in } \Omega, \;\; \text{(ii)} \; -\Delta \overline{u} \geq \lambda\, \beta(\overline{u})\, |\nabla \overline{u}|^2 + \Psi \text{ in } \Omega,$$

$$\text{(iii)} \quad 0 < \underline{u}(x) \leq \overline{u}(x), \quad x \in \Omega, \quad \text{(iv)} \quad \underline{u} = \overline{u} = 0 \;\text{ on }\; \partial\Omega,$$

*then* (1.1) *has a classical solution* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfying* $\underline{u} \leq u \leq \overline{u}$.

## 2 Construction of a Radially Symmetric Upper Solution for (1.6)

Let $R > 0$ such that $\overline{\Omega} \subset B_R$, take a positive continuous extension of $\Psi$ to $\overline{B}_R$ still labeled $\Psi$ and consider the differential inequality problem

$$\begin{cases} -\Delta u \geq \lambda\, h(u)\, |\nabla u|^2 + \Psi_1 \;\text{ in }\; B_R, \\ \\ u > 0 \;\text{ in }\; B_R, \quad u = 0 \;\text{ on }\; \partial B_R, \end{cases} \tag{2.1}$$

where $B_R$ is the ball of radius $R$ centered at the origin of $\mathbf{R}^N$, $h$ was defined in (1.4) and $\Psi_1 := \Psi + 1$. An upper solution of (1.1) will be obtained by first solving (2.1) and subsequently picking a suitable value for $R$. In order to solve (2.1) we shall prove the following result.

**Lemma 2.1.** *There is a positive number* $\Lambda_0 := \Lambda_0(R)$ *such that for each* $\lambda \in (0, \Lambda_0]$, (2.1) *admits at least one radially symmetric solution* $\vartheta \in C^2(B_R) \cap C(\overline{B}_R)$.

The proof of lemma 2.1 will follow as a consequence of some remarks and results to be provided in the sequel. In this regard let $\sigma$, $T$ and $d$ be positive numbers and consider the initial value problem

$$\begin{cases} -\Big(r^{N-1}v'(r)\Big)' = r^{N-1}\Big[h(v(r))\ v'(r)^2+\sigma\Big], \quad 0<r<T, \\ \\ v(0)=d, \quad v'(0)=0. \end{cases} \tag{2.2}$$

**Lemma 2.2.** *There are $T_0>0$ and a non-negative function $v \in C^2([0,T_0)) \cap C([0,T_0])$ satisfying both (2.2) with $T=T_0$ and the conditions*

$$(i) \quad v(r)>0, \quad v'(r)<0 \quad for \quad 0<r<T_0,$$

$$(ii) \quad v(T_0)=0.$$

At this point we remark, (see section 5), that $v \in C^2([0,T)) \cap C([0,T])$ is a solution of (2.2) iff

$$v(r) = d - \frac{\sigma r^2}{2N} - \int_0^r s^{1-N} \int_0^s t^{N-1} h(v(t)) v'(t)^2 dt ds, \quad 0 \le r < T. \tag{2.3}$$

**Proof of lemma 2.2.** Pick $\delta>0$. Denote the norm of $C^1([0,\delta])$ by

$$\|v\|_\delta = \max_{0\le t\le\delta} |v(t)| + \max_{0\le t\le\delta} |v'(t)|.$$

Set

$$E_\delta := \Big\{ v \in C^1([0,\delta]) \mid \|v-d\| \le d/2 \Big\}$$

and consider the operator $\mathcal{F}$ defined by

$$\mathcal{F}v(r) \equiv \mathcal{F}(v(r)) := d - \frac{\sigma r^2}{2N} - \int_0^r s^{1-N} \int_0^s t^{N-1} h(v(t)) v'(t)^2 dt ds, \ 0 \le r \le \delta.$$

It will be shown in section 5 that

$$\begin{aligned} &\text{(i)} \quad \mathcal{F}: E_\delta \to E_\delta \\ &\text{(ii)} \quad \|\mathcal{F}w_2 - \mathcal{F}w_1\|_\delta \le \tfrac{1}{2}\|w_2-w_1\|_\delta, \quad w_1, w_2 \in E_\delta, \end{aligned} \tag{2.4}$$

where $\delta>0$ is suitably small. By (2.4) $\mathcal{F}$ has a fixed point $v \in E_\delta$ which in fact, satisfies (2.2) in $[0,\delta]$. As a consequence (2.2) has a solution $v$.

**Verification of lemma 2.2(i).** Set

$$\text{(i)}\quad \mathcal{A} := \Big\{ r > 0 \mid \text{(2.2) has a positive solution in}(0,r) \Big\},$$

$$\text{(ii)}\quad T_0 := \sup \mathcal{A},$$

We infer by the local existence derived from (2.4) and also by (2.3) that

$$\delta \le T_0 \le \big(2Nd/\sigma\big)^{1/2}. \tag{2.5}$$

Using both the definition of $\mathcal{A}$ and (2.3) we have

$$v(r) > 0 \text{ and } v'(r) < 0, \quad 0 < r < T_0,$$

ending the verification of lemma 2.2(i).

**Verification of lemma 2.2(ii).** Since $v'(r) < 0$ for $0 < r < T_0$, setting $d_0 := \lim_{t \to T_0^-} v(r)$ we claim that $d_0 = 0$. Indeed assume, on the contrary, that $d_0 > 0$.

Let $\delta > 0$ and consider the set

$$E_\delta(d_0) := \Big\{ v \in C^1([T_0, T_0+\delta]) \mid \|v - v_0\| \le d_0/2 \Big\},$$

where

$$\|v\| := \max_{T_0 \le t \le T_0+\delta} |v(t)| + \max_{T_0 \le t \le T_0+\delta} |v'(t)|$$

and

$$v_0(t) := d_0 + \tfrac{\sigma(T_0^2 - t^2)}{2N}, \quad T_0 \le t \le T_0 + \delta.$$

Now, for $r \in [T_0, T_0+\delta]$ we set

$$\widehat{\mathcal{F}}v(r) \equiv \widehat{\mathcal{F}}(v(r)) := v_0(r) - \int_{T_0}^{r} s^{1-N} \int_{T_0}^{s} t^{N-1} h(v(t))v'(t)^2 dtds.$$

It will be shown in an Appendix that

$$\begin{aligned} &\text{(i)} \quad \widehat{\mathcal{F}} : E_\delta(d_0) \to E_\delta(d_0) \\ &\text{(ii)} \quad \|\widehat{\mathcal{F}}w_2 - \widehat{\mathcal{F}}w_1\| \leq \tfrac{1}{2}\|w_2 - w_1\|, \quad w_1, w_2 \in E_\delta(d_0). \end{aligned} \tag{2.6}$$

By (2.6), $\widehat{\mathcal{F}}$ has a fixed point in $v \in E_\delta(d_0)$ and by the very definition of $\widehat{\mathcal{F}}$, $v$ satisfies

$$v(r) = v_0(r) - \int_0^r s^{1-N} \int_{T_0}^{s} t^{N-1} h(v(t))v'(t)^2 dt\ ds, \quad T_0 \leq r \leq T_0 + \delta.$$

As a consequence (2.2) holds for $r \in [0, T_0 + \delta)$, contradicting the definition of $T_0$.

Therefore $d_0 = 0$, showing lemma 2.2(ii) and to finish the proof set $v(T_0) := 0$.

It remains to show that $v \in C^2([0, T_0)) \cap C([0, T_0])$. Integrating from 0 to $r$ in (2.2) once, we get to

$$-v'(r) = \tfrac{1}{r^{N-1}} \int_0^r s^{N-1} h(v(s))(v'(s))^2 ds + \frac{\sigma}{N}\ r. \tag{2.7}$$

Differentiating the expression above we get for $r \in (0, T_0)$,

$$v''(r) = -\tfrac{\sigma}{N} - (1 - N)r^{-N} \textstyle\int_0^r s^{N-1} h(v(s))v'(s)^2 ds - h(v(r))v'(r)^2$$

and so $v \in C^2((0, T_0)) \cap C([0, T_0])$ by some computations we have

$$v''(0) = \lim_{r \to 0} v''(r) = -\frac{\sigma}{N}.$$

showing that $v \in C^2([0, T_0)) \cap C([0, T_0])$. Lemma 2.2 is proved.

□

**Proof of lemma 2.1.** Let $\sigma = \max_{\overline{\Omega}} \Psi_1$, pick $d < \sigma R^2/2N$ and consider the solution $v \in C^2([0,T_0)) \cap C([0,T_0])$ of (2.2) given by lemma 2.2. Reminding that by (2.5), $T_0/R \leq 1$, set

$$\widetilde{\vartheta}(r) := \frac{R^2}{T_0^2}\, v(\frac{rT_0}{R}), \quad 0 \leq r \leq R.$$

It follows that

$$\widetilde{\vartheta}'(r) = \frac{R}{T_0} v'(\frac{T_0 r}{R}), \quad \widetilde{\vartheta}(r) > 0, \quad \widetilde{\vartheta}(0) = \frac{R^2}{T_0^2} d, \qquad \widetilde{\vartheta}(R) = 0.$$

Setting $s := rT_0/R$ and reminding that $T_0/R \leq 1$ we have $0 < s \leq r$. Thus,

$$\begin{aligned} -\big(r^{N-1}\widetilde{\vartheta}'(r)\big)' &= -\frac{d}{dr}\big((r\frac{T_0}{R})^{N-1}\, v'(\frac{rT_0}{R})\big)\frac{R}{T_0}\big(\frac{R}{T_0}\big)^{N-1} \\ &= r^{N-1}\big(\frac{T_0}{R}\big)^{N-1}\big[h(v(\frac{rT_0}{R})\, v'(\frac{rT_0}{R})^2 + \sigma\big]\big(\frac{R}{T_0}\big)^{N-1}. \end{aligned}$$

Using the fact that $h$ is monotone non-increasing we infer that

$$-\Big(r^{N-1}\widetilde{\vartheta}'(r)\Big)' \geq r^{N-1}\Big[\Big(\frac{T_0}{R}\Big)^2\, h(\widetilde{\vartheta}(r))\, \widetilde{\vartheta}'(r)^2 + \sigma\Big].$$

Setting $\Lambda_0 := \Lambda_0(R) = \Big(T_0/R\Big)^2$ and taking $\lambda \in [0, \Lambda_0]$ we infer that

$$-\Big(r^{N-1}\widetilde{\vartheta}'(r)\Big)' \geq r^{N-1}\Big[\lambda\, h(\widetilde{\vartheta}(r))\, \widetilde{\vartheta}'(r)^2 + \sigma\Big], \quad 0 < r < R.$$

Setting $\vartheta(x) := \widetilde{\vartheta}(|x|) := \widetilde{\vartheta}(r)$, where $r := |x|$ we get to

$$-\Delta\vartheta \geq \lambda\, h(\vartheta)\, |\nabla\vartheta|^2 + \Psi_1(x) \ \text{ in } \ B_R,$$

and reminding that $v \in C^2([0, T_0)) \cap C([0, T_0])$ we infer that $\vartheta \in C^2(B_R) \cap C(\overline{B}_R)$, ending the proof of lemma 2.1.

□

## 3 The Auxiliary Problem with a Singular Monotone Term

The main objective of this section is to prove theorem 1.2. We will make use of a well known result on lower and upper solutions, namely theorem 6.5 by Kazdan & Kramer [4]. In this regard consider the family of problems

$$\begin{cases} -\Delta u = \lambda \; h(u) \; |\nabla u|^2 + \Psi_\epsilon(x) \;\; \text{in} \;\; \Omega, \\ \\ \quad u > \epsilon \;\; \text{in} \;\; \Omega, \quad u \geq \epsilon \;\; \text{on} \;\; \partial\Omega, \end{cases} \tag{3.1}$$

where $\epsilon > 0$ is a parameter and $\Psi_\epsilon(x) := \Psi(x) + \epsilon, \; x \in \overline{\Omega}$. We will prove the following basic lemma.

**Lemma 3.1.** *Assume* (1.3) *and* (1.5)(*i*)(*ii*). *Then there are* $\epsilon_0 > 0$ *and a positive number* $\Lambda_\Omega$ *such that for each* $(\epsilon, \lambda) \in (0, \epsilon_0) \times (0, \Lambda_\Omega)$ *problem* (3.1) *has a solution* $u_\epsilon \in C^2(\overline{\Omega})$.

**Proof of lemma 3.1.** At first set

$$R_\Omega := \sup_{x \in \overline{\Omega}} |x|. \tag{3.2}$$

**Construction of an upper solution for** (3.1). Pick $R = (3/2)R_\Omega$ in lemma 2.1, let $\vartheta$ be the solution of (2.1) given by lemma 2.1 and set $\epsilon_0 := \min\{\widetilde{\vartheta}(R), 1\}$. Notice that by the definition of $\vartheta$ in the proof of lemma 2.2 $\epsilon_0$ is positive and $\Lambda_0$ depends on $\Omega$.

Set $\Lambda_\Omega := \Lambda_0(R)$, let $\epsilon \in (0, \epsilon_0)$ and set $u_\epsilon := \vartheta + \epsilon$. If $\lambda \in (0, \Lambda_\Omega]$, by lemma 2.1,

$$-\Delta u_\epsilon \geq \lambda\ h(u_\epsilon)\ |\nabla u_\epsilon|^2 + \Psi_\epsilon \ \text{ in } \ B_R,$$

$$u_\epsilon > \epsilon \ \text{ in } \ B_R, \quad u_\epsilon \geq \epsilon \ \text{ on } \ \partial B_R.$$

Notice that actually $u_\epsilon$ satisfies

$$-\Delta u_\epsilon \geq \lambda\ h(u_\epsilon)\ |\nabla u_\epsilon|^2 + \Psi_\epsilon \ \text{ in } \ \Omega,$$

$$u_\epsilon > \epsilon \ \text{ in } \ \Omega, \quad u_\epsilon \geq \epsilon \ \text{ on } \ \partial\Omega.$$

This shows that for each $(\epsilon, \lambda) \in (0, \epsilon_0) \times (0, \Lambda_\Omega]$, $u_\epsilon \in C^2(\overline{\Omega})$ is an upper solution of (3.1).

**Construction of a lower solution for** (3.1). Take $p > N$ and let $\widetilde{\omega}$ be the unique solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem

$$-\Delta\widetilde{\omega} = \Psi(x) \ \text{ in } \ \Omega, \quad \widetilde{\omega} > 0 \ \text{ in } \ \Omega, \quad \widetilde{\omega} = 0 \ \text{ on } \ \partial\Omega.$$

Take $\epsilon \in (0, \epsilon_0)$. Setting $\omega_\epsilon = \widetilde{\omega} + \epsilon$ it follows that

$$-\Delta\omega_\epsilon \leq \lambda\ h(\omega_\epsilon)\ |\nabla\omega_\epsilon|^2 + \Psi_\epsilon \ \text{ in } \ \Omega,$$

$$\omega_\epsilon > \epsilon \text{ in } \ \Omega, \quad \omega_\epsilon = \epsilon \ \text{ on } \ \partial\Omega,$$

provided $(\epsilon, \lambda) \in (0, \epsilon_0) \times (0, \Lambda_\Omega]$. In addition, notice that

$$-\Delta(u_\epsilon - \omega_\epsilon) > 0 \ \text{ in } \ \Omega, \quad u_\epsilon - \omega_\epsilon \geq 0 \ \text{ on } \ \partial\Omega,$$

which leads by the Maximum Principle to $u_\epsilon \geq \omega_\epsilon$ in $\Omega$. Applying theorem 6.5 of [4] we infer that (3.1) has a solution $u^\epsilon \in W^{2,p}(\Omega)$ provided $(\epsilon, \lambda) \in (0, \epsilon_0) \times (0, \Lambda_\Omega]$.

□

## 4 Proofs of the Main Theorems

At first we proceed to the

**Proof of Theorem 1.2.** Let $(\epsilon, \lambda) \in (0, \epsilon_0) \times (0, \Lambda_\Omega]$ and notice that, by the proof of lemma 3.1, $u^\epsilon > \widetilde{\omega}$ in $\overline{\Omega}$. We claim that

$$u^\epsilon \geq u^\delta \quad \text{in} \quad \Omega \quad \text{if} \quad \epsilon > \delta. \tag{4.1}$$

Indeed, pick $\delta, \epsilon \in (0, \epsilon_0]$ with $\delta < \epsilon$. Assume, on the contrary, that

$$u^\epsilon(x_0) < u^\delta(x_0)$$

for some $x_0 \in \Omega$. Since,

$$(u^\delta - u^\epsilon)(x) = \delta - \epsilon < 0 \quad \text{for} \quad x \in \partial\Omega$$

we have

$$\max_{x \in \overline{\Omega}} (u^\delta - u^\epsilon)(x) = (u^\delta - u^\epsilon)(x_1) \geq (u^\delta - u^\epsilon)(x_0) > 0,$$

for some $x_1 \in \Omega$. It follows that

$$\nabla u^\delta(x_1) = \nabla u^\epsilon(x_1) \quad \text{and} \quad \Delta(u^\delta(x_1) - u^\epsilon(x_1)) \leq 0.$$

Thus

$$\begin{aligned} \Delta u^\delta(x_1) - \Delta u^\epsilon(x_1) &= -\lambda\, h(u^\delta(x_1))\, |\nabla u^\delta(x_1)|^2 - \Psi_\delta(x_1) \\ &+ \lambda\, h(u^\epsilon(x_1))\, |\nabla u^\epsilon(x_1)|^2 + \Psi_\epsilon(x_1) \\ &= \lambda\, |\nabla u^\delta(x_1)|^2 \Big( h(u^\epsilon(x_1)) - h(u^\delta(x_1)) \Big) + (\epsilon - \delta). \end{aligned}$$

Since $h$ is non-decreasing it follows that $0 < \epsilon - \delta \leq 0$, which is impossible. So the claim (4.1) holds true.

Using (4.1), set

$$\overline{u}(x) = \lim_{\epsilon \to 0} u^{\epsilon}(x), \quad x \in \overline{\Omega}. \tag{4.2}$$

The function $\overline{u} : \overline{\Omega} \to [0, \infty)$ so defined satisfies $\overline{u}(x) \geq \widetilde{\omega}(x) > 0$ for $x \in \Omega$. In order to finish the proof of theorem 1.2 we claim that

$$\text{(i) } \overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$$

$$\text{(ii) } -\Delta \overline{u}(x) = \lambda \; h(\overline{u}(x)) \; |\nabla \overline{u}(x)|^2 + \Psi(x), \text{ in } \Omega \tag{4.3}$$

$$\text{(iii) } \overline{u} = 0 \quad \text{on } \partial\Omega.$$

To show (4.3) notice at first that $\overline{u} \in L^{\infty}(\Omega)$. Let $n \geq 1$ be an integer. Setting $\epsilon_n := 1/n$, , $u^{\epsilon_n} := u_n$ and $\Psi_{\epsilon_n} := \Psi_n := \Psi + \frac{1}{n}$ we have

$$\begin{aligned} -\Delta u_n &= \lambda \; h(u_n) \; |\nabla u_n|^2 + \Psi_n \quad \text{in} \quad \Omega, \\ u_n &\in W^{2,p}(\Omega), \quad \widetilde{\omega} < u_n \leq C_0 \quad \text{on} \quad \overline{\Omega}, \end{aligned} \tag{4.4}$$

where $p > N$ and $C_0 > 0$ is some positive constant. As a consequence,

$$-\int_{\Omega} u_n \Delta\phi \; dx = \int_{\Omega} \lambda h(u_n) |\nabla u_n|^2 \phi \; dx + \int_{\Omega} \Psi_n \phi \; dx, \quad \phi \in C_0^{\infty}(\Omega).$$

Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of bounded smooth sub-domains of $\Omega$ such that

$$\overline{\Omega}_k \subset \Omega_{k+1} \quad \text{and} \quad \Omega = \cup \Omega_k.$$

Let $\ell \geq 1$ be an integer. Pick $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi \leq 1$ and $\phi := 1$ on $\overline{\Omega}_\ell$.

Then

$$\lambda \int_{\Omega_\ell} h(u_n)|\nabla u_n|^2 \, dx \leq -\int_\Omega u_n \Delta\phi \, dx - \int_\Omega \Psi_n \phi \, dx$$

which leads to

$$\lambda h(C_0) \int_{\Omega_\ell} |\nabla u_n|^2 \, dx \leq C_0 \int_\Omega |\Delta\phi| \, dx + \int_\Omega (|\Psi|_\infty + 1) dx,$$

showing that $u_n$ is bounded in $H^1(\Omega_\ell)$. Passing to a subsequence we find that

$$u_n \rightharpoonup v \ \text{ in } \ H^1(\Omega_\ell),$$

$$u_n \to v \ \text{ in } \ L^s(\Omega_\ell), \quad 1 \leq s < 2^*,$$

$$u_n \to v \ \text{ a.e. in } \ \Omega_\ell.$$

By (4.2), $v = \overline{u}$ a.e. in $\Omega_\ell$. It follows by a standard diagonal process that, up to subsequences,

$$u_n \rightharpoonup \overline{u} \ \text{ in } \ H^1_{loc}(\Omega) \ \text{ and } \ u_n \to \overline{u} \ \text{ in } \ L^s_{loc}(\Omega), \quad 1 \leq s < 2^*.$$

Now set

$$f_n(x) := \lambda h(u_n(x))|\nabla u_n(x)|^2 + \Psi_n(x), \ x \in \Omega$$

and pick an integer $k \geq 1$. Since $u_n \in C^2(\Omega)$ we get using (4.4) that

$$u_n \in W^{2,p}(\Omega_{k+1}), \quad f_n \in L^\infty(\Omega_{k+1}),$$

$$-\Delta u_n = f_n \ \text{ a.e. in } \ \Omega_{k+2}.$$

By the a priori estimates for elliptic operators there is a constant $C > 0$ such that

$$|u_n|_{W^{2,p}(\Omega_{k+1})} \leq C(|u_n|_{L^p(\Omega_{k+2})} + |f_n|_{L^p(\Omega_{k+2})}).$$

Now, we claim that

$$|f_n|_{L^p(\Omega_{k+2})} \text{ is bounded.} \tag{4.5}$$

Indeed, since $C_{k,0} \leq u_n(x) \leq C_{k,1}$ for $x \in \overline{\Omega}_{k+2}$ and for some constants $C_{k,0}, C_{k,1} > 0$ it follows that

$$|f_n(x)| \leq C_{k,2} \, |\nabla u_n(x)|^2 + C_{k,4}, \quad x \in \overline{\Omega}_{k+2}$$

for $n \geq 1$.

By Ladyzenskaya & Uraltseva [7, theorem 3.1, pg 266] we infer that

$$\max_{x \in \overline{\Omega}_{k+1}} |\nabla u_n(x)|^2 \leq C_{k+1}$$

for some constant $C_{k+1} > 0$.

As a consequence, $|f_n(x)| \leq C_{k,5}$ for $x \in \overline{\Omega}_{k+1}$ and $n \geq 1$ and for some constant $C_{k,5} > 0$, showing (4.5).

Thus by (4.4), $|u_n|_{W^{2,p}(\Omega_{k+1})}$ is bounded and it follows by a standard embedding theorem that for some $\alpha \in (0,1)$, $|u_n|_{C^{1,\alpha}(\overline{\Omega}_{k+1})}$ is bounded, as well.

Recalling that $h$ is Lipschitz continuous we have

$$|h(u_n(x)) - h(u_n(y))| \leq C|u_n(x) - u_n(y)| \leq C|x-y|^\alpha, \quad x, y \in \overline{\Omega}_{k+1},$$

for some positive constant $C$. As a consequence, $f_n$ is bounded in $C^{0,\alpha}(\overline{\Omega}_{k+1})$.

By the Schauder estimates

$$|u_n|_{C^{2,\alpha}(\overline{\Omega}_k)} \leq C(|u_n|_{C^0(\overline{\Omega}_{k+1})} + |f_n|_{C^{0,\alpha}(\overline{\Omega}_{k+1})}),$$

$|u_n|_{C^{2,\alpha}(\overline{\Omega}_k)}$ is bounded. Eventually passing to a subsequence we have

$$u_n \overset{C^2(\overline{\Omega}_k)}{\to} u_k,$$

for some $u_k \in C^2(\overline{\Omega}_k)$. Actually,

$$-\Delta u_k = \lambda h(u_k(x))|\nabla u_k(x)|^2 + \Psi(x),\ x \in \Omega_k.$$

Notice that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $u_k = u_{k+1|\ \Omega_k}$. Using a standard diagonal limit process it follows that $u_k \to \overline{u}$ in $C^2(\overline{U})$ for each subdomain $U$ of $\Omega$ and hence

$$-\Delta \overline{u} = \lambda h(\overline{u})|\nabla \overline{u}(x)|^2 + \Psi(x),\ x \in \Omega. \tag{4.6}$$

To show that $\overline{u} \in C(\overline{\Omega})$ let $x_0 \in \partial\Omega$ and take $\{x_i\} \in \Omega$ such that $x_i \to x_0$. We have $0 \leq \overline{u}(x_i) \leq u_k(x_i)$ so that $\lim_i \overline{u}(x_i) \leq \lim_i u_k(x_i) = u_k(x_0) = 1/k$ and so $\overline{u}(x_0) = 0$. Hence $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$. This shows (4.3). Theorem 1.2 is proved.

□

Next, using the results above we give the proof of theorem 1.1.

**Proof of Theorem 1.1. Step 1.** (Proof of (i)). Let $\overline{u} := \overline{u}_{\Lambda_\Omega}$ be the solution of (1.6) with $\lambda = \Lambda_\Omega$ given by theorem 1.2. From the proof of theorem 1.2 we have both

$$0 < \widetilde{\omega}(x) \leq \overline{u}(x) \leq \vartheta(x), \quad x \in \Omega,$$

and

$$\widetilde{\vartheta}(r) \leq \widetilde{\vartheta}(0) = \frac{R^2 d}{T_0^2}.$$

Hence

$$\overline{u}(x)^\theta \leq \vartheta(x)^\theta = \widetilde{\vartheta}(r)^\theta \leq \frac{R^{2\theta} d^\theta}{T_0^{2\theta}}$$

so that

$$\frac{1}{\overline{u}(x)^\theta} \geq \frac{T_0^{2\theta}}{R^{2\theta} d^\theta}. \tag{4.7}$$

In addition, by (1.4) and (4.7),

$$h(\overline{u}(x)) \geq \frac{T_0^{2\theta}}{R^{2\theta} d^\theta} \; \beta(\overline{u}(x)). \tag{4.8}$$

By (4.6) and (4.8),

$$\begin{aligned} -\Delta \overline{u}(x) &= \Lambda_\Omega \; h(\overline{u}(x)) \; |\nabla \overline{u}(x)|^2 + \Psi(x) \\ &\geq \Lambda_\Omega \; \frac{T_0^{2\theta}}{R^{2\theta} d^\theta} \; \beta(\overline{u}(x)) \; |\nabla \overline{u}(x)|^2 + \Psi(x). \end{aligned}$$

Reminding that

$$\Lambda_\Omega := T_0^2 / R^2$$

we get

$$-\Delta \overline{u} \geq \left( \frac{T_0^{2(\theta+1)}}{R^{2(\theta+1)} d^\theta} \right) \beta(\overline{u}) \; |\nabla \overline{u}|^2 + \Psi(x) \quad \text{in} \quad \Omega.$$

Setting $\lambda^* := T_0^{2(\theta+1)} / R^{2(\theta+1)} d^\theta$, it follows that

$$-\Delta \overline{u} \geq \; \lambda \beta(\overline{u}) \; |\nabla \overline{u}|^2 + \Psi(x) \quad \text{in} \quad \Omega, \quad 0 < \lambda \leq \lambda^*.$$

In conclusion we have shown that

$$\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega}),$$

$$0 < \widetilde{\omega}(x) := \underline{u}(x) \leq \overline{u}(x), \quad x \in \Omega, \quad \underline{u} = \overline{u} = 0 \text{ on } \partial\Omega$$

$$-\Delta\overline{u} \geq \lambda\, \beta(\overline{u})\, |\nabla\overline{u}|^2 + \Psi(x) \text{ in } \Omega, \quad -\Delta\underline{u} \leq \lambda\, \beta(\underline{u})\, |\nabla\underline{u}|^2 + \Psi(x) \text{ in } \Omega.$$

By lemma 1.3 there is a solution $u$ of (1.1), ending the proof of theorem 1.1(i).

**Step 2**. (Proof of (ii)). Assume that (1.1) has a solution, say $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and let

$$M := \max_{\overline{\Omega}} u,$$

$$R(s) := \int_s^M \beta(t/\lambda)dt, \quad 0 < s < \min\{M, \lambda M\} := M_\lambda,$$

$$Q(s) := \int_0^s e^{-R(t)}dt, \quad 0 < s < M_\lambda.$$

By elementary computations we have

$$R'(s) = -\beta(s/\lambda) < 0, \quad 0 < s < M_\lambda,$$

$$Q(0) = 0, \quad Q''(s) = -e^{-R(s)}R'(s) = e^{-R(s)}\beta(s/\lambda).$$

Letting

$$v(x) := Q(\lambda u(x))$$

we find that

$$\frac{\partial v}{\partial x_i} = Q'(\lambda u(x))\lambda \frac{\partial u}{\partial x_i},$$

$$\frac{\partial^2 v}{\partial x_i^2} = Q''(\lambda u(x))\lambda^2 (\frac{\partial u}{\partial x_i})^2 + Q'(\lambda u(x))\lambda \frac{\partial^2 u}{\partial x_i^2}.$$

Hence, $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and for each $x \in \Omega$,

$$\begin{aligned} \Delta v(x) &= Q''(\lambda u(x))\lambda^2 |\nabla u(x)|^2 + Q'(\lambda u(x))\lambda \Delta u(x) \\ &= Q''(\lambda u(x))\lambda^2 |\nabla u(x)|^2 - Q'(\lambda u(x))\lambda^2 \beta(u(x)) |\nabla u(x)|^2 \\ &= |\nabla u(x)|^2 \lambda^2 \Big( e^{-R(\lambda u(x))}\beta(u(x)) - e^{-R(\lambda u(x))}\beta(u(x)) \Big) = 0. \end{aligned}$$

Since also $v = 0$ on $\partial\Omega$ it follows that $v \equiv 0$ and hence $u \equiv 0$ as well. This ends the proof of theorem 1.1(ii).

□

## 5 Appendix

**Verification of** (2.4)**.** To show (2.4)(i), letting $w \in E_\delta$ and $r \in [0, \delta]$ we have

$$d/2 \leq w(r) \leq 3d/2 \quad \text{and} \quad |w'(r)| \leq d/2.$$

So $\mathcal{F}w \in C^1([0, \delta])$. Choosing $\delta$ eventually smaller we have both

$$\begin{aligned} |\mathcal{F}(w(r)) - d| &\leq \frac{\sigma r^2}{2N} + \int_0^r \int_0^r t^{N-1} h(w(t)) w'(t)^2 dt ds \\ &\leq \Big[ \frac{\sigma}{2N} + \frac{d^2}{4} \Big( \max_{d/2 \leq t \leq 3d/2} h(t) \Big) \Big] \delta^2 \end{aligned}$$

and

$$\left|\Big(\mathcal{F}(w(r)) - d\Big)'\right| \leq \frac{\sigma r}{N} + \int_0^r h(w(t))\ w'(t)^2\ dt$$

$$\leq \Big[\frac{\sigma}{N} + \frac{d^2}{4}\Big(\max_{d/2\leq t\leq 3d/2} h(t)\Big)\Big]\delta^2.$$

Picking $\delta$ even smaller it follows that

$$\|\mathcal{F}(w(r)) - d\|_\delta \leq d/2, \quad 0 \leq r \leq \delta,$$

showing (2.4)(i).

□

In order to show (2.4)(ii), let $w_1, w_2 \in E_\delta$ and $0 \leq r \leq \delta$. Then

$$|\mathcal{F}(w_2(r)) - \mathcal{F}(w_1(r))| \leq$$
$$C_d \frac{\delta^2}{2}\Big[\max_{0\leq t\leq\delta} |w_2'(t) - w_1'(t)| + \max_{0\leq t\leq\delta} |w_2(t) - w_1(t)|\Big],$$

where $C_d > 0$ is a constant. Choosing $\delta > 0$ even smaller we get

$$\max\nolimits_{0\leq t\leq\delta} |\mathcal{F}(w_2(r)) - \mathcal{F}(w_1(r))| \leq \tfrac{1}{4}\|w_2 - w_1\|_\delta$$

and estimating in a similar way,

$$\max\nolimits_{0\leq t\leq\delta} |\mathcal{F}(w_2(r))' - \mathcal{F}(w_1(r))'| \leq \tfrac{1}{4}\|w_2 - w_1\|_\delta.$$

Therefore,

$$\|\mathcal{F}w_2 - \mathcal{F}w_1\|_\delta \leq \tfrac{1}{2}\|w_2 - w_1\|_\delta, \ w_2, w_1 \in E_\delta,$$

showing (2.4)(ii).

□

**Verification of** (2.6)**.** To show (2.6)(i), letting $w \in E_\delta(d_0)$ then

$$|\widehat{\mathcal{F}}(w(r)) - v_0(r)| \leq \int_{T_0}^{r} \int_{T_0}^{s} h(w(t)) w'(t)^2 dt ds,$$

where $T_0 \leq t, s \leq T_0 + \delta$. Further on, if $t \in [T_0, T_0 + \delta]$ then

$$v_0(t) - d_0/2 \leq w(t) \leq v_0(t) + d_0/2$$

and

$$\begin{aligned} |w'(t)| &\leq d_0/2 + |v_0(t)| \\ &\leq d_0/2 + \frac{\sigma t}{N}. \end{aligned}$$

Noticing that $v_0(T_0) = d_0$ and picking $\delta > 0$ small enough we have

$$d_0/4 \leq w(t) \leq 2d_0 \quad \text{and} \quad |w'(t)| \leq d_0/2 + \frac{2T_0\sigma}{N}.$$

Hence

$$|\widehat{\mathcal{F}}(w(r)) - v_0(r)| \leq \Big( \max_{d_0/4 \leq t \leq 2d_0} h(t) \Big) \Big( \frac{d_0}{2} + \frac{2T_0\sigma}{N} \Big)^2 (r - T_0)^2$$

so that

$$|\widehat{\mathcal{F}}(w(r)) - v_0(r)| \leq \Big( \max_{d_0/4 \leq t \leq 2d_0} h(t) \Big) \Big( \frac{d_0}{2} + \frac{2T_0\sigma}{N} \Big)^2 \delta^2. \tag{5.2}$$

On the other hand,

$$\begin{aligned} |\widehat{\mathcal{F}}(w(r))' - v_0'(r)| &\leq \int_{T_0}^{r} h(w(t)) w'(t)^2 dt \\ &\leq \Big( \max_{d_0/4 \leq t \leq 2d_0} h(t) \Big) \Big( \frac{d_0}{2} + \frac{2T_0\sigma}{N} \Big) (r - T_0). \end{aligned}$$

and picking $\delta$ smaller we have

$$|\widehat{\mathcal{F}}(w(r))' - v_0'(r)| \leq \Big( \max_{d_0/4 \leq t \leq 2d_0} h(t) \Big) \Big( \frac{d_0}{2} + \frac{2T_0\sigma}{N} \Big) \delta. \tag{5.3}$$

By (5.2) and (5.3) we get to

$$\|\widehat{\mathcal{F}}w - v_0\|_\delta \leq \frac{d_0}{2},$$

showing (2.6)(i) that is

$$\widehat{\mathcal{F}}(E_\delta(d_0)) \subset E_\delta(d_0).$$

In order to show (2.6)(ii), let $w_1, w_2 \in E_\delta(d_0)$ and choose $\delta$ small enough. Then for $T_0 \leq t \leq T_0 + \delta$, we have

$$|\widehat{\mathcal{F}}(w_2(r)) - \widehat{\mathcal{F}}(w_1(r))| \leq C_{d_0}\delta^2\Big[\max_{T_0\leq t\leq T_0+\delta}|w_2'(t) - w_1'(t)| + \max_{T_0\leq t\leq T_0+\delta}|w_2(t) - w_1(t)|\Big],$$

showing (2.6)(ii). One also shows that

$$|\widehat{\mathcal{F}}(w_2(r))' - \widehat{\mathcal{F}}(w_1(r))'| \leq C_{d_0}'\delta\Big[\max_{T_0\leq t\leq T_0+\delta}|w_2'(t) - w_1'(t)| + \max_{T_0\leq t\leq T_0+\delta}|w_2(t) - w_1(t)|\Big],$$

showing (2.6).

□

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