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Global Attractors for Neural Fields in a Weighted Space

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Abstract

In this paper we prove the existence and upper semicontinuity of compact global attractors for the flow of the equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + J * (f \circ u)(x,t) + h, \;\; h \geq 0,$$

in L^2 weighted spaces.

1 Introduction

We consider here the non local evolution equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + J * (f \circ u)(x,t) + h, \;\; h \geq 0, \tag{1.1}$$

where $u(x,t)$ is a real function on $\mathbb{R} \times \mathbb{R}_+$, h is a non negative constant, $J \in C^1(\mathbb{R})$ is a non negative even function supported in the interval $[-1,1]$ and f is a non negative nondecreasing function. The $*$ above denotes convolution product, namely:

$$(J * u)(x) = \int_{\mathbb{R}} J(x-y)u(y)dy. \tag{1.2}$$

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Equation (1.1) was derived in 1972 by Wilson and Cowan, [17], to model a single layer of neurons. The function $u(x,t)$ represents the mean membrane potential of a patch of tissue located at position $x \in (-\infty, \infty)$ at time $t \geq 0$. The connection function J determines the coupling between the elements at positions x and y. The non negative nondecreasing function f gives the neural firing rate, or the average rate at which spikes are generated, corresponding to an activity level u. The neurons at a point x are said to be active at time t if $f(u(x,t)) > 0$. The parameter h denotes a constant external stimulus applied uniformly to the entire neural field, (see [1], [4], [6], [8], [9], [10] and [15]).

An equilibrium of (1.1) is a solution of (1.1) that is constant with respect to t. Thus, if m is an equilibrium for (1.1) then m satisfies

$$m(x) = J * (f \circ m)(x) + h. \tag{1.3}$$

There are already several works in the literature dedicated to the analysis of this model. In [1] lateral inhibition type coupling is studied. Furthermore, when f is a Heaviside step function, [1] also considers the behavior of time dependent periodic solutions as well as traveling waves for systems of equations. Existence and uniqueness of monotone traveling waves was investigated in [6]. Another prove of existence uniqueness and monotonicity of travelling waves is given in [4]. In [8], the existence of a non-homogeneous stationary solution referred to as “bump” is proved. One link between the integral equations given by (1.3) and a system of ODEs is given in [9]. In [10], the existence of “double-bump” stationary solution is proved. In [15] it is proved that “bump” solutions can exist and be linearly stable in neural population models without recurrent excitation.

This paper is organized as follows. In Section 2, we prove that, in the phase space $L^2(\mathbb{R}, \rho) = \{u \in L^1_{loc}(\mathbb{R}) : \int u^2 \rho(x) dx < \infty\}$, the Cauchy problem for (1.1) is well posed with a unique global solution. In Section 3, we prove that the system is dissipative in the same space in the sense of [7], that is, it has a global compact attractor. Our proof uses the Sobolev's compact embedding $H^1([-l,l]) \hookrightarrow L^2([-l,l])$ and some ideas from [12], where the equation $u_t = -u + \tanh(\beta J * u + h)$ is considered (see also [2], [11], [13] and [14] for related work). Finally, in the Section 4 after obtaining some estimates for the flow of (1.1), we prove the upper semicontinuity property of the attractors with respect to function J present in (1.1).

We collect here the additional conditions on f which will are used as hypotheses in our results when necessary.

(H1) The function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, that is, there exists $k_1 > 0$ such that

$$|f(x) - f(y)| \leq k_1|x - y|, \ \ \forall\, x, y \in \mathbb{R}, \tag{1.4}$$

(H2) $|f(x)| \to \textit{constant}$, as $|x| \to \infty$.

It follows from (H2) that there exists $k_2 > 0$ such that

$$|f(x)| \leq k_2, \forall\, x \in \mathbb{R}. \tag{1.5}$$

A prototype for f is $f(x) = \tanh(x)$ which satisfies (H1) and (H2) with $k_1 = k_2 = 1$.

2 Well-posedness in $L^2(\mathbb{R}, \rho)$

In this section we consider the flow generated by (1.1) in the space $L^2(\mathbb{R}, \rho)$ defined by

$$L^2(\mathbb{R}, \rho) = \left\{ u \in L^1_{loc}(\mathbb{R}) \ : \ \int_{\mathbb{R}} u^2(x)\rho(x)dx < +\infty \right\},$$

with norm $\|u\|_{L^2}(\mathbb{R}, \rho) = \left(\int_{\mathbb{R}} u^2(x)\rho(x)dx\right)^{\frac{1}{2}}$. Here ρ is an integrable even function with $\int_{\mathbb{R}} \rho(x)dx = 1$. Note that in this space the constant function equal to 1 has norm 1. The corresponding higher-order Sobolev space $H^k(\mathbb{R}, \rho)$ is the space of functions $u \in L^2(\mathbb{R}, \rho)$ whose distributional derivatives up to order k are also in $L^2(\mathbb{R}, \rho)$, with norm

$$\|u\|_{H^k(\mathbb{R},\rho)} = \left(\sum_{i=1}^{k} \left\| \frac{\partial^i u}{\partial x^i} \right\|^2_{L^2(\mathbb{R},\rho)} \right)^{\frac{1}{2}}.$$

Lemma 2.1. *Suppose that* $\sup_{x\in\mathbb{R}}\{\rho(x) \,:\, y-1 \leq x \leq y+1\} \leq K\rho(y)$, *for some constant* K *and all* $y \in \mathbb{R}$. *Then*

$$\|J * u\|_{L^2(\mathbb{R},\rho)} \leq \sqrt{K}\|J\|_{L^1}\|u\|_{L^2(\mathbb{R},\rho)}.$$

Proof: Since J is bounded and compact supported, $(J * u)(x)$ is well defined for $u \in L^1_{loc}(\mathbb{R})$. Thus, using Holder's inequality (see [3]), we obtain

$$\begin{aligned}
\|J*u\|^2_{L^2(\mathbb{R},\rho)} &= \int_{\mathbb{R}} |(J*u)(x)|^2\rho(x)dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (J(x-y))^{\frac{1}{2}}(J(x-y))^{\frac{1}{2}}|u(y)|dy\right)^2 \rho(x)dx \\
&\leq \int_{\mathbb{R}} \left(\left[\int_{\mathbb{R}} J(x-y)dy\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} J(x-y)|u(y)|^2dy\right]^{\frac{1}{2}}\right)^2 \rho(x)dx \\
&= \|J\|_{L^1}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} J(x-y)|u(y)|^2dy\right)\rho(x)dx \\
&= \|J\|_{L^1}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} J(x-y)\rho(x)dx\right)|u(y)|^2dy \\
&\leq \|J\|_{L^1}\int_{\mathbb{R}} \left(\int_{x=y-1}^{x=y+1} J(x)\rho(x)dx\right)|u(y)|^2dy \\
&\leq \|J\|_{L^1}\int_{\mathbb{R}} \left(K\rho(y)\int_{x=y-1}^{x=y+1} J(x)dx\right)|u(y)|^2dy \\
&\leq K\|J\|^2_{L^1}\int_{\mathbb{R}} |u(y)|^2\rho(y)dy \\
&= K\|J\|^2_{L^1}\|u\|^2_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

This concludes the proof.

□

Remark 2.2. *When $\rho(x)=\frac{1}{\pi}(1+x^2)^{-1}$, the hypothesis $\sup_{x\in\mathbb{R}}\{\rho(x) : y-1\leq x\leq y+1\}\leq K\rho(y)$, from Lemma 2.1, is verified with $K=3$, (see, [12]).*

Proposition 2.3. *Suppose the hypothesis (H1) holds and that $\sup_{x\in\mathbb{R}}\{\rho(x) : y-1\leq x\leq y+1\}\leq K\rho(y)$, for some positive constant K and all $y\in\mathbb{R}$. Then the function*

$$F(u) = -u + J*(f\circ u) + h$$

is globally Lipschitz in $L^2(\mathbb{R},\rho)$.

Proof: From the triangle inequality and Lemma 2.1, it follows that

$$\begin{aligned}
\|F(u)-F(v)\|_{L^2(\mathbb{R},\rho)} &\leq \|v-u\|_{L^2(\mathbb{R},\rho)} + \|J*(f\circ u) - J*(f\circ v)\|_{L^2(\mathbb{R},\rho)} \\
&\leq \|v-u\|_{L^2(\mathbb{R},\rho)} + \sqrt{K}\|J\|_{L^1}\|(f\circ u)-(f\circ v)\|_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

But, using (1.4), we have

$$\begin{aligned}\|(f\circ u)-(f\circ v)\|^2_{L^2(\mathbb{R},\rho)} &\leq \int_{\mathbb{R}} k_1^2|u(x)-v(x)|^2\rho(x)dx\\ &= k_1^2\|u-v\|^2_{L^2(\mathbb{R},\rho)}.\end{aligned}$$

Then

$$\|F(u)-F(v)\|_{L^2(\mathbb{R},\rho)} \leq (1+\sqrt{K}\|J\|_{L^1}k_1)\|u-v\|_{L^2(\mathbb{R},\rho)}.$$

Therefore F is uniformly Lipschitz in $L^2(\mathbb{R},\rho)$.

□

Remark 2.4. *It follows from Proposition 2.3 that the Cauchy problem for (1.1) is well posed in $L^2(\mathbb{R},\rho)$ with a unique global solution, (see [3] and [5]).*

3 Existence of a global attractor in $L^2(\mathbb{R},\rho)$

In this section, we prove the existence of a global maximal invariant compact set $\mathcal{A}\subset L^2(\mathbb{R},\rho)$ for the flow of (1.1), which attracts each bounded set of $L^2(\mathbb{R},\rho)$ ($\mathcal{A}\subset L^2(\mathbb{R},\rho)$ is the global attractor for the flow of (1.1,) see [7] and [16]).

In what follows, we denote by $S(t)$ the flow generated by (1.1).

We recall that a set $\mathcal{B}\subset L^2(\mathbb{R},\rho)$ is an absorbing set for the flow $S(t)$ in $L^2(\mathbb{R},\rho)$ if, for any bounded set $C\subset L^2(\mathbb{R},\rho)$, there is a $t_1>0$ such that $S(t)C\subset\mathcal{B}$ for any $t\geq t_1$, (see [16]).

Lemma 3.1. *Suppose that the hypotheses (H1) and (H2) hold and let $R = k_2\sqrt{K}\|J\|_{L^1}+h$. Then the ball with center at the origin and radius $R+\varepsilon$ is an absorbing set for the flow $S(t)$ in $L^2(\mathbb{R},\rho)$ for any $\varepsilon>0$.*

Proof: Let $u(x,t)$ be the solution of (1.1) with initial condition $u(x,0)\in L^2(\mathbb{R},\rho)$. Then, by the variation of constants formula,

$$u(x,t)=e^{-t}u(x,0)+\int_0^t e^{s-t}[J*(f\circ u)(x,s)+h]ds.$$

Hence

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} &\leq e^{-t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)}+\\ &\quad+\int_0^t e^{s-t}[\|J*(f\circ u)(\cdot,s)\|_{L^2(\mathbb{R},\rho)}+\|h\|_{L^2(\mathbb{R},\rho)}]ds.\end{aligned}$$

Then, using Lemma 2.1,it follows that

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} &\leq e^{-t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)} \\ &\quad + \int_0^t e^{s-t}[\sqrt{K}\|J\|_{L^1}\|f(u(\cdot,s))\|_{L^2(\mathbb{R},\rho)} + \|h\|_{L^2(\mathbb{R},\rho)}]ds.\end{aligned}$$

Now, from (1.5), we have

$$\begin{aligned}\|f(u(\cdot,s))\|^2_{L^2(\mathbb{R},\rho)} &= \int_{\mathbb{R}} |f(u(x,s))|^2\rho(x)dx \\ &\leq k^2\int_{\mathbb{R}}\rho(x)dx \\ &= k_2^2.\end{aligned}$$

Thus

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} &\leq e^{-t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)} + \int_0^t e^{s-t}\left[\sqrt{K}\|J\|_{L^1}k_2 + h\right]ds \\ &\leq e^{-t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)} + R.\end{aligned}$$

Therefore, for any $t > \ln\left(\frac{\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)}}{\varepsilon}\right)$, we have $\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} < \varepsilon + R$, and the proof is complete.

□

Lemma 3.2. *Assume the same hypotheses of Lemma 2.1 and suppose also that the hypotheses (H1) and (H2) hold. Then, for any $\eta > 0$, there exists t_η such that $S(t_\eta)B(0,R+\varepsilon)$ has a finite covering by balls of $L^2(\mathbb{R},\rho)$ with radius smaller than η.*

Proof: From Lemma 3.1, it follows that $B(0,R+\varepsilon)$ is invariant. Now, the solutions of (1.1)
with initial condition $u_0 \in B(0,R+\varepsilon)$ are given, by the variation of constant formula, by

$$u(x,t) = e^{-t}u_0 + \int_0^t e^{-(t-s)}[(J*(f\circ u))(x,s) + h]ds.$$

Write

$$v(x,t) = e^{-t}u_0(x) \text{ and } w(x,t) = \int_0^t e^{-(t-s)}[(J*(f\circ u))(x,s) + h]ds.$$

Let $\eta > 0$ given. We may find $t(\eta)$ such that if $t \geq t(\eta)$ then $\|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq \frac{\eta}{2}$.

Now, using (1.5), we obtain

$$\begin{aligned} |J*(f\circ u)(x,s)| &\leq \int_{\mathbb{R}} J(x-y)|f(u(y,s))|dy \\ &\leq k_2\|J\|_{L^1} \end{aligned}$$

Thus

$$\begin{aligned} |w(x,t)| &\leq \int_0^t e^{-(t-s)}(k_2\|J\|_{L^1}+h)ds \\ &\leq k_2\|J\|_{L^1}+h=R. \end{aligned} \tag{3.6}$$

Hence

$$\|w(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq R. \tag{3.7}$$

Furthermore, differentiating with respect to x, for $t\geq 0$, we have

$$\frac{\partial w}{\partial x}(x,t)=\int_0^t e^{-(t-s)}\left(J'*(f\circ u)\right)(x,s)ds,$$

Thus

$$\left|\frac{\partial w(x,t)}{\partial x}\right| \leq \int_0^t e^{-(t-s)}|J'*(f\circ u)(x,s)|ds.$$

Using that

$$|J'*(f\circ u)(x,s)| \leq k_2\|J'\|_{L^1},$$

it follows that

$$\left|\frac{\partial w(x,t)}{\partial x}\right| \leq k_2\|J'\|_{L^1}. \tag{3.8}$$

Now, let $l>0$ be chosen such that

$$R^2\int_{\mathbb{R}}(1-\chi_l)^4(x)\rho(x)dx \leq \frac{\eta}{4}, \tag{3.9}$$

where χ_l denotes the characteristic functiom of $[-l,l]$. Then, using (3.7) and (3.6) and (3.9), we obtain

$$\begin{aligned} \|(1-\chi_l)w(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)} &\leq \int_{\mathbb{R}}\left[w(x,t)\rho(x)^{\frac{1}{2}}(1-\chi_l)^2(x)w(x,t)\rho(x)^{\frac{1}{2}}\right]dx \\ &\leq \left(\int_{\mathbb{R}}|w(x,t)|^2\rho(x)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(1-\chi_l)^4(x)|w(x,t)|^2\rho(x)dx\right)^{\frac{1}{2}} \\ &= \|w(\cdot,t)\|_{L^2(\mathbb{R},\rho)}\left(\int_{\mathbb{R}}(1-\chi_l)^4(x)|w(x,t)|^2\rho(x)dx\right)^{\frac{1}{2}} \\ &\leq R^2\left(\int_{\mathbb{R}}(1-\chi_l)^4(x)\rho(x)dx\right)^{\frac{1}{2}} \\ &\leq \frac{\eta}{4}. \end{aligned}$$

Also, by (3.6) and (3.8) the restriction of $w(\cdot,t)$ to the interval $[-l,l]$ is bounded in $H^1([-l,l])$ (by a constant independent of $u_0 \in B(0,R+\varepsilon)$ and of t), and therefore the set $\{\chi_l w(\cdot,t)\}$ with $w(\cdot,0) \in B(0,R+\varepsilon)$ is relatively compact subset of $L^2(\mathbb{R},\rho)$ for any $t>0$ and, hence, it can be covered by a finite number of balls with radius smaller than $\frac{\eta}{4}$.

Therefore, since

$$u(\cdot,t) = v(\cdot,t) + \chi_l w(\cdot,t) + (1-\chi_l)w(\cdot,t),$$

it follows that $S(t_\eta)B(0,R+\varepsilon)$ has a finite covering by balls of $L^2(\mathbb{R},\rho)$ with radius smaller than η because

$$\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} = \|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)} + \|\chi_l w(\cdot,t)\|_{L^2(\mathbb{R},\rho)} + \|(1-\chi_l)w(\cdot,t)\|_{L^2(\mathbb{R},\rho)}$$

and the result is proved.

□

We denote by $\omega(C)$ the ω-limit of a set C.

Theorem 3.3. *Assume the same hypotheses of Lemma 3.2. Then $\mathcal{A} = \omega(B(0,R+\varepsilon))$, is a global attractor for the flow $S(t)$ generated by (1.1) in $L^2(\mathbb{R},\rho)$ which is contained in the ball of radius R.*

Proof: Let $\varepsilon > 0$ given. From Lemma 3.1, it follows that $\mathcal{A}$ is contained in the ball of radius $R+\varepsilon$ and center in the origin of $L^2(\mathbb{R},\rho)$. Now, being $\mathcal{A}$ invariant by flow $S(t)$, it follows that $\mathcal{A} \subset S(t)B(0,R+\varepsilon)$, for any $t \geq 0$ and then, from Lemma 3.2, it results that the measure of noncompactness of $\mathcal{A}$ is zero. Hence $\mathcal{A}$ is relatively compact and, since $\mathcal{A}$ is closed,it follows that $\mathcal{A}$ is also compact. Finally, if C is bounded set in $L^2(\mathbb{R},\rho)$ then $S(t_0)C \subset B(0,R+\varepsilon)$ for t_0 big enough and, therefore, $\omega(C) \subset \omega(B(0,R+\varepsilon))$.

□

Theorem 3.4. *Assume the same hypotheses of Theorem 3.3 and that $J \in C^r(\mathbb{R})$, for some integer $r > 0$, then the attractor $\mathcal{A}$ is contained in $H^r(\mathbb{R},\rho)$.*

Proof: Let $u(x,t)$ be a solution of (1.1) in $\mathcal{A}$. Then, by the variation of constants formula

$$u(x,t) = e^{-(t-t_0)}u(x,t_0) + \int_{t_0}^{t} e^{-(t-s)}[J*(f\circ u)(x,s) + h]ds.$$

From Theorem 3.3 follows that $\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq R$.

Since $\|u(\cdot,t_0)\|_{L^2(\mathbb{R},\rho)} \le R$, where $R = k_2\sqrt{K}\|J\|_{L^1} + h$, letting $t_0 \to -\infty$, we obtain

$$u(x,t) = \int_{-\infty}^{t} e^{-(t-s)}[J*(f\circ u)(x,s)+h]ds, \tag{3.10}$$

where the equality in (3.10) is in the sense of $L^2(\mathbb{R},\rho)$.

Using that $J \in C^1(\mathbb{R})$ follows, from (3.10), that $u(x,t)$ is differentiable with respect to x and

$$\frac{\partial u(x,t)}{\partial x} = \int_{-\infty}^{t} e^{-(t-s)} J' * (f\circ u)(x,s)ds. \tag{3.11}$$

Now, using that $J' \in C^1(\mathbb{R})$, follows from (3.11), that $\frac{\partial u(x,t)}{\partial x}$ is differentiable with respect to x and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \int_{-\infty}^{t} e^{-(t-s)} J'' * (f\circ u)(x,s)ds.$$

Repeating the argument, using that $J^{(r-1)} \in C^1(\mathbb{R})$, we have that $\frac{\partial^{r-1} u(x,t)}{\partial x^{r-1}}$ is differentiable with respect to x and

$$\frac{\partial^r u(x,t)}{\partial x^r} = \int_{-\infty}^{t} e^{-(t-s)} J^r * (f\circ u)(x,s)ds. \tag{3.12}$$

Now, since J is bounded and compact supported, it follows that $J^{(r)}$ is also bounded and compact supported. Thus $J^{(r)} * v$ is well defined for $v \in L^1_{loc}(\mathbb{R})$. Hence, proceeding as in the Lemma 2.1, obtain

$$\|J^{(r)} * v\|_{L^2(\mathbb{R},\rho)} \le \sqrt{K}\|J^{(r)}\|_{L^1}\|v\|_{L^2(\mathbb{R},\rho)}.$$

Then, using (H2), obtain

$$\|J^{(r)} * (f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \le k_2\sqrt{K}\|J^{(r)}\|_{L^1},\ \forall\, t\in\mathbb{R}.$$

Hence, from (3.12), follows that

$$\begin{aligned}
\left\|\frac{\partial^r u(x,t)}{\partial x^r}\right\|_{L^2(\mathbb{R},\rho)} &\le \int_{-\infty}^{t} e^{-(t-s)}\|J^{(r)} * (f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)}ds \\
&\le k_2\sqrt{K}\|J^{(r)}\|_{L^1}\int_{-\infty}^{t} e^{-(t-s)}ds \\
&\le k_2\sqrt{K}\|J^{(r)}\|_{L^1}.
\end{aligned}$$

Thus, we can obtain bounded for the derivatives of u of any order, in terms only of J and of the derivatives of J, concluding the proof.

□

4 Upper semicontinuity of the attractors with respect to function J

A natural question to examine is the dependence of this attractors on the function J present in (1.1). We denote by $\mathcal{A}_J$ the global attractor whose existence was proved in the Theorem 3.3

Let us recall that a family of subsets $\{\mathcal{A}_J\}$ in a metric space is upper semicontinuous at J_0 if

$$dist(A_J, A_{J_0}) \longrightarrow 0, \text{ as } J \to J_0,$$

where

$$dist(A_J, A_{J_0}) = \sup_{x \in A_J} dist(x, A_{J_0}) = \sup_{x \in A_J} \inf_{y \in A_{J_0}} dist(x, y).$$

In this section, we prove that the family of attractors is upper semicontinuous, in $L^2(\mathbb{R}, \rho)$, with respect to the function J at J_0 with $J \in C^1(\mathbb{R})$ non negative even and supported in the interval $[-1, 1]$, using the L_1-norm for J.

Lemma 4.1. *Assume the hypotheses of Lemma 2.1 and (H1) and (H2) hold, then the flow $S_J(t)$ is continuous with respect to variations of J in the L_1-norm, uniformly for t in compact and u in bounded sets.*

Proof: As shown above the solutions of (1.1) satisfy the variations of constants formula,

$$S_J(t)u = e^{-t}u + \int_0^t e^{-(t-s)}[J * (f \circ S_J(s)u + h]ds.$$

Let $J_0 \in C^1(\mathbb{R})$ be a non negative even function suported in the interval $[-1, 1]$ and C a bounded set in $L^2(\mathbb{R}, \rho)$, for example the ball $B(0, R)$ (Although R depends on J, it can be uniformly chosen in a neighborhood of J_0). Given $\varepsilon > 0$, we want to find $\delta > 0$ such that $\|J - J_0\|_{L^1} < \delta$ implies

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} < \varepsilon,$$

for $t \geq 0$ and $u \in C$. But

$$\begin{aligned} \|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \quad &\leq \quad \int_0^t e^{-(t-s)}\|J * (f \circ S_J(s)u) \\ &\quad -J_0 * (f \circ S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}ds. \end{aligned}$$

Subtracting and summing the term $J_0 * (f \circ S_J(s)u)$ and using Lemma 2.1, for any $t > 0$, we obtain

$$\begin{aligned}
\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} &\leq \int_0^t e^{-(t-s)}[\|(J - J_0) * (f \circ S_J(s)u)\|_{L^2(\mathbb{R},\rho)} \\
&+ \|J_0 * [f \circ S_J(s)u - f \circ S_{J_0}(s)u]\|_{L^2(\mathbb{R},\rho)}]ds \\
&\leq \int_0^t e^{-(t-s)}[\sqrt{K}\|J - J_0\|_{L^1}\|f \circ S_J(s)u\|_{L^2(\mathbb{R},\rho)} \\
&+ \sqrt{K}\|J_0\|_{L^1}\|f \circ S_J(s)u - f \circ S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}]ds.
\end{aligned}$$

Using (H1) and (H2), we obtain

$$\begin{aligned}
\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} &\leq \int_0^t e^{-(t-s)}k_2\sqrt{K}\|J - J_0\|_{L^1}ds \\
&+ \int_0^t e^{-(t-s)}k_1\sqrt{K}\|J_0\|_{L^1}\|S_J(s)u - S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)} \\
&\leq k_2\sqrt{K}\|J - J_0\|_{L^1} + \int_0^t e^{-(t-s)}k_1\sqrt{K}\|J_0\|_{L^1}\|S_J(s)u \\
&\quad -S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

Therefore, by Gronwall's Lemma, it follows that

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \leq \|J - J_0\|_{L^1}k_2\sqrt{K}e^{(k_1\sqrt{K}\|J_0\|_{L^1}-1)t}.$$

From this, the results follows immediately.

□

Theorem 4.2. *Suppose the same hypotheses of the Lemma 4.1 hold. Then the family of attractors $\mathcal{A}_J$ is upper semicontinuous with respect to J at J_0.*

Proof: From the hypotheses of the theorem, it follows that, for every $J \in C^1(\mathbb{R})$, sufficiently close to J_0 in the L_1-norm, non negative even suported in $[-1, 1]$, the attractor, $\mathcal{A}_J$, given by Theorem 3.3 is in the closed ball $B[0, R]$ in $L^2(\mathbb{R}, \rho)$, with R fixed. Therefore

$$\bigcup_J \mathcal{A}_J \subset B[0, R].$$

Since $\mathcal{A}_{J_0}$ is global attractor and $B[0, R]$ is a bounded set then, for every $\varepsilon > 0$, there exists $t^* > 0$ such that $S_{J_0}(t)B[0, R] \subset \mathcal{A}_{J_0}^{\frac{\varepsilon}{2}}$, for all $t \geq t^*$, where $\mathcal{A}_{J_0}^{\frac{\varepsilon}{2}}$ is $\frac{\varepsilon}{2}$-neighborhood of $\mathcal{A}_{J_0}$.

From Lemma 3.1, it follows that $S_J(t)$ is continuous at J_0, uniformly for t in compacts. Thus, there exists $\delta > 0$ such that

$$\|J - J_0\|_{L^1} < \delta \Rightarrow \|S_J(t^*)u - S_{J_0}(t^*)u\|_{L^2(\mathbb{R},\rho)} < \frac{\varepsilon}{2}, \forall\, u \in B[0,R].$$

We will show that if $\|J - J_0\| < \delta$ then $\mathcal{A}_J \subset \mathcal{A}^{\varepsilon}_{J_0}$. In fact, let $u \in \mathcal{A}_J$. Since $\mathcal{A}_J$ is invariant, $v = S_J(-t^*)u \in \mathcal{A}_J \subset B[0,R]$. Therefore, we obatain

$$S_{J_0}(t^*)v \in \mathcal{A}^{\frac{\varepsilon}{2}}_{J_0}, \tag{4.13}$$

and

$$\|S_J(t^*)v - S_{J_0}(t^*)v\|_{L^2(\mathbb{R},\rho)} < \frac{\varepsilon}{2}. \tag{4.14}$$

From (4.13) and (4.14), it follows that

$$u = S_J(t^*)S_J(-t^*)u = S_J(t^*)v \in \mathcal{A}^{\varepsilon}_{J_0}$$

and the upper semicontinuity of $\mathcal{A}_J$ follows.

□

Remark 4.3. *Similar results can be obtained for the flow of (1.1) in $C_\rho(\mathbb{R})$, where*

$$C_\rho(\mathbb{R}) \equiv \{u : \mathbb{R} \to \mathbb{R} \text{ continuous with the norm } \|\cdot\|_\rho\},$$

with

$$\|u\|_\rho = \sup_{x\in\mathbb{R}}\{|u(x)|\rho(x)\} < \infty,$$

wiht ρ being a positive continuous function on $\mathbb{R}$.

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