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STABILITY OF MINIMAL AND CONSTANT MEAN CURVATURE SURFACES WITH FREE BOUNDARY

A. ROS

Abstract

We prove that stable balance minimal surfaces with free boundary in a centrally symmetric mean-convex region of $\mathbb{R}^3$ are topological disks. For surfaces with constant mean curvature and free boundary, we prove that volume-preserving stability implies that the surface has either genus zero with at most four boundary components or genus one with 1 or 2 curves at its boundary.

1 Introduction

Given a smooth region $W \subset \mathbb{R}^3$, we can consider compact orientable surfaces S of stationary area among surfaces in W whose boundary lies on ∂W and whose interior lies on the interior of W. Then S is a minimal surface (i.e. it has mean curvature zero) and it meets orthogonally ∂W along its boundary. We will then say that S is a minimal surface with free boundary in W. These surfaces have been considered by Courant

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and Davies [4], Meeks and Yau [8], Smyth [24], Jost [7], Tomi [26], Moore and Schulte [10] and other authors. For the case of general dimension and codimension see the survey by Schoen [23]. In applications there is an interest into area minimizing and stable minimal surfaces with free boundary, i.e. stationary surfaces with nonnegative second variation of the area.

We can also consider surfaces with free boundary in a region W and stationary area under other natural geometric constraints. In this paper we will assume that W is a mean-convex region, i. e. ∂W has nonnegative inward mean curvature $H_W \geq 0$, and we will study the stability of the area for surfaces with free boundary in W in two different contexts related with the above: balance minimal surfaces and volume-preserving stationary surfaces.

Assume that the region $W \subset \mathbb{R}^3$ is mean-convex and invariant under the central symmetry $x \mapsto -x$ in $\mathbb{R}^3$. Following Fischer and Koch [5], we say that an embedded proper surface $S \subset W$ with $-S = S$ is a balance surface if it divides W into two regions, $W - S = W_1 \cup W_2$, interchanged by the central symmetry $-W_1 = W_2$. A *balance minimal surface* is a balance surface in W which is minimal and meets the boundary of W orthogonally. This is the same to say that S is a critical point of the area among balance surfaces. More generally, given a group $\mathcal{G}$ of symmetries of W and an index two subgroup $\mathcal{H} \subset \mathcal{G}$, we can consider $(\mathcal{G}, \mathcal{H})$-balance surfaces. These are proper surfaces S invariant under $\mathcal{G}$ and such that the components of $W - S$ are preserved by the symmetries in $\mathcal{H}$ and interchanged by those in $\mathcal{G} - \mathcal{H}$. Balance minimal surfaces appear in geometric crystallography and play a role similar to sphere packing and space filling polyhedra. A number of interesting examples of balance minimal surfaces in classical geometry can be found in [5]. Area minimizing balance surfaces may present singularities at the fixed points of the symmetries of $\mathcal{G}$. These singularities are described by Morgan [11] (Theorem 5.3 and comments below its proof). In this paper we will restrict to the particular case $\mathcal{G} = \{\pm Id\}$ and it follows from [11] that in this situation area minimizing balance surfaces exist and are regular embedded minimal surfaces with free boundary in W. We will prove in Theorem 5 that nonflat stable balance minimal surfaces with free boundary in W are (topological) disks. We will also prove that nonflat stable closed balance minimal surfaces in a 3-torus have genus 3.

If S is a critical point of the area, not for any deformation but just for those preserving the volume enclosed by the surface in W, then S has constant mean curvature and contact angle $\pi/2$ with ∂W. Constant mean curvature surfaces with free boundary appear as solutions of the isoperimetric problem in the region W, see Ros [16] and they have been studied, for instance, in Nitsche [12], Struwe [25], Ros and Vergasta [21] and Bürger and Kuwert [2]. In particular, volume-preserving stability for constant mean curvature surfaces with free boundary was considered in [21] in the case W is convex and Ros and Souam [20] study the stability of capillary surfaces (this is a related situation where the contact angle between S and ∂W is a prescribed constant). We show that volume-preserving stable constant mean curvature surfaces with free boundary in W have either genus 0 and at most four boundary components or genus 1 and at most two components at its boundary, see Theorem 9.

In Theorem 7 we will prove that the results above extend to piecewise smooth regions $W \subset \mathbb{R}^3$.

Stable surfaces with involved topology can be obtained from the Schwarz P minimal surface in Figure 1. Ross [22] proved that this surface is volume-preserving stable in the cubic 3-torus and from that we can deduce that it is a stable balance minimal surface in the 3-torus and that the piece at the right of Figure 1 is stable among balance surfaces in the cube (note that the area-minimizing balance surface in the cube is the flat horizontal planar section, [16]). We can also see that the part of the surface between two consecutive planes of symmetry is volume-preserving with free boundary, genus 1 and 2 boundary components. However, in this case W is not a region in the Euclidean space but a slab in the 3-torus.

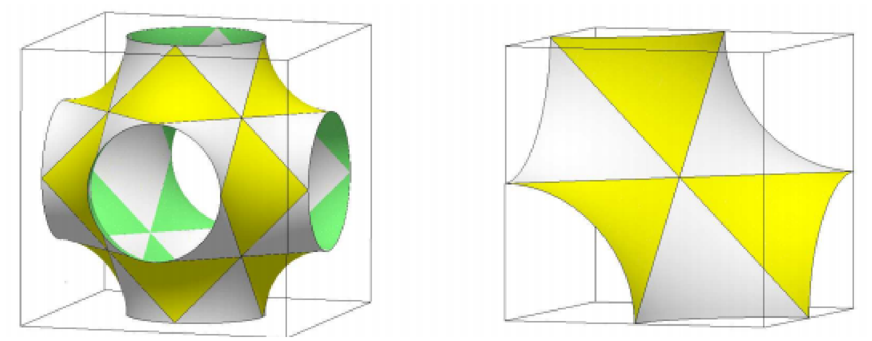

Figure 1: Schwarz P minimal surface provides interesting examples of stable surfaces with free boundary. The piece at the right is both balance stable and volume-preserving stable in the unit cube. The same holds for the whole surface in the cubic 3-torus. The piece of the P surface between two consecutive horizontal planes of symmetry is a volume-preserving stable surface in the flat region $T^2 \times [0, \frac{1}{2}]$, T^2 being the square 2-torus, of genus 1 and two boundary components.

The results of this paper follow by using, as test functions in the second variation, functions which are constructed from harmonic 1-forms on S. From the Hodge Theorem, the existence of these 1-forms depends on the topology of the surface. These test functions where first used in Palmer [13] to study the index of stability of harmonic Gauss maps and then by Ros [18, 19] to obtain several stability properties in classical geometry of surfaces. He proves the following results:

1) *Complete stable minimal surfaces (either orientable or nonorientable) in $\mathbb{R}^3$ are planar.* This extends to the nonorientable case the well-known characterization of the plane given by do Carmo and Peng [3], Fischer-Colbrie and Schoen [6] and Pogorelov [14].

2) If $\Gamma \subset \mathbb{R}^3$ is a discrete group of translations of rank 1 or 2, then *area minimizing surfaces (mod 2) in $\mathbb{R}^3/\Gamma$, are either planar or (a quotient of) the Helicoid or the doubly periodic Scherk surfaces* (in the last two

cases, the total curvature of the surface is -2π), [18]. This result gives the first progress toward the classification of area minimizing surfaces in flat 3-manifolds.

3) If $\Gamma \subset \mathbb{R}^3$ is a group of translations of rank k, then *closed volume-preserving stable surfaces in* $\mathbb{R}^3/\Gamma$ *have genus* $\leq k$. This extends to the periodic context the characterization by Barbosa and do Carmo of the sphere as the unique stable constant mean curvature surface in $\mathbb{R}^3$, [1], and provides the basic theoretical support for some mesoscopic phase separation phenomena appearing in material sciences, [19].

4) *Closed volume-preserving stable constant mean curvature surfaces in a* 3*-manifold of nonnegative Ricci curvature have genus* ≤ 3*, [18].* This improves partial results by several authors and gives the optimal bound for the first time as the Schwarz P minimal surface is volume-preserving stable in the cubic flat 3-torus, Ross [22].

This paper is dedicated to Manfredo do Carmo on his $80th$ Birthday.

2 Preliminaries

Let $W \subset \mathbb{R}^3$ be a smooth region. Denote by σ_W and H_W the second fundamental form (with respect to the inward pointing unit normal vector) and the mean curvature of ∂W. The region is convex if and only if $\sigma_W \geq 0$ and W is said to be mean-convex if $H_W \geq 0$. A *proper surface* in W is an immersed orientable compact surface S with $S \cap \partial W = \partial S$. We assume that the immersion of S is smooth even at the boundary and we denote by D and ∇ the usual derivative in $\mathbb{R}^3$ and the covariant derivative operator in S, respectively. Let N, σ, and A be the unit normal vector, the second fundamental form and the Weingarten endomorphism of the immersion. So, $\sigma(v, w) = \langle Av, w\rangle$ for any v, w tangent vectors to S at $p \in S$. Denote by H, K and κ the mean curvature (normalized so that $H = 1$ for the unit sphere in $\mathbb{R}^3$), the Gauss curvature of S and the inward geodesic curvature of ∂S in S, respectively.

A variation of S is a smooth family of proper surfaces in W given by immersions $\psi_\tau : S \to W$, with $|\tau| < \varepsilon$ and ψ_0 equal to the initial immersion of S. We denote by $A(\tau)$ the area of ψ_τ. The first variation formula of the area is

$$A'(0) = -2\int_S HudA + \int_{\partial S} \langle \frac{d\psi}{d\tau}(0), N\rangle ds. \tag{1}$$

We say that the surface S is a *minimal surface with free boundary* if it is a critical point of the area functional among proper surfaces or, equivalently, if $H = 0$ and S intersects ∂W orthogonally along ∂S. In this case, if we consider a function $u : S \longrightarrow \mathbb{R}$, smooth even at the boundary, then there exists a variation ψ_τ of S by proper surfaces whose velocity vector at $\tau = 0$ is $\frac{d\psi}{d\tau}(0) = uN$. For such a surface S, the second variation formula of the area is

$$A''(0) = Q(u,u) = -\int_S (u\Delta u + |\sigma|^2 u^2)dA + \int_{\partial S} (u\frac{\partial u}{\partial n} - \sigma_W(N,N)u^2)ds, \tag{2}$$

where Δ is the Laplacian of S and n the (outward pointing) unit conormal vector of S along its boundary, [1, 21]. Note that, S being orthogonal to ∂W along ∂S, N is tangent to ∂W and so $\sigma_W(N,N)$ is well defined. The operator $\Delta + |\sigma|^2$ is called the *Jacobi operator* of the surface and the solutions of the equation $\Delta u + |\sigma|^2 u = 0$ are called *Jacobi functions.* After integration by parts, the second variation can be written as

$$Q(u,u) = \int_S (|\nabla u|^2 - |\sigma|^2 u^2)dA - \int_{\partial S} \sigma_W(N,N)u^2 ds. \tag{3}$$

The first term is the usual one in the second derivative of the area functional and the integral along ∂S is the contribution of the free boundary condition. A minimal surface with free boundary is said to be *stable* if it has nonnegative second variation $Q(u,u) \geq 0$, for all u.

If W is symmetric with respect to the origin, $-W = W$ then we can consider a version of the above adapted to the symmetric context. We say that an embedded proper surface S is a balance surface if $-S = S$ and the two components of $W - S$ are interchanged by the central symmetry.

A balance surface has stationary area among balance surfaces if and only if it is a minimal surface with free boundary in W. This kind of surfaces can be constructed by area minimizing arguments and they are free of singularities, see Morgan [11], and they play a important role in surface crystallography, Fischer and Koch [5].

Another natural variational problem for proper surfaces appears when we consider volume preserving variations. It follows form (1) that a proper surface S is a critical point of the area functional, among proper surfaces enclosing a fixed volume in W if and only if the mean curvature H is constant and S meets ∂W orthogonally, see [12, 21]. We say that S is a constant mean curvature surface with free boundary in W. If S is either non embedded or does not enclose any volume, we consider variations ψ_τ by proper surfaces in W which joint with S bound a net oriented volume 0, that is

$$\int_{[0,\tau]\times S} \Omega = 0,$$

where Ω is the pullback of the euclidean volume element by the map $(\tau, p) \mapsto \psi_\tau(p)$, with $|\tau| < \varepsilon$ and $p \in S$. As for minimal surfaces, given a smooth function u on S with $\int_S u\, dA = 0$ there exists a variation of S by proper surfaces enclosing the same volume than S and such that the velocity vector at $\tau = 0$ is given by uN (this follows from a modification of the arguments given in [1]). The second variation formula (2) still holds for constant mean curvature surfaces with free boundary if we consider functions u with mean value zero. The Jacobi operator $\Delta + |\sigma|^2$ and Jacobi functions, functions u satisfying $\Delta u + |\sigma|^2 u = 0$, are defined as in the minimal case. The surface S is said to be *volume-preserving stable* if $Q(u, u) \geq 0$ for all u with $\int_S u\, dA = 0$. This stability notion is related with the isoperimetric problem which consists of studying least area surfaces among the ones enclosing a given volume, see Ros [16, 17].

2.1 Harmonic 1-forms

Let S be a compact, connected and orientable Riemannian surface with smooth boundary and denote by t, n the unit tangent vector and the (outward pointing) conormal vector along ∂S. So t, n is an orthonormal basis of the tangent plane of S at the points of its boundary.

Let ω be an harmonic 1-form on S (smooth even at the boundary). This means that ω is closed and coclosed, i.e. $d\omega = 0$ and $\operatorname{div}\omega = 0$. Thus the covariant derivative $\nabla\omega$ is a symmetric tensor with trace 0 or, equivalently, in a neighborhood of each point of S, ω is the differential of a harmonic function. The conjugate harmonic 1-form of ω is another harmonic 1-form ω^* given by $\omega^*(e_1) = \omega(e_2)$ and $\omega^*(e_2) = -\omega(e_1)$, e_1, e_2 being positive orthonormal basis in the tangent plane of S. The Hodge Theorem gives a relation between the cohomology of S and the space of harmonic 1-forms. When $\partial S \neq \emptyset$ there are several natural boundary conditions for ω. In this paper we will consider the space $\mathcal{H}(S)$ of harmonic 1-forms ω on S with Neumann boundary condition $\omega(n) = 0$.

Lemma 1 (Hodge Theorem)**.** *Given a compact orientable surface with boundary S, there is an isomorphism between the space $\mathcal{H}(S)$ of harmonic 1-forms on S with Neumann boundary condition and the first the Rham cohomology group $H^1(S, \mathbb{R})$ of S.*

Proof. From the divergence theorem we get that for any harmonic function f on S,

$$\int_S |\nabla f|^2 dA = \int_S \operatorname{div}(f\nabla f) dA = \int_{\partial S} f \frac{\partial f}{\partial n} ds.$$

Therefore, if $df(n) = 0$ we get that f is constant. This means that if the cohomology class of $\omega \in \mathcal{H}(S)$ is zero then $\omega = 0$. It follows that the map

$$\mathcal{H}(S) \longrightarrow H^1(S, \mathbb{R})$$

which applies a harmonic form ω into its cohomology class $[\omega]$ is injective. Let α a closed 1-form on S and f be a smooth function given as a solution of the Neumann Problem

$$\Delta f = \operatorname{div}\alpha \ \text{ in } S \qquad \text{and} \qquad \frac{\partial f}{\partial n} = \alpha(n) \text{ in } \partial S. \tag{4}$$

It follows that $\omega = \alpha - df$ lies in $\mathcal{H}(S)$ and defines the same class than α in $H^1(S, \mathbb{R})$. This proves the lemma.

□

Note that in the case $\partial S \neq 0$, from the unique continuation property, it is not possible to have both $\omega \in \mathcal{H}(S)$ and $\omega^* \in \mathcal{H}(S)$, unless $\omega = 0$.

3 Minimal surfaces

In this section S will be a compact orientable minimal surface with free boundary in a mean-convex region $W \subset \mathbb{R}^3$. We first prove the following simple but interesting fact.

Proposition 2. *Let W be a smooth mean-convex region and S a connected stable minimal surface with free boundary in W. Then S is a (topological) disk with total curvature smaller than 2π*

Proof. As S meets ∂W ortogonally, along ∂S the conormal vector of S coincides with the outward pointing normal vector of ∂W and we get

$$\sigma_W(t,t) = \langle D_t\, n, t\rangle = \langle \nabla_t\, n, t\rangle = \kappa, \tag{5}$$

κ being the geodesic curvature of the boundary curve of S. Therefore, from the mean-convexity of W, we have $\sigma_W(N,N) = 2H_W - \sigma_W(t,t) \geq -\kappa$. From the nonnegativity of the second variation (3) we obtain

$$0 \leq \int_S (|\nabla u|^2 - |\sigma|^2 u^2) dA + \int_{\partial S} \kappa\, u^2 ds, \tag{6}$$

for any $u \in C^2(S)$ and the equality holds if and only if $\Delta u + |\sigma|^2 u = 0$ on S and $\frac{\partial u}{\partial n} + \kappa u = 0$ on ∂S. Taking $u = 1$, using the Gauss equation $2K = -|\sigma|^2$ and the Gauss-Bonnet theorem, we conclude that

$$0 \leq 2\int_S K dA + \int_{\partial S} \kappa\, ds = \int_S K dA + 2\pi\chi(S).$$

This implies that S is either a disk or an annulus. In the case of the annulus, the function $u = 1$ gives the equality in (6) and so S is planar and κ vanishes along ∂S, which is impossible.

□

In the proof of our results, we will use dual vector fields of harmonic 1-forms as test functions in the nonnegativity of the second variations formula, see Palmer [13] and Ros [18, 19] for other applications of these functions. For any harmonic 1-form ω on S we consider its dual vector field $X : S \longrightarrow \mathbb{R}^3$, viewed as a vector valued function. Thus $X(p)$ is tangent to S for each point $p \in S$ and $\langle X, v\rangle = \omega(v)$, for any vector v tangent to S at p. The dual vector field of the conjugate harmonic form ω^* will be denoted by X^*. The minimality of S implies that the differentials of the linear coordinates of the immersion dx_1, dx_2, dx_3 and their conjugates dx_1^*, dx_2^*, dx_3^* span two spaces of harmonic 1-forms denoted by $\mathcal{L}(S)$ and $\mathcal{L}^*(S)$, respectively. If $\omega \in \mathcal{L}(S)$, then there is a vector $a \in \mathbb{R}^3$ such that, for any vector v tangent to S, $\omega(v) = \langle a, v\rangle$ and the dual vector fields of ω and ω^* are given by $X = a - \langle N, a\rangle N$ and $X^* = a \wedge N$, respectively. We will need the following result.

Lemma 3 ([18]). *Let S be an orientable minimal surface immersed in $\mathbb{R}^3$, ω a harmonic 1-form on S and X its dual tangent vector field, viewed as a $\mathbb{R}^3$-valued function. Then we have*

$$\Delta X + |\sigma|^2 X = 2\langle \nabla\omega, \sigma\rangle N. \tag{7}$$

Moreover, if S is nonflat, then X is a Jacobi function, i.e. $\langle \nabla\omega, \sigma\rangle \equiv 0$, if and only if $\omega \in \mathcal{L}^*(S)$.

Lemma 4. *Let S be a proper surface immersed in a smooth region $W \subset \mathbb{R}^3$ which intersects ∂W orthogonally along ∂S (no assumption about the mean curvature of the immersion) and $\omega \in \mathcal{H}(S)$ a harmonic 1-form with Neumann boundary condition. If X and X^* are the dual vector fields of ω and ω^* respectively, then, along the boundary of S we have*

$$\langle X, \frac{\partial X}{\partial n}\rangle = \langle X^*, \frac{\partial X^*}{\partial n}\rangle = -\sigma_W(t,t)|X|^2. \tag{8}$$

Proof. Given a local orthonormal basis e_i, $i = 1, 2$, of tangent vector fields to S, we have the duality relation $\omega(e_i) = \langle X, e_i \rangle$ and its derivative $(\nabla\omega)(e_j, e_i) = \langle \nabla_{e_j} X, e_i \rangle$. As $\omega(n) = 0$, the normal derivative of X at the points of ∂S satisfies

$$\langle X, \frac{\partial X}{\partial n} \rangle = \langle X, \nabla_n X \rangle = (\nabla\omega)(n, X) = (\nabla\omega)(n, t)\, \omega(t).$$

Derivating $\omega(n) = 0$ with respect to t, we obtain

$$0 = \frac{d}{dt}\omega(n) = (\nabla\omega)(t, n) + \omega(\nabla_t n) = (\nabla\omega)(t, n) + \omega(t)\, \langle \nabla_t n, t \rangle =$$

$$= (\nabla\omega)(t, n) + \omega(t)\sigma_W(t, t),$$

where we have used (5) and from the above calculation we obtain

$$\langle X, \frac{\partial X}{\partial n} \rangle = -\sigma_W(t, t)|X|^2.$$

The conjugate vector field X^* is obtained from X by a rotation of 90 degrees in the tangent planes of S. As this rotation is parallel, we have

$$\langle X^*, \frac{\partial X^*}{\partial n} \rangle = \langle X^*, \nabla_n X^* \rangle = \langle X, \nabla_n X \rangle = -\sigma_W(t, t)|X|^2,$$

and this proves the lemma.

□

3.1 Balance minimal surfaces

If the region W is symmetric with respect to the origin and S is a balance surface with stationary area among balance surfaces, then it is a minimal surface with free boundary in W. The surface S is stable if it minimizes area up to second order among balance surfaces, i. e., it has nonnegative second variation for odd infinitesimal variations $u \in C^2(S)$ with $u(-p) = -u(p)$ for any point $p \in S$. In the following result we describe the topology of these surfaces.

Theorem 5. *Let $W \subset \mathbb{R}^3$ be a centrally symmetric mean-convex region and $S \subset W$ a nonflat stable balance minimal surface. Then, the connected components of S are disks.*

Proof. If S has a connected component S_0 not equal to $-S_0$, then S_0 is a stable minimal surface with free boundary in W, and Proposition 2 gives that S_0 is a disk. Therefore we can assume that S is connected. Let $\phi : S \longrightarrow S$ be the central symmetry $\phi(p) = -p$ and note that ϕ preserves orientations and the Gauss map N of S satisfies $N \circ \phi = N$. The quotient surface $S' = S/\{Id, \phi\}$ has a structure of Riemann surface and so, harmonic 1-forms are well defined over S'. The Euler characteristic of these two surfaces are related by either $\chi(S) = 2\chi(S') - 1$ (if $0 \in S$) or $\chi(S) = 2\chi(S')$ (when $0 \notin S$). If S' is not a disk, then $\dim H^1(S', \mathbb{R}) \geq 1$. Therefore, from Lemma 1 it admits a nonzero harmonic 1-form with Neumann boundary condition. This 1-form lifts to a harmonic 1-form $\omega \in \mathcal{H}(S)$ such that its pullback image by the involution ϕ satisfy $\phi^*\omega = \omega$. The conjugate harmonic form satisfies $\phi^*\omega^* = \omega^*$, too. As a consequence, the dual tangent vector fields X and X^* of ω and ω^* satisfy $X \circ \phi = -X$ and $X^* \circ \phi = -X^*$. Now we use the linear coordinates of $X^* = (X_1^*, X_2^*, X_3^*)$ as test functions in the stability inequality given by second variation formula (2), and with the notation $Q(X^*, X^*) = \sum_{j=1}^3 Q(X_j^*, X_j^*)$, we conclude that

$$0 \leq Q(X^*, X^*) = -\int_S \langle \Delta X^* + |\sigma|^2 X^*, X^* \rangle dA + \int_{\partial S} (\langle X^*, \frac{\partial X^*}{\partial n} \rangle - \sigma_W(N,N)|X^*|^2) ds. \tag{9}$$

From (7) we have that $\Delta X^* + |\sigma|^2 X^*$ is normal to S and therefore the integral over S in (9) vanishes. Along the boundary of S, the vectors t and N form an ortonormal basis of the tangent plane of ∂W and so $\sigma_W(t,t) + \sigma_W(N,N) = 2H_W$. Hence, using (8), the inequality (9) transforms into

$$0 \leq Q(X^*, X^*) = -2 \int_{\partial S} H_W |X^*|^2 ds \leq 0. \tag{10}$$

As W is mean-convex, this implies that $Q(X^*, X^*) = 0$ and so the linear coordinates of X^* lie in the kernel of Q, i.e. $Q(X^*, Y) = 0$, for all $\mathbb{R}^3$-valued C^2 function Y on S with $Y \circ \phi = -Y$. In particular, from linear elliptic theory, we get

$$\Delta X^* + |\sigma|^2 X^* = 0 \text{ on } S \qquad \text{and} \qquad \frac{\partial X^*}{\partial n} = \sigma_W(N,N)X^* \text{ on } \partial S. \quad (11)$$

As S is nonflat, from Lemma 3 we deduce that $\omega^* \in \mathcal{L}^*(S)$. This implies that $\omega \in \mathcal{L}(S)$. Hence ω is exact and using Lemma 1 we conclude that $\omega = 0$, which is a contradiction. Therefore S' is a disk and the same holds for the surface S.

□

Remark 1. *In the same way, by applying the stability condition to the dual vector field X, from (8) and (7) we obtain*

$$0 \leq Q(X,X) = -2\int_{\partial S} H_W |X|^2 ds \leq 0 \quad (12)$$

and from Lemma 3 we have that $\omega \in \mathcal{L}^*(S)$, too.

Each torus T^3 admits a central symmetry $-Id$ and several classical periodic minimal surfaces are balance minimal surfaces for the case $\mathcal{G} = \{\pm\mathcal{I}\lceil\}$. In particular, minimal surfaces of genus 3 are all balance. The proof of Theorem 5 applies to this case, and gives the following.

Theorem 6. *Let S be a closed balance minimal surface embedded in a flat three torus T^3. If S is stable and nonflat, then genus$(S) = 3$.*

Proof. First we observe that for any closed minimal surface in T^3, genus$(S) \geq 3$. For any balance minimal surface S, denote by ϕ the central symmetry of T^3 restricted to the surface. Then ϕ has 8 fixed points on S. If the Riemann surface $S' = S/\{Id, \phi\}$ is not a sphere, then it admits a non zero harmonic 1-form which lifts to a harmonic 1-form ω on S such that $\phi^*\omega = \omega$ and $\phi^*\omega = \omega$. Following the proof of Theorem 5, we get that $\omega \in \mathcal{L}(S)$. As the 1-forms $\alpha \in \mathcal{L}(S)$ verify $\phi^*\alpha = -\alpha$, we have a contradiction. So S' is the Riemann sphere and, as the projection $S \to S'$ has exactly 8 branch points, it follows that genus$(S) = 3$.

□

3.2 Piecewise smooth regions

We say that W is a piecewise smooth mean-convex region in $\mathbb{R}^3$ if it satisfies the following conditions:

i) ∂W is a union of smooth surfaces with piecewise smooth boundary (the faces of W),

ii) the faces of W have nonnegative mean curvature and the angles at each one of its vertices satisfy $0 < \theta < 2\pi$, and

iii) two of these faces are either disjoint or meet at some vertices and/or along some common edges. If two faces meet along an edge, then their interior angle is everywhere bigger that 0 and smaller than π.

These are natural regions to solve the Plateau Problem, see Meeks and Yau [9]. We say that a proper minimal surface $S \subset W$ is a minimal surface with free boundary if S has piecewise smooth boundary and meets the faces and the edges of W but omits the vertices of ∂W. The edges of S sit orthogonally on the faces of W and its corners lie on the edges of W. Proposition 2 extends trivially to piecewise smooth regions in $\mathbb{R}^3$. In the next result we prove that the same holds for Theorem 5. The argument applies to volume-preserving stability, and so Theorem 9 below extends to piecewise smooth regions, too.

Theorem 7. *Let S be a nonflat balance stable minimal surface with free boundary in a piecewise smooth mean-convex region $W \subset \mathbb{R}^3$. Then S is a topological disk.*

Proof. Given $\varepsilon > 0$, let $\varphi_\varepsilon : S \longrightarrow \mathbb{R}$ be a logarithmic smooth cutoff function vanishing in a neighborhood of the corners of S, equal to 1 at the points whose distance to each vertex is larger than ε, with $0 \leq \varphi_\varepsilon \leq 1$ on S and such that $\int_S |\nabla\varphi_\varepsilon|^2 dA$ converges to zero when $\varepsilon \to 0$.

If u is a smooth function on S minus the vertices and $\varphi = \varphi_\epsilon$, then after integration by parts we get

$$Q(\varphi u, \varphi u) = \int_S (|\nabla(\varphi u)|^2 - |\sigma|^2\varphi^2 u^2) dA - \int_{\partial S} \sigma_W(N,N)\varphi^2 u^2 ds^2 =$$

$$\int_S (|\nabla\varphi|^2 u^2 + |\nabla u|^2\varphi^2 + \frac{1}{2}\langle\nabla\varphi^2, \nabla u^2\rangle - |\sigma|^2\varphi^2 u^2) dA - \int_{\partial S} \sigma_W(N,N)\varphi^2 u^2 ds =$$

$$\int_S (|\nabla\varphi|^2u^2+|\nabla u|^2\varphi^2-\frac{1}{2}\varphi^2\Delta u^2-|\sigma|^2\varphi^2u^2)dA+\int_{\partial S}(\varphi^2 u\frac{\partial u}{\partial n}-\sigma_W(N,N)\varphi^2u^2)ds =$$
$$\int_S (|\nabla\varphi|^2u^2-\varphi^2(\Delta u+|\sigma|^2u)u)dA+\int_{\partial S}\varphi^2(u\frac{\partial u}{\partial n}-\sigma_W(N,N)u^2)ds^2.$$

The surface S is piecewise smooth with angles at most π at its corners. Then it follows that the solution of the Neumann boundary problem (4) is smooth in S minus its corners and C^1 in S, see Wigley [27]. Therefore the harmonic forms $\omega \in \mathcal{H}(S)$ in the Hodge Theorem of Lemma 1 are smooth in S minus the corners and belong to $C^0(S)$ and the same hold for ω^* and the dual vector fields X and X^*.

So, if in the computation above we take u equal to the linear coordinates of X^*, using (7) and (8) we obtain

$$Q(\varphi_\varepsilon X^*,\varphi_\varepsilon X^*)=\int_S|\nabla\varphi_\varepsilon|^2|X^*|^2dA-2\int_{\partial S}\varphi_\varepsilon^2 H_W|X^*|^2ds$$

and taking $\varepsilon\to 0$ we conclude that X^* belongs to the Sobolev space $L^{1,2}(S)$ and $Q(X^*,X^*)\le 0$. Now we finish as in the proof of Theorem 5.

□

In particular, we can take W to be a convex polyhedron in $\mathbb{R}^3$. Minimal surfaces with free boundary in these regions have been constructed by Smyth [24] and Jost [7].

4 Constant mean curvature surfaces.

Let S be a proper surface in a mean-convex region $W\subset\mathbb{R}^3$. Assume that S has constant mean curvature H.

Lemma 8 ([19])**.** *Let ω be a harmonic 1-form on a surface S in $\mathbb{R}^3$ of constant mean curvature H and $X : S\to\mathbb{R}^3$ its dual vector field. Then*

$$\Delta X+|\sigma|^2X=4H^2X-2HAX+2\langle\nabla\omega,\sigma\rangle N, \tag{13}$$

where A denotes the Weingarten endomorphism of S.

In this section we will prove that if S has nonnegative volume-preserving second variation, then the topology of S is controlled. Earlier results in this direction were obtained by Ros and Vergasta [21]. They proved that if W is convex then genus$(S) \in \{0,1\}$ and S has at most three boundary components, or genus$(S) \in \{2,3\}$ and ∂S is connected. In the next theorem we improve that result.

Theorem 9. *Let $W \subset \mathbb{R}^3$ be a smooth mean-convex region and $S \subset W$ a nonflat surface of constant mean curvature with free boundary. If S has nonnegative second variation for volume-preserving variations, then either*

i) genus$(S) = 0$ and S has at most 4 boundaries components, or

ii) S has genus 1 and ∂S has at most two components.

Proof. Let $\omega \in \mathcal{H}(S)$ a harmonic 1-form with Neumann boundary condition. By applying the stability quadratic form (2) to (the linear coordinates of) X, from (13) we have

$$Q(X,X) = -\int_S \langle \Delta X + |\sigma|^2 X, X\rangle dA + \int_{\partial S} (\langle X, \frac{\partial X}{\partial n}\rangle - \sigma_W(N,N)|X|^2) ds =$$
$$-\int_\Sigma (4H^2|X|^2 - 2H\sigma(X,X))dA - 2\int_{\partial S} H_W |X|^2 ds.$$

If we put X^*, the dual tangent vector field of ω^*, in the second variation, (13) and (8) give

$$Q(X^*,X^*) = -\int_\Sigma (4H^2|X^*|^2 - 2H\sigma(X^*,X^*))dA - 2\int_{\partial S} H_W |X^*|^2 ds.$$

As in Palmer [13], from the identities $|X^*| = |X|$ and $\langle X, X^*\rangle = 0$, we obtain

$$Q(X,X) + Q(X^*,X^*) = -4H^2 \int_S |X|^2 dA - 4\int_{\partial S} H_W |X|^2 ds \leq 0. \quad (14)$$

In order to apply the stability assumption in (14) we need that both X and X^* have mean value zero. As ω satisfies the Neumann condition $\omega(n) = 0$, for each vector $a \in \mathbb{R}^3$ we have $\mathrm{div}(\langle p,a\rangle\omega) = \langle X,a\rangle + \langle p,a\rangle \mathrm{div}\,\omega = \langle X,a\rangle$, where p represent a point of S, and therefore

$$\int_S X dA = \int_{\partial S} \omega(n) p ds = 0, \quad (15)$$

and so X has mean value zero. However this does not hold in general for X^*. If $\dim H^1(S,\mathbb{R}) \geq 4$, we can find a nonzero harmonic 1-form $\omega \in \mathcal{H}(S)$ such that $\int_S X^* = 0$. Thus $0 \leq Q(X,X) + Q(X^*,X^*)$ and (14) implies that S is a minimal surface and that $Q(X,X) = Q(X^*,X^*) = 0$. Therefore $Q(X,Y) = Q(X^*,Y) = 0$ for any Y with mean value zero. Thus, there exist $a, b \in \mathbb{R}^3$ such that

$$\Delta X + |\sigma|^2 X = a \qquad \text{and} \qquad \Delta X^* + |\sigma|^2 X^* = b$$

and using (7) we conclude that $2\langle \nabla\omega^*, \sigma\rangle N = b$.

As S is nonflat, the Gauss map of S is an open map which implies that $b = 0$ and $\langle \nabla\omega^*, \sigma\rangle = 0$. Therefore Lemma 3 gives that ω^* belongs to $\mathcal{L}^*(S)$. Then $\omega \in \mathcal{L}(S)$, ω is exact and Lemma 1 gives that $\omega = 0$, a contradiction. Hence $\dim H^1(S,\mathbb{R}) \leq 3$, which means that S is either a genus zero surface with at most 4 boundary components or a surface of genus 1 with 1 or 2 boundary components.

$\square$

If W is convex, then the case genus$(S) = 0$ and four components at the boundary cannot hold by Theorem 5 in Ros and Vergasta [21].

A wide vertical annulus in a horizontal slab give an example of a volume-preserving stable surface with nontrivial topology. In fact it follows from the results in Ros [19], that a volume-preserving stable constant mean curvature surface with free boundary immersed in a horizontal slab is either a half-sphere or a flat vertical cylinder (for the embedded case, see also [15]). A more involved example in a related situation if given by Schwarz P minimal surface in the cubic 3-torus $\mathbb{R}^3/\mathbb{Z}^3$, see Figure 1. Ross [22] proved that the Schwarz P surface is volume-preserving stable. So, if we take the piece of the surface between two consecutive horizontal planes of symmetry, we get a volume preserving stable surface in $T^2 \times [0, 1/2]$, where T^2 is the flat 2-torus generated by $(1,0), (0,1)$. This surface has genus 1 and two boundary components. We remark that part of the arguments in the proof of Theorem 9 do not apply to surfaces in $T^2 \times [0,a]$ because in this case, the mean value property $\int_S X = 0$ in (15) is no longer valid.

Remark 2. *If W is a ball, then a volume-preserving stable surface with free boundary S must be either a planar equator, a spherical cap or a surface of genus* 1 *with at most two boundary components, see [21]. We don't know examples of surfaces S of genus* 1 *with free boundary in a ball and nonnegative volume-preserving second variation, but it is worth noticing that although the reasoning in [21] is different from the one used in the proof of Theorem 9, both arguments give the same bound on the topology of S.*

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Departamento de Geometría y Topología
Facultad de Ciencias - Universidad de Granada
18071 Granada Spain
E-mail: aros@ugr.es