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A NOTE ON THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN NONCONVEX PLANAR DOMAINS

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

Given a bounded smooth domain Ω of $\mathbb{R}^2$ satisfying the exterior circle condition with radius r and a smooth boundary data φ on $\partial\Omega$, we prove that if r is bigger than a constant (explicitly calculated) depending only on the C^2 norm of φ then the Dirichlet problem for the minimal surface equation for Ω and φ has a solution. Since the condition on r is trivially satisfied if the domain is convex, our result generalizes the classical theorem of R. Finn [F].

Let Ω be a $C^{2,\alpha}$ bounded open domain in $\mathbb{R}^2$ satisfying the exterior circle condition of radius r, that is, given $p \in \partial\Omega$ there is a circle of radius r tangent to $\partial\Omega$ at p and contained in $\mathbb{R}^2\backslash\Omega$.

Given $\varphi \in C^{2,\alpha}\left(\overline{\Omega}\right)$, set

$$\begin{aligned} M &= \max_{x\in\Omega}\varphi(x) - \min_{x\in\Omega}\varphi(x) \\ B &= \sup_{x\in\Omega}\left|D\varphi(x)\right| \\ A &= \sup_{x\in\Omega}\left|D^2\varphi(x)\right| \end{aligned}$$

where $\left|D^2\varphi\right| = |D_{11}\varphi| + |D_{22}\varphi| + |D_{12}\varphi|$.

We prove:

Theorem *If*

$$r \geq \max\left\{e^{4M\left(52A+50AB^2+10B^2+5\right)} - 1, 1\right\} \tag{1}$$

then the Dirichlet problem for the minimal surface equation

$$\begin{cases} M[u] := \left(1+|Du|^2\right)\Delta u - \sum\limits_{i,j=1}^{2} D_i u D_j u D_{ij} u = 0 \\ u|_{\partial\Omega} = \varphi. \end{cases}, \tag{2}$$

has an unique solution.

Since the minimality of a graph is invariant by homotheties, we may apply the above result to treat the case $0 < r < 1$ after rescaling the problem by the factor $1/r$.

If Ω is convex then we may take $r = \infty$ so that the above theorem recovers the classical result of R. Finn (see [F]) for smooth boundary data.

Proof. We may assume that $\min\varphi = 0$. Choose $p \in \partial\Omega$. Let $p_0 = (p_1, p_2)$ be the center of the circle tangent to $\partial\Omega$ at p, with radius r and contained in $\mathbb{R}^2\backslash\Omega$, and set

$$d(x) = |x - p_0| - r.$$

Define $w \in C^{2,\alpha}\left(\overline{\Omega}\right)$ by

$$w(x) = \varphi(x) + \psi(d(x)),$$

where

$$\psi(d) = \delta \ln(bd + 1) \tag{3}$$

and δ, b are positive constants to be determined. We will only prove that w is an upper barrier at p for some choices of δ and b. For a lower barrier, note that if w is an upper barrier for $-\varphi$ at p, then $-w$ is a lower barrier for φ at p.

We have

$$\begin{aligned} D_i d(x) &= \frac{x_i - p_i}{|x - p_0|}, \\ D_{ij} d &= \frac{1}{|x - p_0|} \left(\delta_{ij} - \frac{(x_i - p_i)(x_j - p_j)}{|x - p_0|^2} \right), \\ \Delta d(x) &= \mathsf{div}\left(D_1 d(x), D_2 d(x)\right) = \frac{1}{|x - p_0|} \end{aligned}$$

so that

$$D_i \psi(d(x)) = \psi'(d(x)) \frac{x_i - p_i}{|x - p_0|},$$

$$\begin{aligned} D_{ij} \psi(d(x)) &= \psi''(d(x)) D_j d(x) D_i d(x) + \psi'(d(x)) D_{ij} d(x) \\ &= \psi''(d(x)) \frac{(x_i - p_i)(x_j - p_j)}{|x - p_0|^2} \\ &\quad + \frac{\psi'(d(x))}{|x - p_0|} \left(\delta_{ij} - \frac{(x_i - p_i)(x_j - p_j)}{|x - p_0|^2} \right) \end{aligned}$$

and

$$\Delta \psi(d(x)) = \psi''(d(x)) + \frac{\psi'(d(x))}{|x - p_0|}.$$

Using the above equalities we obtain

$$\begin{aligned} M[w] &= \Delta\varphi + (D_1 w)^2 D_{22}\varphi + (D_2 w)^2 D_{11}\varphi - 2 D_1 w D_2 w D_{12}\varphi \\ &\quad + \psi''(d)) + \frac{\psi'(d)}{|x - p_0|} + |Dw|^2 \psi''(d) \\ &\quad - \left(\psi''(d) - \frac{\psi'(d)}{|x - p_0|} \right) \left\langle (D_1 w, D_2 w), \left(\frac{x_1 - p_1}{|x - p_0|}, \frac{x_2 - p_2}{|x - p_0|} \right) \right\rangle^2 \end{aligned}$$

and the estimate

$$M[w] \leq 2A\left(1 + B^2 + \psi'(d)^2\right) + \frac{\psi'(d)}{r}\left(1 + 2B^2\right) + \frac{2\psi'(d)^3}{r} + \psi''(d).$$

From (3) we obtain

$$\begin{aligned} M[w] &\leq 2A\left(1+B^2+\delta^2\frac{b^2}{(bd+1)^2}\right)+\frac{\delta b}{r(bd+1)}\left(1+2B^2\right) \\ &+\frac{2\delta^3 b^3}{r(bd+1)^3}-\frac{\delta b^2}{(bd+1)^2}\leq 2A\left(1+B^2+\frac{\delta^2 b^2}{(bd+1)^2}\right) \\ &+\frac{\delta b}{r(bd+1)}\left(1+2B^2\right)-\frac{\delta b^2}{(bd+1)^2}\left(1-\frac{2\delta^2 b}{r(bd+1)}\right). \end{aligned}$$

The last term above is negative if and only if

$$\frac{b}{bd+1}\leq\frac{r}{2\delta^2}.$$

This inequality is satisfied if one chooses $b=r/\left(4\delta^2\right)$. With this choice of b we obtain

$$\begin{aligned} M[w] &\leq 2A\left(1+B^2+\frac{1}{16\delta^2}\frac{r^2}{\left(\frac{1}{4\delta^2}dr+1\right)^2}\right)+\frac{1}{4\delta}\frac{1}{\frac{1}{4\delta^2}dr+1}\left(1+2B^2\right) \\ &-\frac{1}{32\delta^3}\frac{r^2}{\left(\frac{1}{4\delta^2}dr+1\right)^2} \\ &= \frac{1}{2\left(4\delta^2+dr\right)^2}\left(4AB^2d^2r^2+32AB^2dr\delta^2+4B^2dr\delta\right. \\ &+64AB^2\delta^4+16B^2\delta^3+4Ad^2r^2+32Adr\delta^2+2dr\delta \\ &\left.+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3\right) \end{aligned}$$

We then have $M[w]\leq 0$ if

$$\begin{aligned} &4AB^2d^2r^2+32AB^2dr\delta^2+4B^2dr\delta+64AB^2\delta^4+16B^2\delta^3+4Ad^2r^2 \\ &+32Adr\delta^2+2dr\delta+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3 \\ =\ &\left(4AB^2r^2+4Ar^2\right)d^2+\left(32ArB^2\delta^2+4rB^2\delta+32Ar\delta^2+2r\delta\right)d \\ &+64AB^2\delta^4+16B^2\delta^3+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3\leq 0 \end{aligned}$$

For $0 < d \leq \delta$ we have

$$\begin{aligned}
& \left(4AB^2r^2 + 4Ar^2\right) d^2 + \left(32ArB^2\delta^2 + 4rB^2\delta + 32Ar\delta^2 + 2r\delta\right) d \\
& +64AB^2\delta^4 + 16B^2\delta^3 + 4Ar^2\delta^2 - r^2\delta + 64A\delta^4 + 8\delta^3 \\
\leq\ & \left(4AB^2r^2 + 4Ar^2\right) \delta^2 + \left(32ArB^2\delta^2 + 4rB^2\delta + 32Ar\delta^2 + 2r\delta\right) \delta \\
& +64AB^2\delta^4 + 16B^2\delta^3 + 4Ar^2\delta^2 - r^2\delta + 64A\delta^4 + 8\delta^3 \\
=\ & \left(64AB^2 + 64A\right) \delta^4 + \left(32Ar + 16B^2 + 32AB^2r + 8\right) \delta^3 \\
& + \left(4AB^2r^2 + 4B^2r + 8Ar^2 + 2r\right) \delta^2 + \left(-r^2\right) \delta \\
=\ & \delta \left[\left(64AB^2 + 64A\right) \delta^3 + \left(32Ar + 16B^2 + 32AB^2r + 8\right) \delta^2 \right. \\
& \left. + \left(4AB^2r^2 + 4B^2r + 8Ar^2 + 2r\right) \delta + \left(-r^2\right)\right]
\end{aligned}$$

Choosing $\delta \leq 1$, we obtain that $M[w] \leq 0$ for $0 < d \leq \delta$ if

$$\begin{aligned}
& \left(64AB^2 + 64A\right) \delta^3 + \left(32Ar + 16B^2 + 32AB^2r + 8\right) \delta^2 \\
& + \left(4AB^2r^2 + 4B^2r + 8Ar^2 + 2r\right) \delta + \left(-r^2\right) \\
\leq\ & \left(64AB^2 + 64A\right) \delta + \left(32Ar + 16B^2 + 32AB^2r + 8\right) \delta \\
& + \left(4AB^2r^2 + 4B^2r + 8Ar^2 + 2r\right) \delta - r^2 \leq 0.
\end{aligned}$$

It follows that $M[w] \leq 0$ for $0 < d \leq \delta$ if

$$\delta \leq \frac{r^2}{(4AB^2 + 8A)\, r^2 + (32A + 32AB^2 + 4B^2 + 2)\, r + 64A + 64AB^2 + 16B^2 + 8}.$$

Noting that the function

$$f(r) = \frac{r^2}{(4AB^2 + 8A)\, r^2 + (32A + 32AB^2 + 4B^2 + 2)\, r + 64A + 64AB^2 + 16B^2 + 8}$$

is increasing on r and $r \geq 1$, we have

$$\frac{1}{104A + 100AB^2 + 20B^2 + 10} = f(1) \leq f(r)$$

so that $M[w] \leq 0$ if $0 < d \leq \delta$ for

$$\delta = \frac{1}{104A + 100AB^2 + 20B^2 + 10}. \tag{4}$$

In sum: Defining δ by (4), taking $b = r/\left(4\delta^2\right)$, we have $M[w] \leq 0$ on $\mathcal{N}_p$ where

$$\mathcal{N}_p = \{x \in \Omega | 0 \leq d(x) \leq \delta\}.$$

Thus, to guarantee that w is a local barrier from above for M on $\mathcal{N}_p$ the function w must satisfy the a priori height estimate

$$w|_{\partial\mathcal{N}_p} \geq u|_{\partial\mathcal{N}_p} \tag{5}$$

where u is a solution of $M[u] = 0$ and $u|_{\partial\Omega} = \varphi$.

Note that with the choices above

$$\psi(d) = \frac{1}{104A + 100AB^2 + 20B^2 + 10} \ln\left(\frac{r\left(104A + 100AB^2 + 20B^2 + 10\right)^2 d}{4} + 1\right)$$

At $\partial\mathcal{N}_p \cap \partial\Omega$ we have $u = \varphi$ so that (5) is satisfied at these points. By the maximum principle $\sup|u| \leq M$ so that, at $\partial\mathcal{N}_p \backslash \partial\Omega$ we have

$$w(x) = \psi(\delta) + \varphi(x) \geq \psi(\delta) - M.$$

Then (5) is satisfied at $\partial\mathcal{N}_p \backslash \partial\Omega$ if $\psi(\delta) \geq 2M$, which is the case if r satisfies (1).

Observing now that if condition (1) is satisfied for a given φ it is also satisfied for $t\varphi$ for any $t \in [0,1]$, we may conclude the proof of the theorem using the continuity method: Setting

$$V = \{t \in [0,1] | \exists u_t \in C^{2,\alpha}\left(\overline{\Omega}\right) \text{ such that } M[u_t] = 0,\ u_t|_{\partial\Omega} = t\varphi\}$$

we have $V \neq \varnothing$ since $t = 0 \in V$; moreover, V is open by the implicit function theorem. From the barriers above we obtain a priori uniform C^1 estimates for the family of Dirichlet problems $M[u_t] = 0$, $u_t|_{\partial\Omega} = t\varphi$, guaranteeing that V is closed ([GT]), that is, $V = [0,1]$.

The uniqueness of the solution is a consequence of the maximum principle for the difference of two solutions of (2).

This concludes with the proof of the theorem.

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