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SINGULARITIES OF THE RICCI FLOW ON 3-MANIFOLDS

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

We present an overview of the singularity formation of the Ricci flow on 3-manifolds. The article, is the written version of the talks I gave at the BIRS Workshop on Geometric Flows in Mathematics and Physics (Banf, April 13-18, 2008), and at the XV Brazilian School of Differential Geometry (Fortaleza, July 14-18, 2008) in honor of the 80th birthday of Manfredo do Carmo.

In 1982, Hamilton [23] introduced the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

to study compact three-manifolds with positive Ricci curvature. The Ricci flow, which evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. As a consequence, the curvature tensors evolve by a system of diffusion-reaction equations which tends to distribute the curvature uniformly over the manifold. Hence, one expects that the initial metric be improved and evolve into a more canonical metric, thereby leading to a better understanding of the topology of

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the underlying manifold. Indeed, in the celebrated paper [23] Hamilton showed that on any compact three-manifold with an initial metric of positive Ricci curvature, the Ricci flow, after rescaling to keep the constant volume, converges to a metric of positive constant sectional curvature, implying that the underlying manifold is diffeomorphic to the three-sphere $\mathbb{S}^3$ or a quotient of $\mathbb{S}^3$ by a linear group of isometries. However, on general 3-manifolds the Ricci flow could develop singularities. Understanding the singularities of the Ricci flow is not only an essential step in proving the geometrization of 3-manifolds, but also of great significance in geometric analysis and nonlinear PDEs in general. In this article, we shall outline Hamilton's theory of singularity formation and Perelman's singularity structure theorem in the Ricci flow on 3-manifolds.

1 The Ricci Flow

Given a complete Riemannian manifold (M^n, g_{ij}), Hamilton's Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t), \tag{1.1}$$

with the initial metric $g_{ij}(0) = g_{ij}$, is a system of *second order, nonlinear, weakly parabolic* partial differential equations. The degeneracy of the system is caused by the diffeomorphism group of the underlying manifold M^n which acts as the gauge group of the equation: if $g_{ij}(t)$ is a solution to (1.1) and ϕ is a diffeomorphism of M^n, then the pull-back $\tilde{g}_{ij}(t) = \phi^* g_{ij}(t)$ is also a solution to (1.1).

1.1 Short time existence and uniqueness

If M^n is compact, then for any initial metric g_{ij} the Ricci flow (1.1) admits a unique solution $g_{ij}(t)$ for a short time. This short-time existence and uniqueness theorem was first proved by Hamilton [23] using the sophisticated Nash-Moser implicit function theorem. One year later, a simpler proof was given by De Turck [19] (see also an improved version by him in [9]) using the idea of gauge fixing. When M^n is noncompact, W.-X Shi [42] established the short-time existence under the assumption that the initial metric g_{ij} has bounded curvature $|Rm| \le C$, while B.-L. Chen and X.-P. Zhu [12] recently showed the uniqueness in the class of complete solutions with bounded curvature.

1.2 The evolution of curvatures

Since the most important geometric quantity is the curvature tensors, it is important to know how the curvature tensors evolve and behave under the Ricci flow.

The (Riemannian) curvature tensor $Rm = \{R_{ijkl}\}$, when viewed as an endomorphism $Rm : \Lambda^2(M) \to \Lambda^2(M)$ on the space of 2-forms, satisfies the evolution equation

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#}, \qquad (1.2)$$

where Rm^2 is the matrix square of Rm, and $Rm^{\#}$ is a certain square of Rm involving the structure constants of the Lie algebra $so(n)$. The evolution equations for the Ricci tensor R_{ij} and the scalar curvature R are given, respectively, by

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2R_{ijkl}R_{jl},$$

and

$$\frac{\partial R}{\partial t} = \Delta R + 2|R_{ij}|^2.$$

For example, for $n = 3$, if we diagonalize

$$Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix},$$

so that $\lambda \ge \mu \ge \nu$ are the principal sectional curvatures, then the Ricci tensor

$$Rc = \begin{pmatrix} \lambda+\mu & & \\ & \lambda+\nu & \\ & & \mu+\nu \end{pmatrix}$$

and the scalar curvature

$$R = 2(\lambda+\mu+\nu).$$

Moreover,

$$Rm^2 = \begin{pmatrix} \lambda^2 & & \\ & \mu^2 & \\ & & \nu^2 \end{pmatrix} \quad \text{and} \quad Rm^{\#} = \begin{pmatrix} \mu\nu & & \\ & \lambda\nu & \\ & & \lambda\mu \end{pmatrix}.$$

In particular, the ODE corresponding to the evolution PDE (1.2) of Rm (in the space of 3×3 matrices) has a relatively simple form

$$\begin{cases} \frac{d}{dt}\lambda = \lambda^2 + \mu\nu, \\ \frac{d}{dt}\mu = \mu^2 + \lambda\nu, \\ \frac{d}{dt}\nu = \nu^2 + \lambda\mu. \end{cases}$$

The parabolic nature of the Ricci flow yields many nice properties, such as **Shi's local derivative estimate** [42] (cf. Theorem 1.4.2 in [10]), which *gives the bound on the derivatives of the evolved curvature* $|\nabla^k Rm|(x_0,t_0)$ *at any point* (x_0,t_0) *in space-time in terms of the bound of the curvature* $|Rm|(x,t)$ *in a parabolic neighborhood of* (x_0,t_0). Hamilton [23, 24] also developed important maximum principles for the Ricci flow, including **the maximum principle for tensors** (cf. Lemma 2.1.3 in [10]), **the strong maximum principle** (cf. Theorem 2.2.1 in [10]) and **the advanced maximum principle** (cf. Theorem 2.3.1 in [10]). In particular, the positivity of various curvatures are preserved. For example:

(a) The scalar curvature $R\geq 0$ is preserved in all dimensions.

(b) The Ricci curvature $Rc\geq 0$ is preserved in dimension three [23].

(c) The curvature operator $Rm\geq 0$ is preserved in all dimensions [24].

1.3 Long time convergence results in $n = 2$ and $n = 3$

On any compact Riemann surface M^2, Hamilton [25] showed that the solution to the normalized Ricci flow (i.e., after rescaling to keep constant area) exists for all time, and converges to a metric of constant Gaussian curvature provided the Euler number $\chi(M^2) \leq 0$ or the initial metric has positive curvature $R > 0$ when $\chi(M^2) > 0$. Subsequently, B. Chow [15] proved that starting with any metric on the (topological) 2-sphere, the Ricci flow will evolve it to a metric of positive curvature $R > 0$ after a short time. Thus, the works of Hamilton [25] and Chow [15] imply that *on any compact Riemann surface, the solution to the normalized Ricci flow converges to a metric of constant curvature.*

For dimension $n = 3$, as we mentioned in the beginning, Hamilton [23] showed that for any compact three-dimensional Riemannian manifold (M^3, g_{ij}) with positive Ricci curvature $Rc > 0$, the Ricci flow exists on a maximal time interval $0 \leq t < T$ $(T < \infty)$ such that Ricci curvature remains positive $Rc(t) > 0$ and $(M^3, g_{ij}(t))$ shrinks to a point as $t \to T$. If we rescale $g_{ij}(t)$ to keep constant volume, then the normalized solution exists for all time and converges to a metric of positive constant (sectional) curvature.

Remark 1.1. *It was not known before Hamilton's work [23] that a 3-manifold of positive Ricci curvature admits a metric of constant positive sectional curvature (which is equivalent to Einstein in $n = 3$). So in this case the Ricci flow actually finds the canonical metric without a priori knowing it exists or not. This illustrates the sheer power of the Ricci flow.*

Remark 1.2. *Various higher dimensional differentiable sphere theorems have also been proved by using the Ricci flow, e.g., Hamilton [24], H. Chen [14], and Börm-Wilking [2] for manifolds with positive curvature operator and 2-positive curvature operator; Brendle-Schoen [3] for manifolds with 1/4-pinched sectional curvatures, proving a long standing conjecture in Riemannian geometry.*

2 Special Solutions: Einstein Metrics and Ricci Soltons

To help the readers to develop some feel of the Ricci flow, we examine two special classes of solutions: Einstein metrics and Ricci solitons.

2.1 Exact solutions I: Einstein metrics

When the initial metric g_{ij} is an Einstein metric, i.e.,

$$R_{ij} = \rho g_{ij}$$

for some constant ρ, the solution $g_{ij}(t)$ to the Ricci flow is given by

$$g_{ij}(t) = (1 - 2\rho t) g_{ij}.$$

For simplicity, we can normalize $\rho = 0, 1/2$ or $-1/2$. Thus we see that

- *Ricci flat metrics are stationary solutions*

  If $\rho = 0$, i.e., the initial metric g_{ij} is Ricci flat, then

  $$g_{ij}(t) = g_{ij} \tag{2.1}$$

  for all $t \in (-\infty, \infty)$. This happens, for example, on a flat torus or a K3-surface with a Calabi-Yau metric.

- *Positive Einstein metrics shrink homothetically*

  If $\rho = 1/2$, then

  $$g_{ij}(t) = (1 - t) g_{ij}, \tag{2.2}$$

  which shrinks homothetically as t increases. Moreover, *the curvature blows up like* $1/(1-t)$ *as* $t \to 1$. Note that $g_{ij}(t)$ goes back in time all the way to $-\infty$, an **ancient solution**. This happens, for example, on round spheres S^n which shrink to a point in finite time.

- *Negative Einstein metrics expand homothetically*

  By contrast, if $\rho = -1/2$ then

$$g_{ij}(t) = (1+t)g_{ij}$$

  expands homothetically as t increases and the curvature falls back to zero like $-1/t$ as $t \to \infty$. Note that now $g_{ij}(t)$ only goes back in time to -1, when the metric explodes out of a single point in a "big bang". This happens, for example, on hyperbolic spaces.

In particular we see that *under the Ricci flow, metrics expand in directions of negative Ricci curvature and shrink in directions of positive Ricci curvature.*

2.2 Exact solutions II: gradient Ricci solitons

A complete metric g_{ij} on M^n is called a **gradient Ricci soliton** if there exists a smooth function f such that the Ricci tensor R_{ij} of g_{ij} satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some constant ρ, where $\nabla_i \nabla_j f$ is the Hessian of f and f is called a **potential function**. For $\rho = 0$, the Ricci soliton is **steady**, for $\rho > 0$ it is **shrinking**, and for $\rho < 0$ it is **expanding**. If f is a constant function, then we get an Einstein metric. As we shall see later, shrinking and steady Ricci solitons often arise as limits of dilations of singularities in the Ricci flow.

Similar to Einstein metrics, Ricci solitons give rise to special solutions to the Ricci flow. For example,

- Let g_{ij} be a steady gradient Ricci soliton and φ_t the one-parameter group of diffeomorphisms generated by the vector field ∇f. Then

$$g_{ij}(t) = \phi_t^* g_{ij} \tag{2.3}$$

is a solution to the Ricci flow. Thus $g_{ij}(t)$ moves along the Ricci flow simply under a one-parameter subgroup of diffeomorphisms, the symmetry group of the Ricci flow, hence the name steady Ricci soliton.

- Similarly, a shrinking gradient Ricci soliton g_{ij}, with $\rho = 1/2$, gives rise to the self-similar solution

$$g_{ij}(t) = (1-t)\varphi_t^* g_{ij} \qquad (t < 1) \tag{2.4}$$

to the Ricci flow, where φ_t is the 1-parameter family of diffeomorphisms generated by $\nabla f/(1-t)$.

Remark 2.1. *The reader may wish to compare (2.3) with (2.1), and (2.4) with (2.2).*

2.3 Examples of 3-D shrinking Ricci solitons

- *Quotients of round 3-sphere* $\mathbb{S}^3/\Gamma$

  The round 3-sphere $\mathbb{S}^3$, or any its metric quotient $\mathbb{S}^3/\Gamma$, is clearly a (compact) shrinking soliton. Note that under the Ricci flow it shrinks to a point in finite time.

- *The round cylinder* $\mathbb{S}^2 \times \mathbb{R}$

  The round cylinder $\mathbb{S}^2 \times \mathbb{R}$ (or either of its two $\mathbb{Z}_2$ quotients) is a (noncompact) shrinking soliton. Under the Ricci flow, it shrinks to a line in some finite time.

Remark 2.2. *According to Hamilton [25] ($n = 2$) and Ivey [31] ($n = 3$), there are no compact shrinking solitons in dimensions $n = 2$ and $n = 3$ other than the round sphere (and its quotients). Also, as will be described in Section 5, the only complete noncompact nonflat 3-dimensional gradient shrinking solitons are the round cylinder $\mathbb{S}^2 \times \mathbb{R}$ and its $\mathbb{Z}_2$ quotients [8].*

Remark 2.3. *There do exist both compact and noncompact (non-Einstein) shrinking gradient solitons for $n \geq 4$. So far they are all Kähler.*

Koiso [33] and the author [6] independently constructed a $U(2)$-invariant (compact) gradient shrinking Kähler soliton on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$, the below-up of the complex projective plane at one point. Later, X.-J Wang and X.H. Zhu [43] found a (compact) gradient Kähler shrinking soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ which has $U(1)\times U(1)$-symmetry. More generally, Koiso and the author found $U(m)$-invariant Kähler shrinking solitons on twisted projective line bundle over $\mathbb{C}P^{m-1}$ ($m \geq 2$), while Wang-Zhu proved the existence of gradient Kähler shrinking solitons on all Fano toric varieties of complex dimension $m \geq 2$ with non-vanishing Futaki invariant. Feldman-Ilmanen-Knopf [21] found complete **noncompact** *$U(m)$-invariant gradient shrinking Kähler solitons on certain twisted complex line bundles over $\mathbb{C}P^{m-1}$ ($m \geq 2$) which are cone-like at infinity. See also the recent work of Dancer-Wang [18] for further examples.*

2.4 Examples of steady Ricci solitons

- *The cigar soliton Σ*

  The cigar soliton Σ is defined on $\mathbb{R}^2$ with the metric

$$g_0 = ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}.$$

  It has positive curvature,

$$R = 1/(1+x^2+y^2) > 0,$$

  the linear volume growth, and is asymptotic to the flat cylinder at infinity. The corresponding solution $g(t)$, $-\infty < t < \infty$, is given by

$$g(t) = \frac{dx^2+dy^2}{e^{2t}+x^2+y^2} = \phi_t^* g_0,$$

  where $\phi_t(x,y) = (e^{-t}x, e^{-t}y)$. Thus, under the Ricci flow the whole picture of the cigar soliton looks the same at each time, but a point

$p_0 = (x_0, y_0) \in \Sigma$ at time t_0 will be moved radially towards the origin to $p_1 = (e^{-(t_1-t_0)}x_0, e^{-(t_1-t_0)}y_0) \in \Sigma$ at a later time $t_1 > t_0$.

- *The product* $\Sigma \times \mathbb{R}$

  Clearly, the product $\Sigma \times \mathbb{R}$ of the cigar soliton with the real line is a three-dimensional steady Ricci soliton with *nonnegative curvature.* Note that the volume has quadratic growth and the curvatures decay to zero at infinity.

- *The Bryant soliton*

  Bryant found a complete rotationally symmetric gradient steady Ricci soliton on $\mathbb{R}^n$ $(n \geq 3)$ which has *positive curvature* and *opens like a paraboloid*: the sphere $\mathbb{S}^{n-1}$ at geodesic distance s from the origin has radius on the order $\sqrt{s}$. The geodesic ball $B(O, s)$ centered at the origin with radius r has volume growth on the order of $s^{(n+1)/2}$, and the curvature decays to zero like $1/s$ as $\to \infty$.

Remark 2.4. *The author [6] found a complete $U(m)$-invariant gradient steady Kähler-Ricci soliton on the complex Euclidean space $\mathbb{C}^m$ ($m \geq 2$) with positive curvature. He also found a complete $U(m)$-symmetric gradient steady Kähler- Ricci soliton on the blow-up of $\mathbb{C}^m/\mathbb{Z}_m$ at the origin which is the same underlying space that Eguchi-Hanson [20] and Calabi [5] constructed ALE Hyper-Kähler metrics.*

Remark 2.5. *Of course, any Ricci flat metric is a steady soliton. On the other hand, it is known that there are no compact steady solitons other than Ricci flat ones.*

3 Formation of singularities in the Ricci flow

In all dimensions, Hamilton [28] showed that any solution $g_{ij}(t)$ to the Ricci flow on a complete manifold M^n, with the initial metric of bounded curvature, will exist on a maximal time interval [0, T), where either $T =$

∞, or $0 < T < \infty$ and the maximal curvature $|Rm|_{\max}(t)$ at t becomes unbounded as $t \to T$. We call such a solution $g_{ij}(t)$ a **maximal solution**.

If $T < \infty$ and $|Rm|_{\max}(t) \to \infty$ as $t \to T$, we say the maximal solution $g(t)$ **develops singularities** at time T and T is a **singular time**. Furthermore, such a solution $g_{ij}(t)$ is either of

- Type I: $\limsup_{t\to T} (T-t)\ |Rm|_{\max}(t) < \infty$; or
- Type II: $\limsup_{t\to T} (T-t)\ |Rm|_{\max}(t) = \infty$.

Remark 3.1. *The round sphere and round cylinder shrink to a point in finite time. They are both of Type I.*

We mentioned earlier that the Ricci flow on **any** closed Riemann surface will converge to a constant metric after normalizing to keep constant area. This will no longer be so when the dimension $n \geq 3$ and the initial metric is not of positive curvature. In fact, in mid-1980s S.-T. Yau first pointed out the neck-pinch singularity could form in dimension three, and Hamilton further noted that the degenerate neck-pinch could also occur. Now we describe three typical examples of singularity formation in the Ricci flow on compact 3-manifolds, in particular the neck-pinch and the degenerate neck-pinch.

3.1 Examples of 3-D singularities

(i) *Compact 3-manifold with positive curvature*

According to Hamilton [23], the Ricci flow on a compact 3-manifold with positive (Ricci) curvature will shrink to a round point (and the curvature becomes infinite!) in finite time. This is a Type I singularity. (Note that if we rescale to keep the constant volume, then the normalized solution exists for all time and converges to a quotient of the round 3-sphere.)

(ii) *The neck-pinch*

Imagine we take a rotationally symmetric dumbbell metric on a topological $\mathbb{S}^3$ with a neck-like $\mathbb{S}^2 \times I$, where I is some interval, then we expect the neck in the middle will shrink under the Ricci flow because the positive curvature in the $\mathbb{S}^2$ direction will dominate the slightly negative curvature in the direction of interval I. We also expect the neck will pinch off in finite time. Note that the dumbbell metric may have positive scalar curvature but is not of nonnegative Ricci curvature.

(iii) *The degenerate neck-pinch*

Hamilton noted that one could also pinch off a small sphere from a big one. Imagine we choose the size of the little one to be just right, then we expect a degenerate neck-pinching: there is nothing on the other side.

Remark 3.2. *We want to point out that when one makes connected sums of 3-manifolds one creates necks. Thus one could view the neck-pinch as a geometric process in the Ricci flow trying to locate necks so one can undo the connected sum operation and perform the prime decomposition by surgery.*

Remark 3.3. *The above intuitive pictures of neck-pinch and degenerate neck-pinch have been confirmed by Angnents-Knopf [1], and Gu-Zhu [22] respectively. Also, the neck-pinch is of Type I, while the degenerate neck-pinch is of Type II.*

3.2 The rescaling argument and Hamilton's compactness theorem

Starting from early 1990s', Hamilton systematically studied the formation of singularities in the Ricci flow. The parabolic rescaling (or blow-up) method, inspired by the theory of minimal surfaces and harmonic maps, was developed by him since then to understand the structure of singularities. Now we describe this argument. (See the schematic picture shown in Fig. 1 for an illustration.)

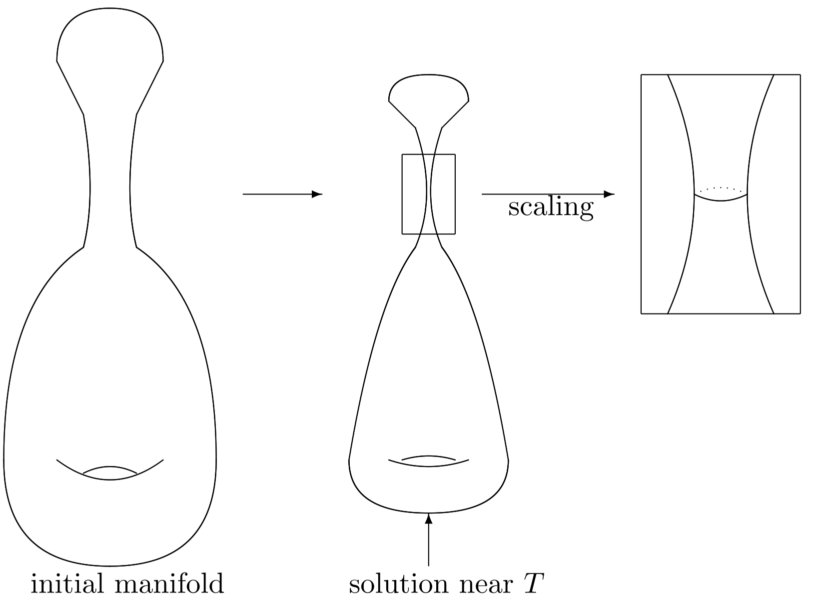

Figure 1: Rescaling

The Rescaling Argument:

- **Step 1**: Take a sequence of (almost) maximum curvature points $\{(x_k, t_k)\}_{k=1}^{\infty}$, where $t_k \to T$ and $x_k \in M$, such that for all $(x,t) \in M \times [0, t_k]$, we have
$$|Rm|(x,t) \leq CQ_k,$$
where $Q_k = |Rm|(x_k, t_k)$.

- **Step 2**: rescale $g(t)$ around (x_k, t_k) by the factor Q_k and shift t_k to new time zero to get the rescaled solution $\tilde{g}_k(t) = Q_k g(t_k + Q_k^{-1} t)$ for $t \in [-Q_k t_k, Q_k(T - t_k))$ with $|Rm|(x_k, 0) = 1$, and
$$|Rm|(x,t) \leq C \quad \text{on } M \times [-Q_k t_k, 0]. \tag{3.1}$$

Remark 3.4. *Hamilton's original rescaling argument is more careful, according to whether a singularity is of Type I or Type II (cf. [28] or [10]).*

Naturally, in the above rescaling argument one would hope to obtain a limit (smooth) solution by letting $k \to \infty$.

Remark 3.5. *The reader may wish to pause for a moment here and try to imagine what kind of limits one would get as $k \to \infty$ if one carries out this rescaling procedure to the neck-pinch and the degenerate neck-pinch singularities described in Section 3.1. (Hint: review the Ricci soliton examples in Sections 2.3 and 2.4. Also, shrinking solitons are often tied to Type I singularity models, while steady solitons to Type II singularity models.)*

In [27], Hamilton proved a Cheeger-type compactness theorem for the Ricci flow (see also Theorem 16.1 in [28]). Roughly speaking, **Hamilton's compactness theorem** says that *for any sequence of marked solutions $(M_k^n, g_k(t), x_k)$ $(k = 1, 2, \ldots)$ to the Ricci flow defined on some time interval $(A, \Omega]$, if there exist positive constants $C > 0$ and $\delta > 0$ such that for all k we have*

$$|Rm|_{g_k(t)} \leq C \tag{3.2}$$

and

$$inj(M_k^n, x_k, g_k(0)) \geq \delta > 0, \tag{3.3}$$

then a subsequence of $(M_k^n, g_k(t), x_k)$ converges in the C^∞_{loc} topology to a complete solution $(M_\infty^n, g_\infty(t), x_\infty)$ to the Ricci flow defined on the same time interval $(A, \Omega]$.

Note that the two conditions in Hamilton's compactness theorem are necessary: if the curvatures $|Rm|_{g_k(t)}$ of $(M_k, g_k(t))$ are not uniformly bounded for all k then the limit manifold would have "corners" and hence would not be smooth; on the other hand, if the injectivity radii of $(M_k, g_k(0))$ at x_k are not uniformly bounded away from zero then the sequence of marked manifolds $(M_k, g_k(0), x_k)$ would collapse and the limit manifold would be of lower dimensional.

Now back to our rescaled marked solutions $(M^n, \tilde{g}_k(t), x_k)$. Clearly, by (3.1), the curvatures of $(M^n, \tilde{g}_k(t))$ are uniformly bounded on the time interval $[-Q_k t_k, 0]$ (which tends to $(-\infty, 0]$ as $k \to \infty$). However, to be able to apply Hamilton's compactness, one still needs the uniform positive lower bound on the injectivity radii of $(M^n, \tilde{g}_k(0))$ at x_k which is not obvious at all.

In [28], by imposing an injectivity radius condition on the maximal solution $(M^n, g(t))$ which ensures (3.3) holds, Hamilton obtained the following singularity structure result at maximal curvature points (cf. Theorem 26.5 in [28]):

Type I Limit: spherical or necklike;

Type II Limit: either a steady Ricci soliton with positive curvature; or $\Sigma \times \mathbb{R}$*, the product of the cigar soliton with the real line.*

This implies that a (maximal) singularity would look like one of the three examples in Section 3.1, plus a possible 4th type whose singularity mode is $\Sigma \times \mathbb{R}$. Hamilton also conjectured that $\Sigma \times \mathbb{R}$ cannot occur as a limit solution of dilations of singularities (such unwanted singularities, if exist, could not be removed by the surgery!). He further conjectured the "Little Loop Lemma" (cf. Lemma 15.1 in [28]), which would rule out $\Sigma \times \mathbb{R}$ as a limit solution.

3.3 Perelman's no Local collapsing theorem

Clearly, two important issues remained in completing the (maximal) singularity formation picture painted by Hamilton:

(a) To verify the *injectivity radius condition* (3.3) for all maximal solutions;

(b) To exclude the possibility of forming a singularity modelled on $\Sigma \times \mathbb{R}$.

Both obstacles were removed by the following Perelman's non-collapsing theorem, which is a major breakthrough in the Ricci flow.

No Local Collapsing Theorem (Perelman [38]). *Given any solution $g_{ij}(t)$ on $M^n \times [0,T)$, with M^n compact and $T < \infty$, there exist constants $\kappa > 0$ and $\rho_0 > 0$ such that for any point $(x_0, t_0) \in M \times [0,T)$, $g_{ij}(t)$ is κ-noncollapsed at (x_0, t_0) on scales less than ρ_0 in the sense that, for any $0 < r < \rho_0$, whenever*

$$|Rm| \le r^{-2} \quad \text{on } B_{t_0}(x_0, r) \times [t_0 - r^2, t_0], \tag{3.4}$$

we have

$$Vol_{t_0}(B_{t_0}(x_0, r)) \ge \kappa r^n.$$

Corollary (Little Loop Lemma). *If $|Rm| \le r^{-2}$ on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then*

$$inj(M, x_0, g(t_0)) \ge \delta r$$

for some positive constant δ.

Remark 3.6. *Perelman proved the above non-collapsing theorem using the monotonicity of the* **reduced volume**

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(q,\tau)} dV_\tau(q)$$

associated to the **reduced distance** *l, a space-time distance function obtained by path integral analogous to what Li-Yau did in [34]. Here, the reduced distance l is defined as follows: for any space path $\gamma(s)$ $(0 \le s \le \tau)$ joining p to q, define its $\mathcal{L}$-length by $\mathcal{L}(\gamma) = \int_0^\tau \sqrt{s}(R(\gamma(s), t_0 - s) + |\dot{\gamma}(s)|^2_{g(t_0-s)})ds$, and denote by $L(q,\tau)$ the L-length of $\mathcal{L}$-shortest curve from p to q. Then the reduced distance is defined as $l(q,\tau) = \frac{1}{2\sqrt{\tau}} L(q,\tau)$.*

In addition, the theorem can be extended to the case when M^n is non-compact or Ricci flow with surgery.

Remark 3.7. *Using his* **$\mathcal{W}$-functional**

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$

and the monotonicity of the associated μ-**entropy**

$$\mu(g,\tau) = \inf\left\{\mathcal{W}(g,f,\tau) \mid f \in C^\infty(M), \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1\right\},$$

Perelman also proved a stronger version of the non-collapsing theorem for Ricci flow on **compact** *manifolds which only requires the curvature bound* $|Rm| \le r^{-2}$ *on the geodesic ball* $B_{t_0}(x_0, r)$ *only. Very recently, Q. Zhang [44] has proven a strong non-collapsing result for Ricci flow with surgery on 3-manifolds using Parelman's* μ*-entropy.*

Now, we can conclude from (3.1) and Perelman's non-collapsing theorem (and the Little Loop lemma) that the rescaled solutions $(M^n, \tilde{g}_k(t), x_k)$ also satisfy the injectivity condition (3.3) in Hamilton's compactness theorem. Thus $(M^n, \tilde{g}_k(t), x_k)$ converges to some limit marked solution $(\tilde{M}^\infty, \tilde{g}_\infty(t), x_\infty)$, which is *complete ancient with bounded curvature* and is κ*-noncollapsed on all scales.*

Moreover, since the product $\Sigma \times \mathbb{R}$ of the cigar with the real line is almost flat at large distances, *it is not* κ*-noncollapsed on large scales for any* $\kappa > 0$ *hence cannot occur in the limit of such rescalings.*

3.4 A magic of the 3-D Ricci flow: the Hamilton-Ivey pinching theorem

Recall that, as we described in Section 1.2, in dimension $n = 3$ we can express the curvature operator $Rm : \Lambda^2(M) \to \Lambda^2(M)$ as

$$Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix},$$

where $\lambda \ge \mu \ge \nu$ are the principal sectional curvatures, and the scalar curvature $R = 2(\lambda + \mu + \nu)$.

The Hamilton-Ivey Pinching Theorem ([28, 31]). *Suppose we have a solution* $g_{ij}(t)$ *to the Ricci flow on a 3-manifold* M^3 *which is complete* *with bounded curvature for each $t \geq 0$. Assume at $t = 0$ the eigenvalues $\lambda \geq \mu \geq \nu$ of Rm at each point are bounded below by $\nu \geq -1$. Then at all points and all times $t \geq 0$ we have the pinching estimate*

$$R \geq (-\nu)[\log(-\nu) + \log(1 + t) - 3]$$

whenever $\nu < 0$.

This means in 3-D Ricci flow whenever the curvature tensor $|Rm|$ blows up, the positive sectional curvature blows up faster than the (absolute value of) negative sectional curvature. It follows that *any limit of dilations around a (almost) maximal singularity of the Ricci flow on an* 3*-manifold necessarily has nonnegative sectional curvature* (which is equivalent to nonnegative curvature operator $Rm \geq 0$ for $n = 3$.) In particular, the limit solution $(\tilde{M}^{\infty}, \tilde{g}_{\infty}(t), x_{\infty})$ described in Section 3.2 is a *complete ancient solution with bounded and nonnegative curvature* and is *κ-noncollapsed on all scales*, called an **ancient κ-solution.**

3.5 A key a priori estimate: the Li-Yau-Hamilton inequality

Having nonnegative curvature in the rescaling limits has tremendous advantage in the classification of singularities of the Ricci flow. For one, we can make use of the powerful Li-Yau-Hamilton inequality (also known as the differential Harnack inequality) for such rescaling limits.

The Li-Yau-Hamilton Inequality (Hamilton [26]). *Let $g_{ij}(t)$ be a complete ancient solution to the Ricci flow with bounded and nonnegative curvature operator $0 \leq Rm \leq C$. Then for any vector field V we have*

$$\frac{\partial R}{\partial t} + 2\nabla R \cdot V + 2Rc(V, V) \geq 0.$$

Corollary. $\frac{\partial R}{\partial t} > 0$ *and hence the scalar curvature* $R(\cdot, t)$ *is pointwise nondecreasing in* t.

As a consequence, *for any ancient* κ*-solution the upper bound on the curvature at the current time gives the upper bound on the curvature at any earlier time.* This is especially useful when combined with Shi's local derivative estimate mentioned in Section 1.2. Moreover, in this case the curvature assumption (3.4) over the parabolic cylinder in Perelman's non-collapsing theorem is reduced to only over the geodesic ball at time t_0.

4 Structure of Singularities in 3-D Ricci Flow

In this section we describe the singularity structure theorem, due to Perelman, for (almost) maximal singularities as well as general singularities in the Ricci flow on 3-manifolds.

4.1 Structure of 3-D ancient κ-solutions

Recall that whenever a 3-D maximal solution $g(t)$ on $M^3 \times [0, T)$ ($T < \infty$) develops singularities, the parabolic dilations around (almost) maximum curvature points (x_k, t_k) converges to a limit non-flat ancient κ-solution $(\tilde{M}^3, \tilde{g}(t), \tilde{x})$, i.e., a complete ancient solution with nonnegative and bounded curvature and is κ-noncollapsed on all scales.

Therefore, to understand the singularity structure of $(M^3, g(t))$ near the maximal time T it is important to understand the geometry of ancient κ-solutions. As claimed by Perelman in [39], there holds a universal non-collapsing property (cf. Proposition 6.4.2 in [10]) for ancient κ-solutions. Namely, there exists a positive constant κ_0 such that for any nonflat 3-dimensional ancient κ-solution (for some $\kappa > 0$), either it is κ_0-noncollapsed on all scales, or it is a metric quotient of the round three-sphere S^3. By using this universal noncollapsing property and the Li-Yau-Hamilton inequality, one can derive the following important elliptic type estimates, which were implicitly given by Perelman [38], for the evolving scalar curvature R of a three-dimensional ancient κ-solution $(\tilde{M}, \tilde{g}_{ij}(t))$ (cf. Theorem 6.4.3 in [10]):

There exist a positive constant η and a positive increasing function $\omega : [0, +\infty) \to (0, +\infty)$ such that

(i) *for every $x, y \in \tilde{M}$ and $t \in (-\infty, 0]$, we have*

$$R(x,t) \leq R(y,t) \cdot \omega(R(y,t) d_t^2(x,y));$$

(ii) *for all $x \in \tilde{M}$ and $t \in (-\infty, 0]$, we have*

$$|\nabla R|(x,t) \leq \eta R^{\frac{3}{2}}(x,t) \quad \text{and} \quad \left|\frac{\partial R}{\partial t}\right|(x,t) \leq \eta R^2(x,t).$$

Based on these two estimates, one obtains a rather complete picture of the geometry of three-dimensional orientable ancient κ-solution in terms of the canonical neighborhood theorem given by Perelman [38] which can be roughly described as follows (cf. Theorem 6.4.6 in [10] for a more detailed statement of the theorem).

Canonical Neighborhood Theorem (Perelman). *$\forall\ \varepsilon > 0$, every point (x_0, t_0) on an orientable nonflat ancient κ-solution $(\tilde{M}^3, \tilde{g}(t))$ has an open neighborhood B, which falls into one of the following three categories:*

(a) *B is an ε-neck of radius $r = R^{-1/2}(x_0, t_0)$; (i.e., after scaling by the factor $R(x_0, t_0)$, B is ϵ-close, in $C^{[\epsilon^{-1}]}$-topology, to $\mathbb{S}^2 \times [-\epsilon^{-1}, \epsilon^{-1}]$ of scalar curvature 1.)*

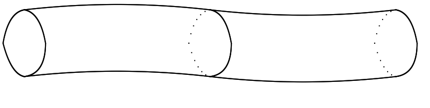

Figure 2: ε-neck

(b) *B is an ε-cap; (i.e., a metric on $\mathbb{B}^3$ or $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ and the region outside some suitable compact subset is an ε-neck).*

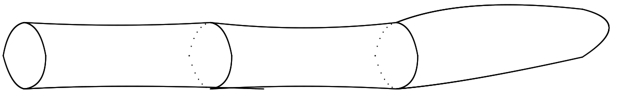

Figure 3: ε-cap

(c) *B is compact (without boundary) with positive sectional curvature (hence diffeomorphic to a space form by Hamilton [23]).*

Consequently, for arbitrarily given $\varepsilon > 0$, the maximal solution $(M^3, g(t))$ around x_k and at time t_k near T has a canonical neighborhood which is either an ε-neck, or an ε-cap, or a compact positively curved manifold (without boundary). *Thus, Perelman's canonical neighborhood theorem confirms the picture of the structure of (maximal) singularities developed by Hamilton and asserts that the (maximal) singularities of the Ricci flow on 3-manifolds look just like one of the three examples we described in Section 3.1.*

4.2 Structure of 3-D singularities

Furthermore, Perelman was able to extend his canonical neighborhood theorem for almost maximal curvature points to points of high curvature which are not necessary almost maximal. Note that, however, the same argument as before would not work for (non-maximal) singularities coming from a sequence of points (y_k, s_k) with $s_k \to T$ and $|Rm|(y_k, s_k) \to +\infty$ *but $|Rm|(y_k, s_k)$ is not comparable with the maximal curvature at time s_k*, since the uniform curvature bound assumption (3.2) fails for the rescaled solutions around (y_k, s_k) so we cannot take a limit directly. Nevertheless, one can generalize Hamilton's compactness theorem to allow local limits for smooth solutions over geodesic balls (cf. Theorem 4.1.5 in [10]), and taking local limits and applying Hamilton's strong maximum principle, as done by Perelman [39], to derive the following general singularity structure theorem (cf. Theorem 51.3 in [32] or Theorem 7.1.1 in [10] for a more precise statement).

Singularity Structure Theorem (Perelman). *Given $\varepsilon > 0$ and $T_0 > 1$, one can find $r_0 > 0$ such that for any maximal solution $g(t)$, $0 \le t < T$ with $1 < T \le T_0$, to the Ricci flow on a compact orientable M^3 with normalized initial metric, each point (x_0, t_0) with $t_0 > 1$ and $R(x_0, t_0) \ge r_0^{-2}$ admits a canonical neighborhood B, which is either*

(a) *an ε-neck, or*

(b) *an ε-cap, or*

(c) *a closed 3-manifold with positive sectional curvature.*

Here, a metric is *normalized* if at every point the principal sectional curvatures satisfy $\frac{1}{10} \ge \lambda \ge \mu \ge \nu \ge -\frac{1}{10}$, and every geodesic ball of radius one has volume at least one. (This condition in particular implies $T > 1$). We remark that one can always make any given metric normalized by a suitable scaling.

4.3 Solutions near the singular time T

Based on the singularity structure theorem of Perelman, we can see a clear picture of a maximal solution near its singular time T as follows.

For any given $\varepsilon > 0$ and a given maximal solution $(M^3, g(t))$ on $[0, T)$ ($T < \infty$) with normalized initial metric, we can find $r_0 > 0$ depending only on ε and T such that each point (x, t) of high curvature, with $R(x, t) \ge r_0^{-2}$, admits a canonical neighborhood B given in the singularity structure theorem.

Let Ω denote the set of all points in M^3 where the curvature stays bounded as $t \to T$.

If Ω is empty, then the solution $g(t)$ becomes **extinct** at time T. In this case, either M^3 is compact and positively curved, or it is entirely covered by ε-necks and ε-caps shortly before the maximal time T. It follows that M^3 is diffeomorphic to either $\mathbb{S}^3$, or a metric quotient of $\mathbb{S}^3$, or $\mathbb{S}^2 \times \mathbb{S}^1$, or $\mathbb{RP}^3 \# \mathbb{RP}^3$.

We now consider the case when Ω is nonempty. By using Shi's local derivative estimates, we see that, as $t \to T$, $g(t)$ has a smooth limit $\bar{g}$ on Ω. Let $\bar{R}$ denote the scalar curvature of $\bar{g}$. For any $0 < \rho < r_0$, let us consider the set

$$\Omega_\rho = \{x \in \Omega \mid \bar{R}(x) \leq \rho^{-2}\}.$$

First, we need some terminologies:

A metric on $\mathbb{S}^2 \times \mathbb{I}$, such that each point is contained in some ε-neck, is called an **ε-tube**, or an **ε-horn**, or a **double ε-horn**, if the scalar curvature stays bounded on both ends, or stays bounded on one end and tends to infinity on the other end, or tends to infinity on both ends, respectively; A metric on $\mathbb{B}^3$ or $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ is called an **capped ε-horn** if each point outside some compact subset is contained in an ε-neck and the scalar curvature tends to infinity on the end.

Now take any ε-neck in $(\Omega, \bar{g})$ and consider a point x on one of its boundary components. If $x \in \Omega \setminus \Omega_\rho$, then there is either an ε-cap or an ε-neck adjacent to the initial ε-neck. In the latter case we can take a point on the boundary of the second ε-neck and continue. This procedure will either terminate when we get into Ω_ρ or an ε-cap, or go on indefinitely, producing an ε-horn. The same procedure can be repeated for the other boundary component of the initial ε-neck. Therefore, taking into account that Ω has no compact components, we conclude that each ε-neck of $(\Omega, \bar{g})$ is contained in one of the following types of subsets of Ω:

(a) an ε-tube with boundary components in Ω_ρ, or

(b) an ε-cap with boundary in Ω_ρ, or

(c) an ε-horn with boundary in Ω_ρ, or

(d) a capped ε-horn, or

(e) a double ε-horn.

Similarly, each ε-cap of $(\Omega, \bar{g}_{ij})$ is contained in a subset of type (b), or type (d).

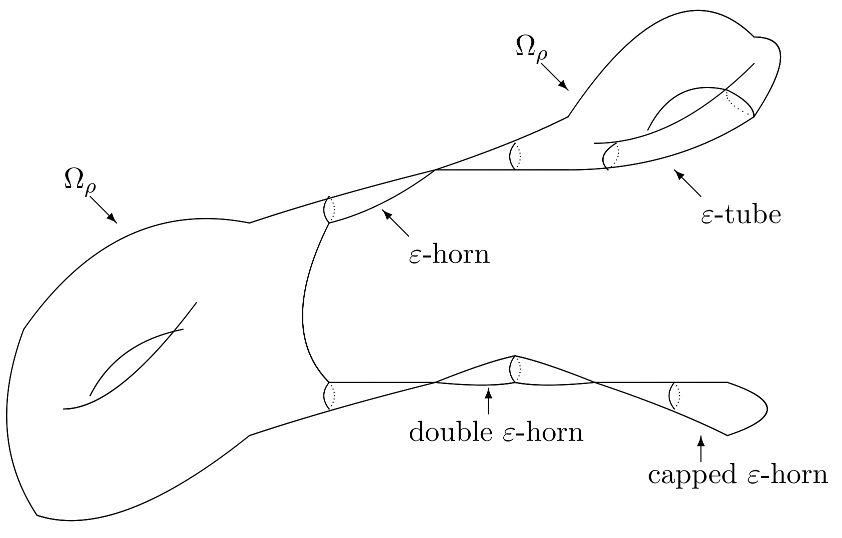

Figure 4:

It is clear that there is a definite lower bound (depending on ρ) on the volumes of subsets of type (a), (b), and (c). So there can be only finitely many of them. Hence we conclude that there are only finitely many components of Ω containing points of Ω_ρ, and every such component has a finite number of ends, each being an ε-horn. On the other hand, every component of Ω containing no points of Ω_ρ is either a capped ε-horn, or a double ε-horn. If we look at the solution $g(t)$ at a time slightly before T, the above argument shows that each ε-neck or ε-cap of $(M, g(t))$ is contained in a subset of type (a) or (b), while the ε-horns, capped ε-horns and double ε-horns (at the maximal time T) are connected together to form ε-tubes and ε-caps at any time t shortly before T.

Let us denote by Ω_j, $1 \leq j \leq m$, the connected components of Ω which contain points of Ω_ρ. Then *the initial three-manifold M^3 is diffeomorphic to the connected sum of $\bar{\Omega}_j$, $1 \leq j \leq m$, with finitely many copies of* $\mathbb{S}^2 \times \mathbb{S}^1$ (*which correspond to gluing a tube to two boundary components of the same Ω_j*) *and finitely many copies of* $\mathbb{RP}^3$. Here $\bar{\Omega}_j$, $j = 1, 2, \ldots, m$, is the compact manifold (without boundary) obtained from Ω_j by taking some ε-neck in each ε-horn of Ω_j, cutting it along the middle two-sphere, removing the horn-shaped end, and gluing back a cap (or more precisely, a differentiable three-ball).

5 Further remarks

I. To capture the topology of M^3, one only needs to understand the topology of each compact orientable three-manifold $\bar{\Omega}_j$, $1 \leq j \leq m$, described above. To do so, one evolves $\bar{\Omega}_j$, $j = 1, 2, \ldots, m$, by the Ricci flow simultaneously. If the new solution develops singularities, one performs the surgeries again as described above and continue with the Ricci flow. By repeating this procedure, one obtains a "weak" solution, called a solution to **the Ricci flow with surgery** or a **surgically modified solution** to the Ricci flow.

In order to extract the topological information of the initial three-manifold M^3 from the Ricci flow with surgery, one needs to make sure that there are at most finitely many surgeries on any finite time interval, and the surgically modified solution admits a well-understood long-time behavior. For this purpose, the surgeries, which are topologically trivial, need to be performed rather carefully geometrically (in a 3-step geometric surgery procedure designed by Hamilton in [29]) and in a controlled way to make sure the geometry of the surgically modified solution is well controlled after the surgeries so that there is always "enough recovery time" before the next surgery. The key to prevent the surgery times from accumulating is to construct the surgically modified solution to the Ricci flow in such a way that one can recognize canonical neighborhoods at high curvature points in some uniform manner on any finite time interval. We refer the interested readers to the recent survey articles [8, 46] for an outline of this part, and [10] for the details.

II. Now that the singularity formation in 3-D Ricci flow is well understood, naturally one would ask what happens in higher dimensions, particularly in dimension $n = 4$ and in the Kähler case.

For the Ricci flow on 4-manifolds with **positive isotropic curvature (PIC)**, the singularity formation is now also well understood, thanks to the works of Hamilton [29] and Chen-Zhu [13]. In particular, Chen-Zhu [13] extended Perelman's singularity structure theorem to 4-D Ricci flow with positive isotropic curvature (see also the very recent extension by Chen-Tang-Zhu [11] to the orbifold case). The Kähler-Ricci flow on Fano manifolds has attracted more attention in recent years and various progress has been made. The most significant advance is Perelman's uniform estimates for the scalar curvature R, the diameter, and the C^1-norm of the Ricci potential function of the normalized Kähler-Ricci flow on Fano manifolds. These will be discussed elsewhere. One would hope to gain a good understanding of the singularity formation both in some more general cases in 4-D and in the Kähler case, and much remains to be done.

III. In proving the canonical neighborhood theorem of 3-D ancient κ-solution, Perelman [39] derived the following classification of (nonflat) 3-dimensional κ-shrinking gradient solitons (i.e., shrinking gradient solitons with bounded and nonnegative curvature and κ-noncollapsed on all scales for some $\kappa > 0$).

Classification of 3-D κ-Shrinking Gradient Solitons (Perelman).

A 3-dimensional complete nonflat κ-shrinking gradient soliton is either

(i) a quotient of the round sphere $\mathbb{S}^3$; or

(ii) a quotient of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.

The key step in the proof is to show that *there is no 3-dimensional complete noncompact nonflat* κ*-shrinking gradient solitons with positive (sectional) curvature.* Part of Perelman's proof of this last assertion used the Gauss-Bonnet theorem on the level surfaces of the potential function f which only works in dimension two.

Very recently, various works have been done to improve Perelman's classification result in dimension three and to extend it to higher dimensions under certain assumptions. We list some of those progress below, and refer the reader to a recent article [7] of the author for more information on geometry of gradient shrinking solitons.

3-D :

- *A complete noncompact non-flat shrinking gradient soliton with nonnegative Ricci curvature* $Rc \geq 0$ *and with curvature growing at most by* $|Rm|(x) \leq Ce^{ar(x)}$ *is necessarily a quotient of* $\mathbb{S}^2 \times \mathbb{R}$. (Ni-Wallach [36])

- *A complete noncompact non-flat shrinking gradient soliton is necessarily a quotient of* $\mathbb{S}^2 \times \mathbb{R}$. (Cao-Chen-Zhu [8])

4-D :

- *A complete gradient shrinking soliton with* $Rm \geq 0$ *and positive isotropic curvature (PIC), and satisfying certain additional assumptions, is either a quotient of* $\mathbb{S}^4$ *or a quotient of* $\mathbb{S}^3 \times \mathbb{R}$. (Ni-Wallach [37])

- *A non-flat complete noncompact shrinking Ricci soliton with bounded and nonnegative curvature operator* $0 < Rm \geq C$ *is a quotient of either* $\mathbb{S}^3 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times \mathbb{R}^2$. (Naber [35])

In addition, complete noncompact gradient shrinking solitons in $n \geq 4$ which are locally conformally flat are classified recently, see the works of Ni-Wallach [36], Petersen-Wylie [40], and Z.-H. Zhang [45].

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