

# C-TOTALLY REAL SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN $\lambda$ -SASAKIAN SPACE FORMS

A. Brasil

G. A. Lobos 

M. Mariano

*Dedicated to Professor Manfredo do Carmo on the occasion of his 80<sup>th</sup> birthday*

## Abstract

In this paper, we prove a generalized integral inequality for an  $n$ -dimensional oriented closed  $C$ -totally real submanifold  $M$  with parallel mean curvature vector  $h$  in a  $(2m+1)$ -dimensional closed  $\lambda$ -Sasakian space form  $\tilde{M}(c)$  of constant  $\varphi$ -sectional curvature  $c$  with  $0 < c \leq \lambda$ ,  $n \geq 2$  and if a tensor  $\phi$  related to  $h$  and the second fundamental form satisfies a certain inequality. As a consequence we obtain that  $M$  is totally umbilic or minimal with  $S = (n(c + 3\lambda) + (c - \lambda))/6$ , which generalize the Theorem 3 of [10]. Finally, we prove that if  $M$  is  $f$ -pseudo-parallel in a  $(2n+1)$ -dimensional  $\lambda$ -Sasakian space form with  $f \geq (n(c + 3\lambda) + (c - \lambda))/4n$ , then  $M$  is totally geodesic, which generalize the Theorem 1 of [13], when  $\lambda = 1$ .

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# 1 Introduction

Let  $\tilde{M}$  be a  $(2m + 1)$ -dimensional manifold and  $\Gamma(\tilde{M})$  the Lie algebra of vector fields on  $\tilde{M}$ . An almost contact structure on  $\tilde{M}$  is defined by a  $(1,1)$ -tensor  $\varphi$ , a vector field  $\xi$  and a 1-form  $\eta$  on  $\tilde{M}$  such that for any  $p \in \tilde{M}$ , we have

$$\varphi_p^2 = -I + \eta_p \otimes \xi_p, \quad \eta_p(\xi_p) = 1,$$

where  $I$  denote the identity transformation of the tangent space  $T_p\tilde{M}$  at  $p$ . Then  $\varphi(\xi) = 0$  and  $\eta \circ \varphi = 0$ . Manifolds equipped whit an almost contact structure are called almost contact manifolds. A Riemannian manifold  $\tilde{M}$  with metric tensor  $\langle \cdot, \cdot \rangle$  and an almost contact structure  $(\varphi, \xi, \eta)$  such that

$$\langle \varphi X, \varphi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),$$

or equivalently

$$\langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle \quad \text{and} \quad \langle X, \xi \rangle = \eta(X),$$

for all  $X, Y \in \Gamma(\tilde{M})$ , is an almost contact metric manifold. The existence of an almost contact metric structure on  $\tilde{M}$  is equivalent with the existence of a reduction of the structural group to  $\mathcal{U}(m) \times 1$ , i. e. all the matrices of  $\mathcal{O}(2m + 1)$  of the form

$$\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $A$  and  $B$  are real  $(n \times n)$ -matrices. The fundamental 2-form  $\Psi$  of an almost contact metric manifold  $(\tilde{M}, \varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$  is defined by

$$\Psi(X, Y) = \langle X, \varphi Y \rangle,$$

for all  $X, Y \in \Gamma(\tilde{M})$ , and this form satisfies  $\eta \wedge \Psi^m \neq 0$ . When  $\Psi = \frac{1}{\lambda} d\eta$ ,  $\lambda \neq 0$  the associated structure is a contact structure and  $\tilde{M}$  is an almost

$\lambda$ -Sasakian manifold. An almost  $\lambda$ -Sasakian manifold  $(\tilde{M}, \varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$  is called a  $\lambda$ -Sasakian manifold if

$$[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$$

for all  $X, Y \in \Gamma(\tilde{M})$ . A necessary and sufficient condition for an almost contact metric manifold  $(\tilde{M}, \varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$  to be a  $\lambda$ -Sasakian manifold is

$$\left(\tilde{\nabla}_X \varphi\right) Y = \lambda\{\langle X, Y \rangle \xi - \eta(Y)X\}, \tag{1.1}$$

for all  $X, Y \in \Gamma(\tilde{M})$ , where  $\tilde{\nabla}$  is the Levi-Civita connection of the Riemannian metric  $\langle \cdot, \cdot \rangle$ . Moreover, a  $\lambda$ -Sasakian manifold satisfies:

$$\tilde{\nabla}_X \xi = -\lambda\varphi X, \tag{1.2}$$

see [6]. If  $\lambda = 1$  a  $\lambda$ -Sasakian manifold is a Sasakian manifold [4].

An  $n$ -dimensional Riemannian manifold  $M$  isometrically immersed in  $\tilde{M}$  is said to be anti-invariant in  $\tilde{M}$  if  $\varphi T_p M \subset T_p M^\perp$  for each  $p$  of  $M$ , where  $T_p M$  and  $T_p M^\perp$  denote respectively the tangent and the normal space to  $M$  at  $p$ . Thus, for any vector  $X$  tangent to  $M$ ,  $\varphi X$  is normal to  $M$ . In this case,  $\varphi$  is necessarily of rank  $2m$  and hence  $n \leq m + 1$ . An  $n$ -dimensional Riemannian manifold  $M$  isometrically immersed in  $\tilde{M}$  is said to be  $C$ -totally real if  $\xi$  is a normal vector field to  $M$ . Recall that a direct consequence of this definition is that  $M$  is a anti-invariant submanifold in  $\tilde{M}$  and  $n \leq m$ . A plane section  $\sigma$  in  $T_p \tilde{M}$  of a  $\lambda$ -Sasakian manifold is called a  $\varphi$ -section if it is spanned by  $X$  and  $\varphi X$ , where  $X$  is a unit tangent vector field orthogonal to  $\xi$ . The sectional curvature  $\tilde{k}(\sigma)$  with respect a  $\varphi$ -section  $\sigma$  is called a  $\varphi$ -sectional curvature. In this paper a  $\lambda$ -Sasakian manifold  $\tilde{M}$  complete simply connected with constant  $\varphi$ -sectional curvature  $c$  is called a  $\lambda$ -Sasakian space form and is denoted by  $\tilde{M}(c)$ . The curvature tensor  $\tilde{R}$  of  $\tilde{M}(c)$  is given by [9]:

$$\begin{aligned} \tilde{R}(X, Y)Z = & \frac{c + 3\lambda}{4}(X \wedge Y)Z + \frac{c - \lambda}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + \langle X, Z \rangle \eta(Y)\xi - \langle Y, Z \rangle \eta(X)\xi \\ & + \langle \varphi Y, Z \rangle \varphi X - \langle \varphi X, Z \rangle \varphi Y - 2\langle \varphi X, Y \rangle \varphi Z\}, \end{aligned} \tag{1.3}$$

where  $X \wedge Y$  is the operator defined by  $(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$ .

**Example 1.1.** [4] Let  $\mathbb{R}^{2m+1}$  be a Euclidean space with cartesian coordinates  $(x^i, y^i, z)$ . Then a 1-Sasakian structure on  $\mathbb{R}^{2m+1}$  is defined by  $(\varphi_0, \xi, \eta, g)$  such that

$$\xi = 2 \frac{\partial}{\partial z}, \quad \eta = \frac{1}{2} \left( dz - \sum_{i=1}^m y^i dx^i \right), \quad g = \frac{1}{4} \left( \eta \otimes \eta + \sum_{i=1}^m ((dx^i)^2 + (dy^i)^2) \right)$$

and the tensor field  $\varphi_0$  is given by matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

With such a structure,  $\mathbb{R}^{2m+1}$  is of constant  $\varphi$ -sectional curvature  $-3$  and denoted by  $\mathbb{R}^{2m+1}(-3)$ .

**Example 1.2.** [1] For  $\theta \in (0, \pi/2)$ , the immersion

$$F(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta, v \sin \theta, s \cos \theta s \sin \theta, t),$$

defines a 5-dimensional submanifolds  $M$  in  $\mathbb{R}^9(-3)$ . We consider on  $M$  the induced almost contact structure  $(\varphi, \xi, \eta, g)$ , where  $\varphi = (\sec \theta)T$ ,  $T$  being the tangential component of  $\varphi_0$ . It can be checked that  $(\nabla_X \varphi)Y = \cos \theta (g(X, Y)\xi - \eta(Y)X)$ , for any vector fields  $X, Y$  tangent to  $M$ , which means that  $M$  is a  $\lambda$ -Sasakian manifold with  $\lambda = \cos \theta \in (0, 1)$ .

For other examples, we refer to [2].

The purpose of present paper is to study  $n$ -dimensional  $C$ -totally real submanifolds  $M$ , with parallel mean curvature in  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ .

It is we need consider  $\Phi : T_p M \times T_p M \rightarrow T_p M^\perp$  a bilinear map defined as follows: choose an orthonormal frame  $\{e_{n+1}, \dots, e_{2m+1}\}$  of  $T_p M^\perp$  and for each  $\alpha = n+1, \dots, 2m+1$ , define maps  $\Phi_\alpha : T_p M \rightarrow T_p M$  by

$$\Phi_\alpha X = \langle h, e_\alpha \rangle X - A_{e_\alpha} X, \quad (1.4)$$

where  $h$  is the mean curvature vector and  $A_{e_\alpha}$ 's are the shape operators. Then  $\Phi$  is given by

$$\Phi(X, Y) = \sum_{\alpha} \langle \Phi_{\alpha} X, Y \rangle e_{\alpha}. \tag{1.5}$$

Therefore both  $\Phi$  and  $|\Phi|$  not depend on the choice of  $\{e_{\alpha}\}$ , moreover, if  $S$  be the squared norm of the second fundamental form of  $M$ , then

$$|\Phi|^2 = \sum_{\alpha} tr (\Phi_{\alpha})^2 = S - nH^2, \tag{1.6}$$

where  $H = |h|$ . We recall that  $|\Phi|^2 \equiv 0$  if and only if  $M$  is totally umbilic;  $H \equiv 0$  if and only if  $M$  is minimal; and  $S \equiv 0$  if and only if  $M$  is totally geodesic. We remark that the immersion  $F$  in the example (1.2) defines a 5-dimensional minimal submanifold  $M$  in an 1-Sasakian space form  $\mathbb{R}^9(-3)$ .

Now, for any  $H \in \mathbb{R}$ , we define the polynomial  $P_{H,c,\lambda}$  by

$$P_{H,c,\lambda}(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - \left( \frac{n(c+3\lambda) + c - \lambda}{4} + nH^2 \right). \tag{1.7}$$

Denoting by  $\vartheta_H$  the square of the positive root of  $P_{H,c,\lambda}(x) = 0$ , our results can be stated as:

**Theorem 1.1.** *Let  $M$  be an  $n$ -dimensional oriented complete closed  $C$ -totally real submanifold with parallel mean curvature vector in a closed  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ ,  $n \geq 2$  and  $0 < c \leq \lambda$ . If  $|\Phi|^2 \leq \vartheta_H$  on  $M$ , then*

$$\int_M |\Phi|^2 P_{H,c,\lambda}(|\Phi|) dM \geq 0. \tag{1.8}$$

As a consequence Theorem 1.1 , we get:

**Theorem 1.2.** *Let  $M$  be an  $n$ -dimensional oriented complete closed  $C$ -totally real submanifold with parallel mean curvature vector in a closed  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ ,  $n \geq 2$  and  $0 < c \leq \lambda$ . If  $|\Phi|^2 \leq \vartheta_H$  on  $M$ ,*

then either  $M$  is totally umbilical or  $m = n$  and  $M$  is minimal, non-totally geodesic. In this case,

$$S = \frac{1}{6}\{n(c + 3\lambda) + c - \lambda\}.$$

In particular, if  $c = \lambda = 1$ , then  $M$  is either a totally geodesic submanifold or a Veronese surface.

A submanifold  $M$  is  $f$ -pseudo-parallel if its second fundamental form  $\sigma$  satisfies the following condition

$$\bar{R}(X, Y) \cdot \sigma = f X \wedge Y \cdot \sigma,$$

for some real valued smooth function  $f$  on  $M$  and for any  $X$  and  $Y$  vectors tangent to  $M$ , where  $\bar{R}(X, Y)$  is the curvature operator of the Van der Waerden-Bortolotti connection  $\bar{\nabla}$  of  $M$ , which with the operator  $X \wedge Y$  act on  $\sigma$  as a derivation [3]. We prove a result that generalize the Theorem 1 of [13].

**Theorem 1.3.** *Let  $M$  be an  $n$ -dimensional  $C$ -totally real submanifold with parallel mean curvature vector in a  $(2n + 1)$ -dimensional  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ . If  $M$  is  $f$ -pseudo-parallel and  $f \geq (n(c+3\lambda)+c-\lambda)/4n$ , then  $M$  is totally geodesic.*

Finally, we get the following results for closed  $f$ -pseudo-parallel submanifolds with parallel mean curvature vector in a  $\lambda$ -Sasakian space form.

**Theorem 1.4.** *Let  $M$  be an  $n$ -dimensional closed  $C$ -totally real submanifold with parallel mean curvature vector in a  $(2m + 1)$ -dimensional  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ . If  $M$  is  $f$ -pseudo-parallel and  $f \geq 0$ , then  $M$  is parallel, i.e.  $\bar{\nabla}\sigma = 0$ .*

**Corollary 1.1.** *Let  $M$  be an  $n$ -dimensional closed  $C$ -totally real submanifold with parallel mean curvature vector in a  $(2n + 1)$ -dimensional  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ . If  $M$  is  $f$ -pseudo-parallel and  $f > 0$ , then  $M$  is totally geodesic.*

## 2 Preliminaries

Let  $\tilde{M}(c)$  be a  $(2m + 1)$ -dimensional  $\lambda$ -Sasakian space form with structure  $(\varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$  and  $M$  an  $n$ -dimensional  $C$ -totally real submanifold ( $n \leq m$ ). As usual,  $\tilde{\nabla}$  (resp.  $\nabla$ ) be the Riemannian connection with respect to  $\langle \cdot, \cdot \rangle$  (resp.  $\langle \cdot, \cdot \rangle|_M$ ) and  $\nabla^\perp$  the connection in the normal bundle on  $M$ . These connections are related by the Gauss and the Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for any  $X, Y$  vectors tangent to  $M$  and any  $N$  vector normal to  $M$ , where  $A_N$  is the shape operator (which is auto-adjunt) in the direction  $N$  and  $\sigma$  is the second fundamental form on  $M$ . The shape operator and second fundamental form are related by

$$\langle A_N X, Y \rangle = \langle \sigma(X, Y), N \rangle. \tag{2.2}$$

Let  $R, \tilde{R}$  and  $R^\perp$  the curvature tensors of  $\nabla, \tilde{\nabla}$  and  $\nabla^\perp$ , respectively. Then, the Gauss and the Ricci equations are given by

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle \tilde{R}(X, Y)Z, W \rangle + \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &\quad - \langle \sigma(X, Z), \sigma(Y, W) \rangle, \end{aligned} \tag{2.3}$$

$$\langle R^\perp(X, Y)N_1, N_2 \rangle = \langle \tilde{R}(X, Y)N_1, N_2 \rangle + \langle [A_{N_1}, A_{N_2}], Y \rangle. \tag{2.4}$$

The Codazzi-Mainardi equation is

$$(\overline{\nabla}\sigma)(X, Y, Z) = (\overline{\nabla}\sigma)(X, Z, Y), \tag{2.5}$$

where  $\overline{\nabla}\sigma$  is the first covariant derivative of  $\sigma$  is defined by

$$\begin{aligned} (\overline{\nabla}\sigma)(X, Y, Z) &= (\overline{\nabla}_Z\sigma)(X, Y) \\ &= \nabla_Z^\perp[\sigma(X, Y)] - \sigma(\nabla_Z Y, X) - \sigma(Y, \nabla_Z X), \end{aligned} \tag{2.6}$$

and the second covariant derivative is defined by

$$\begin{aligned}
 (\bar{\nabla}^2\sigma)(X, Y, Z, W) &= (\bar{\nabla}_W\bar{\nabla}_Z\sigma)(X, Y) \\
 &= \nabla_W^\perp[(\bar{\nabla}_Z\sigma)(X, Y)] - (\bar{\nabla}_Z\sigma)(\nabla_W X, Y) \quad (2.7) \\
 &\quad - (\bar{\nabla}_Z\sigma)(X, \nabla_W Y) - (\bar{\nabla}_{\bar{\nabla}_W Z}\sigma)(X, Y).
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 R^\perp(X, Y)[\sigma(Z, W)] &= (\bar{\nabla}_X\bar{\nabla}_Y\sigma)(Z, W) - (\bar{\nabla}_Y\bar{\nabla}_X\sigma)(Z, W) \\
 &\quad + \sigma(R(X, Y)Z, W) + \sigma(Z, R(X, Y)W). \quad (2.8)
 \end{aligned}$$

In this work we use the following convention of index:

$$\begin{aligned}
 1 \leq A, B, C, \dots \leq 2m + 1, \\
 1 \leq i, j, k, \dots \leq n, \quad i^* = m + i, \\
 n + 1 \leq \alpha, \beta, \gamma, \dots \leq 2m + 1.
 \end{aligned}$$

As  $M$  is a  $C$ -totally real submanifold, we can choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m, e_{1^*} = \varphi e_1, \dots, e_{(n+1)^*} = \varphi e_{n+1}, \dots, e_{m^*} = \varphi e_m, e_{2m+1} = \xi\}$  in  $\tilde{M}(c)$  such that  $\{e_i\}$  at each point of  $M$  span the tangent space of  $M$ .

Let  $\{\omega_A\}$  be the dual of  $\{e_A\}$  and let  $\{\omega_{AB}\}$  be the connection 1-forms of  $\tilde{M}(c)$ . Then the structure equations of Cartan are given by

$$d\omega_A = - \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad (2.9)$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} \tilde{R}_{ABCD} \omega_C \wedge \omega_D. \quad (2.10)$$

The  $(\omega_{AB})$  is a real representation of a skew-Hermitian matrix. Hence

$$\omega_{i^*j} = \omega_{j^*i}. \quad (2.11)$$



Moreover,

$$\omega_{ij} = \omega_{i^*j^*} \quad \text{and} \quad \omega_{i^*} = -\omega_{i(2m+1)}. \quad (2.12)$$

Thus, we have along  $M$  that

$$\omega_\alpha = 0,$$

which implies  $0 = d\omega_\alpha = -\sum_i \omega_{\alpha i} \wedge \omega_i$  along  $M$ . From Cartan's Lemma, we write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \quad (2.13)$$

where  $h_{ij}^\alpha$  denoted the components of second fundamental form  $\sigma$ , that is

$$h_{ij}^\alpha = \langle A_{e_\alpha} e_i, e_j \rangle = \langle \sigma(e_i, e_j), e_\alpha \rangle. \quad (2.14)$$

Therefore, from (2.11) and (2.2) we have

$$h_{jk}^{i^*} = h_{ik}^{j^*} = h_{ij}^{k^*}, \quad h_{ij}^{2m+1} = 0. \quad (2.15)$$

From (1.3), we get

$$\tilde{R}_{ijkl} = \frac{c + 3\lambda}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad (2.16)$$

and

$$\tilde{R}_{\alpha\beta kl} = \begin{cases} \frac{c - \lambda}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), & \text{if } \alpha = i^*, \beta = j^*; \\ 0, & \text{otherwise,} \end{cases} \quad (2.17)$$

where  $\langle e_i, e_j \rangle = \delta_{ij}$ . Using (2.16) in (2.3), we obtain

$$R_{ijkl} = \frac{c + 3\lambda}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \quad (2.18)$$

and substituting (2.17) in (2.4), we get

$$R_{\alpha\beta kl}^\perp = \begin{cases} \frac{c - \lambda}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_r (h_{rk}^\alpha h_{rl}^\beta - h_{rl}^\alpha h_{rk}^\beta), & \text{if } \alpha = i^*, \beta = j^*; \\ \sum_r (h_{rk}^\alpha h_{rl}^\beta - h_{rl}^\alpha h_{rk}^\beta), & \text{otherwise.} \end{cases} \quad (2.19)$$

Let  $S$  be the squared norm of second fundamental form,  $h$  denote the mean curvature vector field and  $H$  the mean curvature of  $M$ , that is

$$S = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2, \quad h = \frac{1}{n} \sum_{\alpha} \left( \sum_i h_{ii}^\alpha \right) e_\alpha, \quad H = |h|. \quad (2.20)$$

The Ricci curvature tensor  $\{R_{kl}\}$  and the scalar curvature  $K$  are expressed, respectively, as follows:

$$R_{kl} = \frac{c + 3\lambda}{4} (n - 1) \delta_{kl} + \sum_{\alpha} \left( \sum_i h_{ii}^\alpha \right) h_{kl}^\alpha - \sum_{\alpha, i} h_{ki}^\alpha h_{il}^\alpha, \quad (2.21)$$

$$K = \frac{c + 3\lambda}{4} n(n - 1) + (n^2 H^2 - S). \quad (2.22)$$

The components of the covariant derivative of  $\sigma$  are given by

$$h_{ijk}^\alpha = \langle (\bar{\nabla}_{e_k} \sigma)(e_i, e_j), e_\alpha \rangle = \bar{\nabla}_{e_k} h_{ij}^\alpha, \quad (2.23)$$

hence, the square of the length of third fundamental form of  $M$  is given

$$|\bar{\nabla} \sigma|^2 = \sum_{\alpha, i, j, k} (h_{ijk}^\alpha)^2. \quad (2.24)$$

The components of the second covariant derivative of  $\sigma$  are given by

$$h_{ijkl}^\alpha = \langle (\bar{\nabla}_{e_l} \bar{\nabla}_{e_k} \sigma)(e_i, e_j), e_\alpha \rangle = \bar{\nabla}_{e_l} h_{ijk}^\alpha = \bar{\nabla}_{e_l} \bar{\nabla}_{e_k} h_{ij}^\alpha. \quad (2.25)$$

Hence, we get

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_r h_{jr}^\alpha \omega_{ri} - \sum_r h_{ir}^\alpha \omega_{rj} + \sum_\beta h_{ij}^\beta \omega_{\alpha\beta}, \quad (2.26)$$

$$\begin{aligned} \sum_l h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha - \sum_r h_{rjk}^\alpha \omega_{ri} - \sum_r h_{irk}^\alpha \omega_{rj} \\ &\quad - \sum_r h_{ijr}^\alpha \omega_{rk} + \sum_\beta h_{ijk}^\alpha \omega_{\alpha\beta}. \end{aligned} \quad (2.27)$$

From (2.5), we have

$$h_{ijk}^\alpha - h_{ikj}^\alpha = 0, \quad (2.28)$$

and by (2.8), we obtain the following Ricci formula

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_r h_{rj}^\alpha R_{rikl} + \sum_r h_{ri}^\alpha R_{rjkl} - \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}^\perp. \quad (2.29)$$

From (2.12), (2.11) and (2.26), we get

$$h_{ijk}^{2m+1} = -h_{ij}^{k*}. \quad (2.30)$$

The Laplacian  $\Delta h_{ij}^\alpha$  of  $h_{ij}^\alpha$  is defined by  $\Delta h_{ij}^\alpha = \sum_k h_{ijk}^\alpha = \sum_k h_{kij}^\alpha$ . Using (2.28) and (2.29), we obtain

$$\begin{aligned} \Delta h_{ij}^\alpha &= \sum_{k,r} h_{kr}^\alpha R_{rijk} + \sum_{kr} h_{ri}^\alpha R_{rkjk} - \sum_{k,\beta} h_{ki}^\beta R_{\alpha\beta kj}^\perp \\ &= \sum_{k,r} (h_{kr}^\alpha \tilde{R}_{rijk} + h_{ri}^\alpha \tilde{R}_{rkjk}) + \sum_{k,\beta} h_{ki}^\beta \tilde{R}_{\alpha\beta kj} \\ &\quad + \sum_{r,k,\alpha} (h_{ri}^\beta h_{rj}^\beta h_{kk}^\beta + 2h_{kr}^\alpha h_{rj}^\beta h_{ik}^\beta - h_{kr}^\alpha h_{kr}^\beta h_{ij}^\beta \\ &\quad - h_{ri}^\alpha h_{kr}^\beta h_{kj}^\beta - h_{rj}^\alpha h_{ki}^\beta h_{kr}^\beta). \end{aligned} \quad (2.31)$$

Since

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha + \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2, \quad (2.32)$$

we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j,k,r} (h_{ij}^\alpha h_{kr}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) \\ &\quad + \sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta \tilde{R}_{\alpha\beta kj} - \sum_{\alpha,\beta,i,j,k,r} h_{ij}^\alpha h_{kr}^\alpha h_{ij}^\beta h_{kr}^\beta \\ &\quad + \sum_{\alpha,\beta,i,j,k,r} h_{ij}^\alpha h_{ir}^\alpha h_{jr}^\beta h_{kk}^\beta \\ &\quad - \sum_{\alpha,\beta,i,j,k,r} (h_{rj}^\alpha h_{kr}^\beta - h_{kr}^\alpha h_{rj}^\beta)(h_{ij}^\alpha h_{ki}^\beta - h_{ki}^\alpha h_{ij}^\beta). \end{aligned} \quad (2.33)$$

We remark that (2.31) and (2.33) can be found by specialising the result of [8] to this case.

### 3 Estimates and proofs of Theorems 1.2 and 1.3

Now, we assume that the mean curvature vector  $h$  of  $M$  is parallel (i.e.,  $\nabla^\perp h = 0$ ), and  $M$  is a complete submanifold in  $\tilde{M}(c)$ .

In this section  $\Phi_\alpha$  denoted the matrix  $(\Phi_{ij}^\alpha)$ , where  $\Phi_{ij}^\alpha = \langle \Phi_\alpha e_i, e_j \rangle$ . Note that to  $H = 0$  (i.e.,  $M$  is minimal submanifold), we get  $\Phi_\alpha = -H_\alpha$ , for all  $\alpha$ , where  $H_\alpha$  is the matrix  $(h_{ij}^\alpha)$ . If  $H \neq 0$ , we choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m, \dots, e_{2m+1}\}$  such that  $e_{n+1} = \frac{h}{H}$ . With this choose

$$\Phi_{n+1} = HI - H_{n+1}, \quad \Phi^\alpha = H_\alpha, \quad \alpha \neq n+1, \quad (3.1)$$

where  $I = (\delta_{ij})$ . Since  $e_{n+1}$  is a parallel direction,

$$H_\alpha H_{n+1} = H_{n+1} H_\alpha, \quad \omega_{\alpha(n+1)} = 0 \quad \text{and} \quad \sum_k h_{kki}^\alpha = 0. \quad (3.2)$$

In this case, we obtain

$$\text{tr } H_{n+1} = nH, \quad \text{tr } H_\alpha = 0, \quad \alpha \neq n+1 \quad \text{and} \quad R_{(n+1)\alpha ij}^\perp = 0. \quad (3.3)$$

Furthermore,

$$|\Phi_{n+1}|^2 = \text{tr } H_{n+1}^2 - nH^2, \quad (3.4)$$

$$\sum_{\alpha \neq n+1} |\Phi_\alpha|^2 = \sum_{\beta \neq n+1} \left( h_{ij}^\beta \right)^2, \quad (3.5)$$

and

$$\text{tr } \Phi_\alpha = 0, \quad (3.6)$$

for all  $\alpha$ . Thus,

$$S = \sum_\alpha |\Phi_\alpha|^2. \quad (3.7)$$

Now, we need the following algebraic lemmas:

**Lemma 3.1.** [11] *If  $A$  and  $B$  are two symmetric linear maps of  $\mathbb{R}^n$  with  $AB - BA = 0$  and  $\text{tr } A = \text{tr } B = 0$ . Then*

$$|\text{tr } A^2 B| \leq \frac{(n-2)}{\sqrt{n(n-1)}} \text{tr } A^2 \sqrt{\text{tr } B^2} \quad (3.8)$$

and the equality holds if only if  $n - 1$  of eigenvalues  $x_i$  of  $A$  and the corresponding eigenvalues  $y_i$  of  $B$  satisfy

$$|x_i| = \sqrt{\frac{\operatorname{tr} A^2}{n(n-1)}}, \quad x_i x_j \geq 0,$$

$$y_i = \sqrt{\frac{\operatorname{tr} B^2}{n(n-1)}} \quad \left( \text{resp. } y_i = -\sqrt{\frac{\operatorname{tr} B^2}{n(n-1)}} \right).$$

**Lemma 3.2.** [5, 10]. Let  $A_1, A_2, \dots, A_k$  be symmetric  $(n \times n)$ -degree matrices, where  $k \geq 2$ . Denote  $L_{ij} = \operatorname{tr} A_i A_j^t$  and  $L = L_{11} + L_{22} + \dots + L_{kk}$ . Then

$$\sum \{N(A_i A_j - A_j A_i) + (L_{ij})^2\} \leq \frac{3}{2} L^2, \quad (3.9)$$

where  $N(A) = \operatorname{tr} A A^t$ , for all matrix  $A$ .

The ideas used for proving the following lemmas are analogous to that found in [8].

**Lemma 3.3.**

$$\sum_{\alpha, i, j, k, r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) = \frac{c + 3\lambda}{4} n |\Phi|^2. \quad (3.10)$$

**Proof:** Fix a vector  $e_\alpha$  and let  $\{e_i\}$  be a local orthogonal frame on  $M$  such that the matrix  $H_\alpha$  (resp.  $\Phi_\alpha$ ) takes the diagonal form with  $h_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$  (resp.  $\Phi_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ , where  $\lambda_i^\alpha = \langle h, e_\alpha \rangle - \mu_i^\alpha$ ). Then, of (2.16) we get

$$\begin{aligned} \sum_{i, j, k, r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) &= \sum_{i, k} (\mu_i^\alpha \mu_k^\alpha \tilde{R}_{kiik} + (\mu_i^\alpha)^2 \tilde{R}_{ikik}) \\ &= \sum_{i, k} ((\mu_i^\alpha)^2 - \mu_i^\alpha \mu_k^\alpha) \tilde{R}_{ikik} \\ &= \sum_{i, k} ((\lambda_i^\alpha)^2 - \lambda_i^\alpha \lambda_k^\alpha) \tilde{R}_{ikik} \\ &= \frac{c + 3\lambda}{4} n \operatorname{tr} \Phi_\alpha^2 \\ &= \frac{c + 3\lambda}{4} n |\Phi_\alpha|^2. \end{aligned}$$

Hence

$$\sum_{\alpha, i, j, k, r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) = \frac{c + 3\lambda}{4} n |\Phi|^2.$$

□

**Lemma 3.4.** *If  $c \leq \lambda$ , then*

$$\sum_{\alpha, \beta, i, j, k} h_{ij}^\alpha h_{ki}^\beta \tilde{R}_{\alpha\beta kj} \geq \frac{c - \lambda}{4} |\Phi|^2.$$

**Proof:** As  $M$  is a  $C$ -totally real submanifold, we can choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m, e_{1^*} = \varphi e_1, \dots, e_{(n+1)^*} = \varphi e_{n+1}, \dots, e_{m^*} = \varphi e_m, e_{2m+1} = \xi\}$  in  $\tilde{M}(c)$ . If  $\alpha \neq r^*$  or  $\beta \neq s^*$ , then from (2.17) we have

$$\sum_{\alpha, \beta, i, j, k} h_{ij}^\alpha h_{ki}^\beta \tilde{R}_{\alpha\beta kj} = 0.$$

If  $\alpha = r^*$  and  $\beta = s^*$ , from (2.17) we obtain

$$\begin{aligned} \sum_{r^*, s^*, i, j, k} h_{ij}^{r^*} h_{ki}^{s^*} \tilde{R}_{r^* s^* kj} &= \sum_{r^*, s^*, i, k} h_{jr}^{i^*} h_{ks}^{i^*} \tilde{R}_{r^* s^* kj} \\ &= \sum_{r, s, i} \frac{c - \lambda}{4} \left( (h_{sr}^{i^*})^2 - h_{rr}^{i^*} h_{ss}^{i^*} \right) \\ &= \frac{c - \lambda}{4} \sum_i \text{tr } \Phi_{i^*}^2 = \frac{c - \lambda}{4} \sum_i |\Phi_{i^*}|^2 \geq \frac{c - \lambda}{4} |\Phi|^2. \end{aligned}$$

and the lemma is proved. □

**Lemma 3.5.**

$$- \sum_{\alpha, \beta, i, j, k, l} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta = - \sum_{\alpha, \beta} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2.$$

**Proof:** If  $H = 0$ , we have  $\Phi_\alpha = -H_\alpha$  for all  $\alpha$ . Hence,

$$- \sum_{\alpha, \beta, i, j, k, l} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta = - \sum_{\alpha, \beta} (\text{tr } H_\alpha H_\beta)^2 = - \sum_{\alpha, \beta} (\text{tr } \Phi_\alpha \Phi_\beta)^2,$$

which proves the lemma in this case. If  $H \neq 0$ , choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$  such that  $e_{n+1} = \frac{h}{H}$ , and thus

$$\begin{aligned}
- \sum_{\alpha, \beta, i, j, k, l} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta &= - \sum_{\alpha, \beta} (\text{tr } H_\alpha H_\beta)^2 \\
&= - \sum_{\alpha, \beta > n+1} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - 2 \sum_{\alpha > n+1} (\text{tr } (HI - \Phi_{n+1}) \Phi_\alpha)^2 \\
&\quad - (\text{tr } (HI - \Phi_{n+1}))^2 \\
&= - \sum_{\alpha, \beta > n+1} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - 2 \sum_{\alpha > n+1} (H \text{tr } (\Phi_\alpha) - \text{tr } \Phi_{n+1} \Phi_\alpha)^2 \\
&\quad - (\text{tr } (H^2 I - 2H\Phi_{n+1} + \Phi_{n+1}^2))^2 \\
&= - \sum_{\alpha, \beta > n+1} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - 2 \sum_{\alpha > n+1} (\text{tr } \Phi_{n+1} \Phi_\alpha)^2 \\
&\quad - (nH^2 + \text{tr } \Phi_{n+1}^2)^2 \\
&= - \sum_{\alpha, \beta} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - n^2 H^4 - 2nH^2 \text{tr } \Phi_{n+1}^2 \\
&= - \sum_{\alpha, \beta} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2.
\end{aligned}$$

□

**Lemma 3.6.**

$$\sum_{\alpha, \beta, i, j, k, l} h_{ij}^\alpha h_{il}^\alpha h_{jl}^\beta h_{kk}^\beta \geq - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|^3 + 2nH^2 |\Phi_{n+1}|^2 + nH^2 |\Phi|^2 + n^2 H^4.$$

**Proof:** Note that the inequality is obvious if  $H = 0$ . If  $H \neq 0$ , we obtain

$$\begin{aligned}
\sum_{\alpha,\beta,i,j,k,l} h_{ij}^\alpha h_{il}^\alpha h_{jl}^\beta h_{kk}^\beta &= \sum_{\alpha,\beta} \operatorname{tr} H_\alpha \operatorname{tr} H_\alpha H_\beta^2 \\
&= nH \sum_{\alpha} \operatorname{tr} H_{n+1} H_\alpha^2 \\
&= nH^2 \sum_{\alpha>n+1} \operatorname{tr} (HI - \Phi_{n+1}) \Phi_\alpha^2 + nH \operatorname{tr} (HI - \Phi)^3 \\
&= nH^2 \sum_{\alpha>n+1} \operatorname{tr} \Phi_\alpha^2 - nH \sum_{\alpha>n+1} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 \\
&\quad + nH \operatorname{tr} (H^3 I - 3H^2 \Phi_{n+1} + 3H \Phi_{n+1}^2 - \Phi_{n+1}^3) \\
&= nH^2 \sum_{\alpha>n+1} \operatorname{tr} \Phi_\alpha^2 - nH \sum_{\alpha} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 \\
&\quad + n^2 H^4 + 3nH^2 \operatorname{tr} \Phi_{n+1}^2 \\
&= nH^2 |\Phi|^2 - nH \sum_{\alpha} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 + n^2 H^4 + 2nH^2 |\Phi_{n+1}|^2
\end{aligned}$$

Using lemma 3.1, we have

$$\operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 \leq \frac{n-2}{\sqrt{n(n-1)}} |\Phi_{n+1}| |\Phi_\alpha|^2, \quad (3.11)$$

and so

$$\sum_{\alpha} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 \leq \frac{n-2}{\sqrt{n(n-1)}} |\Phi_{n+1}| |\Phi|^2. \quad (3.12)$$

Hence,

$$\sum_{\alpha,\beta,i,j,k,l} h_{ij}^\alpha h_{il}^\alpha h_{jl}^\beta h_{kk}^\beta \geq -\frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|^3 + 2nH^2 |\Phi_{n+1}|^2 + nH^2 |\Phi|^2 + n^2 H^4.$$

□

**Lemma 3.7.**

$$\begin{aligned}
\sum_{\alpha,\beta,i,j,k,r} (h_{rj}^\alpha h_{kr}^\beta - h_{kr}^\alpha h_{rj}^\beta) (h_{ij}^\alpha h_{ki}^\beta - h_{ki}^\alpha h_{ij}^\beta) - \sum_{\alpha,\beta,i,j,k,r} h_{ij}^\alpha h_{kr}^\alpha h_{ij}^\beta h_{kr}^\beta \\
\geq -\frac{3}{2} |\Phi|^4 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2.
\end{aligned}$$

**Proof:** Note that

$$\sum_{\alpha,\beta,i,j,k,r} (h_{rj}^\alpha h_{kr}^\beta - h_{kr}^\alpha h_{rj}^\beta) (h_{ij}^\alpha h_{ki}^\beta - h_{ki}^\alpha h_{ij}^\beta) = -\sum_{\alpha,\beta} N(\Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha),$$



and

$$- \sum_{\alpha, \beta, i, j, k, r} h_{ij}^\alpha h_{kr}^\alpha h_{ij}^\beta h_{kr}^\beta = - \sum_{\alpha, \beta} (tr(\Phi_\alpha \Phi_\beta))^2 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2.$$

From lemma 3.2, we have

$$- \sum_{\alpha, \beta} N(\Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha) - \sum_{\alpha, \beta} (tr(\Phi_\alpha \Phi_\beta))^2 \geq -\frac{3}{4} |\Phi|^4,$$

and so

$$\begin{aligned} - \sum_{\alpha, \beta, i, j, k, l} (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta) - \sum_{\alpha, \beta, i, j, k, l} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta \\ \geq -\frac{3}{2} |\Phi|^4 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2. \end{aligned}$$

□

### 3.1 Proof of the Theorem 1.1

Now, using lemmas 3.3, 3.4, 3.5, 3.6 and 3.7, we get the following result:

**Proposition 3.1.** *Let  $\tilde{M}(c)$  an  $(2m + 1)$ -dimensional  $\lambda$ -Sasakian space form with structure  $(\varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$  and  $M$  an  $n$ -dimensional  $C$ -totally real submanifold with parallel mean curvature vector in  $\tilde{M}(c)$ . If  $c \leq \lambda$ , then*

$$\begin{aligned} \frac{1}{2} \Delta S \geq |\bar{\nabla} \sigma|^2 - \frac{3}{2} |\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|^3 \\ + \left( \frac{n(c+3\lambda) + c - \lambda}{4} + H^2 \right) |\Phi|^2. \end{aligned} \tag{3.13}$$

Suppose now that  $M$  is a closed  $n$ -dimensional  $C$ -totally real submanifold with parallel mean curvature vector in  $\tilde{M}(c)$ . From proposition 3.1, we have

$$0 \leq \int_M |\bar{\nabla} \sigma|^2 dM \leq \int_M |\Phi|^2 P_{H,c,\lambda}(|\Phi|) dM, \tag{3.14}$$

where

$$P_{H,c,\lambda}(x) = \frac{3}{2} x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - \left( \frac{n(c+3\lambda) + c - \lambda}{4} + nH^2 \right).$$

This proves the Theorem 1.1.

### 3.2 Proof of the Theorem 1.2

If  $|\Phi|^2 \leq \vartheta_H$ , we have that  $P_{H,c}(|\Phi|) \leq 0$ . Then, follows from Theorem 1.1 that

$$0 \leq \int_M |\Phi|^2 P_{H,c}(|\Phi|) dM \leq 0. \quad (3.15)$$

Thus,  $|\Phi|^2 P_{H,c}(|\Phi|) \equiv 0$ . Therefore,  $|\Phi|^2 = 0$  and  $M$  is totally umbilical or  $|\Phi|^2 = \vartheta_H$ .

If  $|\Phi|^2 = \vartheta_H$ , from (3.15) we have that in all the inequalities of the lemmas above become equalities. Then, from lemma 3.4, we obtain  $\sum_{i=1}^n |\Phi_{i^*}|^2 = |\Phi|^2$  and  $m = n$ . Hence  $M$  is minimal by Theorem 1.1 given in [12]. Note that, in this case

$$P_{H,c,\lambda}(|\Phi|) = \frac{3}{2}|\Phi|^2 - \frac{n(c+3\lambda)+c-\lambda}{4},$$

and

$$S = |\Phi|^2 = \frac{n(c+3\lambda)+c-\lambda}{6}.$$

In particular, if  $c = \lambda = 1$ , then  $\tilde{M}(c)$  is the Sakakian unit sphere  $S^{2n+1}(1) \subset \mathbb{C}^{m+1}$  with contact structure induced and  $S = \frac{2n}{3}$ . Hence, from Theorem 3 in [10],  $M$  is a Veronese surface in  $S^4(1) \subset S^{2m+1}(1)$ .

## 4 Proofs of the Theorems 1.3 and 1.4

### 4.1 Proof of theorem 1.3

Let  $M$  be a  $n$ -dimensional  $C$ -totally real submanifold in a  $(2n+1)$ -dimensional  $\lambda$ -Sasakian space form  $\tilde{M}(c)$ . We choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, e_1^* = \varphi e_1, \dots, e_{(n+1)}^* = \varphi e_{n+1}, \dots, e_n^* = \varphi e_n, e_{2n+1} = \xi\}$ . From [4] follows that

$$\begin{aligned} \frac{1}{2} \Delta S &= \sum_{i,j,\alpha} h_{ij}^\alpha \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} (\text{tr } H_\alpha) + \frac{n(c+3\lambda)+c-\lambda}{4} S \\ &\quad - \sum_{\alpha,\beta} [(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2 - \text{tr } H_\beta \text{tr } H_\alpha H_\beta H_\alpha] + |\bar{\nabla} \sigma|^2. \end{aligned} \quad (4.1)$$

And the other hand, we have that  $f$  is pseudo-parallel if and only if

$$h_{ijkl}^\alpha = h_{ijlk}^\alpha - f \{ \delta_{ki} h_{lj}^\alpha - \delta_{li} h_{kj}^\alpha + \delta_{kj} h_{il}^\alpha - \delta_{lj} h_{ik}^\alpha \}, \quad (4.2)$$

where  $i, j, k, l = 1, \dots, n$  and  $\alpha = n+1, \dots, 2n+1$ , see [3]. Using (4.2), (2.16), (2.17), (2.18) and Codazzi equation in (2.33), we get

$$\frac{1}{2} \Delta S = \sum_{i,j,\alpha} h_{ij}^\alpha \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} (\text{tr } H_\alpha) + nf|\Phi|^2 + |\bar{\nabla}\sigma|^2. \tag{4.3}$$

Therefore, for a  $C$ -totally real  $f$ -pseudo-parallel submanifold of a  $\lambda$ -Sasakian space form of  $\varphi$ -sectional curvature  $c$ , we have:

$$0 = \sum_{\alpha,\beta} [(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2 - \text{tr } H_\beta \text{tr } H_\alpha H_\beta H_\alpha] + nf|\Phi|^2 - \frac{n(c + 3\lambda) + c - \lambda}{4} S$$

Now, the condition  $\nabla^\perp h = 0$  in an  $n$ -dimensional  $C$ -totally real submanifold  $M$  of a  $(2n + 1)$ -dimensional  $\lambda$ -Sasakian space form  $\tilde{M}(c)$  is equivalent to the condition  $H = 0$ . This follows by taking the trace of (2.30), see also [7] in the special case that  $\lambda = 1$ . Hence, we have that  $\text{tr } H_\alpha = 0$ , for all  $\alpha$  and we get:

$$0 = \left( nf - \frac{n(c + 3\lambda) + c - \lambda}{4} \right) S + \sum_{\alpha,\beta} [(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2].$$

If  $f \geq (n(c + 3\lambda) + c - \lambda)/4n$ , then  $\text{tr } (H_\alpha H_\beta) = 0$ , for all  $\alpha, \beta$ . In particular  $|A_\alpha|^2 = \text{tr } H_\alpha^2 = 0$ , hence  $\sigma = 0$ . This proves Theorem 1.3.

### 4.2 Proof of Theorem 1.4

If  $M$  is  $f$ -pseudo-parallel and  $\nabla^\perp h = 0$ , then we obtain

$$\frac{1}{2} \Delta S = nf|\Phi|^2 + |\bar{\nabla}\sigma|^2.$$

If  $f \geq 0$ , we get  $\frac{1}{2} \Delta S \geq 0$ . Hence, if  $M$  is compact, then we have  $\bar{\nabla}\sigma = 0$ . This proves our result.

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Departamento de Matemática  
 Universidade Federal do Ceará  
 Campus do Pici  
 60455-760, Fortaleza - CE, Brazil  
*E-mail:* aldir@mat.ufc.br

Departamento de Matemática  
 Universidade Federal de São Carlos  
 Rod. Washington Luís, Km 235  
 13565-905, São Carlos - SP, Brazil  
 \protect\relax\itshapeE-mail: lobos@dm.ufscar.br

Departamento de Matemática  
 Universidade Federal do Maranhão  
 Campus do Bacanga  
 65085-580, São Luís - MA, Brazil  
*E-mail:* mxwbarros@gmail.com