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A LOCAL ESTIMATE FOR MAXIMAL SURFACES IN LORENTZIAN PRODUCT SPACES

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

In this paper we introduce a local approach for the study of maximal surfaces immersed into a Lorentzian product space of the form $M^2 \times \mathbb{R}_1$, where M^2 is a connected Riemannian surface and $M^2 \times \mathbb{R}_1$ is endowed with the product Lorentzian metric. Specifically, we establish a local integral inequality for the squared norm of the second fundamental form of the surface, which allows us to derive an alternative proof of our Calabi-Bernstein theorem given in [1].

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1 Introduction

Maximal surfaces in 3-dimensional Lorentzian manifolds, that is, spacelike surfaces with zero mean curvature, have become a research field of increasing interest in recent years, both from mathematical and physical points of view. In fact, one of the most relevant global results for maximal surfaces in Lorentzian geometry is the well-known Calabi-Bernstein theorem, which states that the only complete maximal surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1$ are the spacelike planes.

This result was firstly proved by Calabi [4] and extended later to arbitrary dimension by Cheng and Yau [5]. After that, several extensions and generalizations of the Calabi-Bernstein theorem have been given, and several alternatives proofs have been provided. In particular, in [3] the second author jointly with Palmer introduced a new approach to the Calabi-Bernstein theorem in the Lorentz-Minkowski space $\mathbb{R}^3_1$ based on a local integral inequality for the Gaussian curvature of a maximal surface in $\mathbb{R}^3_1$ which involved the local geometry of the surface and the image of its Gauss map. As an application of it, they provided a new proof of the Calabi-Bernstein theorem in $\mathbb{R}^3_1$. In this paper, we generalize this local approach to the case of maximal surfaces in a product space $M^2 \times \mathbb{R}$, where M^2 is a connected Riemannian surface and $M^2 \times \mathbb{R}$ is endowed with the product Lorentzian metric

$$\langle , \rangle = \pi_M^*(\langle , \rangle_M) - \pi_{\mathbb{R}}^*(dt^2).$$

Here π_M and $\pi_{\mathbb{R}}$ stand for the projections from $M^2 \times \mathbb{R}$ onto each factor and $\langle , \rangle_M$ is the Riemannian metric on M. For simplicity, we will simply write

$$\langle , \rangle = \langle , \rangle_M - dt^2,$$

and we will denote by $M^2 \times \mathbb{R}_1$ the 3-dimensional Lorentzian product manifold obtained in that way. Specifically, we will prove the following extension of [3, Theorem 1].

Theorem 1. *Let M^2 be an analytic Riemannian surface with non-negative Gaussian curvature, $K_M \geq 0$, and let $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$ be a maximal sur-*

face in $M^2 \times \mathbb{R}_1$*. Let* p *be a point of* Σ *and* $R > 0$ *be a positive real number such that the geodesic disc of radius* R *about* p *satisfies* $D(p, R) \subset\subset \Sigma$*. Then for all* $0 < r < R$ *it holds that*

$$0 \leq \int_{D(p,r)} \|A\|^2 \mathrm{d}\Sigma \leq c_r \frac{L(r)}{r \log (R/r)}, \tag{1}$$

where $L(r)$ *denotes the length of the geodesic circle of radius* r *about* p*, and*

$$c_r = \frac{\pi^2(1+\alpha_r^2)^2}{4\alpha_r \arctan \alpha_r} > 0.$$

Here

$$\alpha_r = \sup_{D(p,r)} \cosh \theta \geq 1,$$

where θ *denotes the hyperbolic angle between* N *and* ∂_t *along* Σ*.*

In particular, when Σ is complete then the local integral inequality (1) provides an alternative proof of the following parametric version of the Calabi-Bernstein type result for complete maximal surfaces in Lorentzian product spaces given by the authors in [1, Theorem 3.3].

Corollary 2. *Let* M^2 *be a (necessarily complete) analytic Riemannian surface with non-negative Gaussian curvature,* $K_M \geq 0$*. Then any complete maximal surface* Σ^2 *in* $M^2 \times \mathbb{R}_1$ *is totally geodesic. In addition, if* $K_M > 0$ *at some point on* M*, then* Σ *is a slice* $M \times \{t_0\}$*,* $t_0 \in \mathbb{R}$*.*

As another application of Theorem 1, at points of a maximal surface where the second fundamental form does not vanish, we are able to estimate the maximum possible geodesic radius in terms of a local positive constant.

Corollary 3. *Let* M^2 *be an analytic Riemannian surface with non-negative Gaussian curvature and let* $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$ *be a maximal surface in* $M^2 \times \mathbb{R}_1$ *which is not totally geodesic. Assume that* $p \in \Sigma$ *is a point with* $\|A\|(p) \neq 0$ *and let* $r > 0$ *be a positive real number such that* $D_r = D(p, r) \subset\subset \Sigma$*. Then*

$$R \leq r e^{C_r}$$

for every $R > r$ with $D(p,R) \subset\subset \Sigma$, where

$$C_r = \frac{c_r L(r)}{r \int_{D_r} \|A\|^2} > 0$$

is a local positive constant depending only on the geometry of $f|_{D(p,r)}$.

A similar estimate for stable minimal surfaces in 3-dimensional Riemannian surfaces with non-negative Ricci curvature was given by Schoen in [6]. See also [2] for another similar estimate given by the second author and Palmer for the case of non-flat spacelike surfaces with non-negative Gaussian curvature and zero mean curvature in a flat 4-dimensional Lorentzian space.

2 Preliminaries

A smooth immersion $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$ of a connected surface Σ^2 is said to be a spacelike surface if the induced metric via f is a Riemannian metric on Σ, which as usual is also denoted by $\langle , \rangle$. Observe that

$$\partial_t = (\partial/\partial_t)_{(x,t)}, \quad x \in M, t \in \mathbb{R},$$

is a unitary timelike vector field globally defined on the ambient spacetime $M^2 \times \mathbb{R}_1$. This allows us to consider the unique unitary timelike normal field N globally defined on Σ which is in the same time-orientation as ∂_t, so that

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on} \quad \Sigma.$$

We will refer to N as the future-pointing Gauss map of Σ, and we will denote by $\Theta : \Sigma \to (-\infty, -1]$ the smooth function on Σ given by $\Theta = \langle N, \partial_t \rangle$. Observe that the function Θ measures the hyperbolic angle θ between the timelike future-pointing vector fields N and ∂_t along Σ, since $\cosh \theta = -\Theta$.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $M^2 \times \mathbb{R}_1$ and Σ, respectively, and let $A : T\Sigma \to T\Sigma$ stands for the shape operator (or second fundamental form) of Σ with respect to its future-pointing Gauss map N. It is well known that the Gauss and Weingarten formulae for the spacelike surface $f:\Sigma^2\to M^2\times\mathbb{R}_1$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N \tag{2}$$

and

$$AX = -\overline{\nabla}_X N, \tag{3}$$

for any tangent vector fields $X, Y \in T\Sigma$. The mean curvature of a spacelike surface $f:\Sigma^2\to M^2\times\mathbb{R}_1$ is defined by $H = -(1/2)\mathrm{tr}A$, and $f:\Sigma^2\to M^2\times\mathbb{R}_1$ is said to be a maximal surface when H vanishes on Σ.

The Gauss equation of a spacelike surface Σ describes its Gaussian curvature K in terms of the shape operator and the curvature of the ambient space and it is given by

$$K = \overline{K} - \det A, \tag{4}$$

where $\overline{K}$ denotes the sectional curvature in $M^2\times\mathbb{R}_1$ of the plane tangent to Σ. On the other hand, if $\overline{R}$ stands for the curvature tensor of the Lorentzian product $M^2\times\mathbb{R}_1$, then the Codazzi equation of Σ describes the tangent component of $\overline{R}(X,Y)N$, for any tangent vector fields $X, Y \in T\Sigma$, in terms of the derivative of the shape operator. Specifically, it is given by

$$(\overline{R}(X,Y)N)^\top = (\nabla_X A)Y - (\nabla_Y A)X, \tag{5}$$

where $\nabla_X A$ denotes the covariant derivative of A, that is,

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y).$$

In the particular case where $f:\Sigma^2\to M^2\times\mathbb{R}_1$ is a maximal surface, it is not difficult to see that the Gauss (4) and Codazzi (5) equations for Σ become

$$K = \kappa_M \Theta^2 + \frac{1}{2}\|A\|^2 \tag{6}$$

and

$$(\nabla_X A)Y = (\nabla_Y A)X + \kappa_M\Theta(\langle X, \partial_t^\top\rangle Y - \langle Y, \partial_t^\top\rangle X), \tag{7}$$

for any tangent vector fields $X, Y \in T\Sigma$, respectively. Here $\|A\|^2 = \mathrm{tr}(A^2)$ and κ_M stands for the Gaussian curvature of M along the surface Σ, that is, $\kappa_M = K_M \circ \Pi \in \mathcal{C}^\infty(\Sigma)$ where K_M is the Gaussian curvature of M and $\Pi = \pi_M \circ f : \Sigma \to M$ denotes the projection of Σ onto M. Here and in what follows, $Z^\top \in T\Sigma$ denotes the tangential component of a vector field Z along the immersion $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$, that is

$$Z = Z^\top - \langle N, Z\rangle N.$$

Thus, in particular,

$$\partial_t^\top = \partial_t + \Theta N, \tag{8}$$

(for the details see [1]). Taking norms in the last expression we get

$$\|\partial_t^\top\|^2 = \Theta^2 - 1. \tag{9}$$

It is well known that a spacelike surface $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$ is locally a spacelike graph over M (see for instance [1, Lemma 3.1]), that is, for any given point $p \in \Sigma$, there exists an open subset Ω on M containing $\Pi(p)$, $\Pi(p) \in \Omega \subset M$, and a function $u \in \mathcal{C}^\infty(\Omega)$ such that the surface Σ is locally given in a neighborhood of p by $\Sigma(u) = \{(x, u(x)) : x \in \Omega\} \subset M^2 \times \mathbb{R}_1$. Therefore, the metric induced on $\Sigma(u)$ from the Lorentzian metric on the ambient space is given by

$$\langle , \rangle = \langle , \rangle_M - du^2. \tag{10}$$

The condition that $\Sigma(u)$ is spacelike becomes $|Du|^2 < 1$ on $\Omega \subset M$, where Du denotes the gradient of u in M and $|Du|$ denotes its norm. Finally, it is not difficult to see that the mean curvature function H of $\Sigma(u)$ is given by

$$2H = \mathrm{Div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right),$$

on Ω, where Div stands for the divergence operator on M with respect to the metric $\langle , \rangle_M$. In particular, a spacelike immersion $f : \Sigma^2 \to M^2 \times \mathbb{R}_1$

is a maximal surface if and only if it is locally given as the graph of a function u satisfying the following partial differential equation,

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \quad |Du|^2<1. \tag{11}$$

3 Proof of the results

The proof of Theorem 1 is inspired by the ideas in [3], and it is an application of the following intrinsic property.

Lemma 4. *[3, Lemma 3] Let Σ be an analytic Riemannian surface with non-negative Gaussian curvature $K \geq 0$. Let ψ be a smooth function on Σ which satisfies*

$$\psi\Delta\psi \geq 0$$

on Σ. Then for $0<r<R$

$$\int_{D_r}\psi\Delta\psi \leq \frac{2L(r)}{r\log(R/r)}\sup_{D_R}\psi^2,$$

where D_r denotes the geodesic disc of radius r about a fixed point in Σ, $D_r \subset D_R \subset\subset \Sigma$, and $L(r)$ denotes the length of ∂D_r, the geodesic circle of radius r.

Proof of Theorem 1. Observe that since M is analytic and Σ is locally given by the maximal surface equation (11), then Σ, endowed with the induced metric, is also an analytic Riemannian surface. Besides, from (6) we also know that the Gaussian curvature of Σ is non-negative, $K \geq 0$. Therefore, we may apply Lemma 4 to an appropriate smooth function ψ. Let us consider $\psi = \arctan\Theta$.

Since ∂_t is parallel on $M^2 \times \mathbb{R}_1$ we have that

$$\overline{\nabla}_X\partial_t = 0 \tag{12}$$

for any tangent vector field $X \in T\Sigma$. Thus,

$$X(\Theta)=\langle\overline{\nabla}_X N,\partial_t\rangle = -\langle AX,\partial_t^\top\rangle = -\langle X, A\partial_t^\top\rangle$$

for every $X \in T\Sigma$, and then the gradient of Θ on Σ is given by

$$\nabla\Theta = -A\partial_t^\top. \tag{13}$$

Therefore, from (13) and (9) we obtain

$$\|\nabla\Theta\|^2 = \frac{1}{2}\|A\|^2(\Theta^2 - 1), \tag{14}$$

since for a maximal surface it holds $A^2 = (1/2)\|A\|^2 I$.

On the other hand, taking into account (8), and using Gauss (2) and Weingarten (3) formulae, (12) also yields

$$\nabla_X\partial_t^\top = -\Theta AX \tag{15}$$

for every $X \in T\Sigma$. Therefore, using Codazzi equation (7) and equations (9) and (15) we get

$$\begin{aligned}
\nabla_X\nabla\Theta &= -(\nabla_X A)(\partial_t^\top) - A(\nabla_X\partial_t^\top)\\
&= -(\nabla_{\partial_t^\top} A)(X) - \kappa_M\Theta\left(\langle X, \partial_t^\top\rangle\partial_t^\top - \|\partial_t^\top\|^2 X\right) + \Theta A^2 X\\
&= -(\nabla_{\partial_t^\top} A)(X) + \kappa_M\Theta\left((\Theta^2 - 1)X - \langle X, \partial_t^\top\rangle\partial_t^\top\right) + \Theta A^2 X,
\end{aligned}$$

for every $X \in T\Sigma$. Thus, the Laplacian of Θ is given by

$$\Delta\Theta = \Theta(\kappa_M(\Theta^2 - 1) + \|A\|^2), \tag{16}$$

since

$$\mathrm{tr}(\nabla_{\partial_t^\top} A) = \nabla_{\partial_t^\top}(\mathrm{tr}A) = 0.$$

Using (16) and (14) we can compute

$$\Delta\psi = \frac{\Delta\Theta}{1+\Theta^2} - \frac{2\Theta\|\nabla\Theta\|^2}{(1+\Theta^2)^2} = \frac{2\Theta}{(1+\Theta^2)^2}\|A\|^2 + \frac{(\Theta^2-1)\Theta}{1+\Theta^2}\kappa_M,$$

and therefore, taking into account that $\Theta\arctan\Theta \geq 0$, $\Theta \leq -1$ and $\kappa_M \geq 0$, we obtain

$$\psi\Delta\psi = \frac{2\Theta\arctan\Theta}{(1+\Theta^2)^2}\|A\|^2 + \frac{(\Theta^2-1)\Theta\arctan\Theta}{1+\Theta^2}\kappa_M \geq \phi(\Theta)\|A\|^2, \tag{17}$$

where

$$\phi(s) = \frac{2s \arctan s}{(1+s^2)^2}.$$

Observe that the function $\phi(s)$ is strictly increasing for $s \leq -1$. Since $-\alpha_r \leq \Theta \leq -1$ on $D(p,r)$, we get

$$\phi(\Theta) \geq \phi(-\alpha_r) = \frac{2\alpha_r \arctan \alpha_r}{(1+\alpha_r^2)^2} \quad \text{on} \quad D(p,r),$$

which, jointly with (17), yields

$$\psi \Delta \psi \geq \frac{2\alpha_r \arctan \alpha_r}{(1+\alpha_r^2)^2} \|A\|^2 \quad \text{on} \quad D(p,r).$$

Integrating now this inequality over $D(p,r)$ and using Lemma 4 we conclude that

$$0 \leq \frac{2\alpha_r \arctan \alpha_r}{(1+\alpha_r^2)^2} \int_{D(p,r)} \|A\|^2 \mathrm{d}\Sigma \leq \int_{D(p,r)} \psi \Delta \psi \leq \frac{\pi^2}{2} \frac{L(r)}{r \log (R/r)},$$

which yields (1).

□

Proof of Corollary 2. Since Σ is complete, then R can approach to infinity in (1) for a fixed arbitrary $p \in \Sigma$ and a fixed r, which gives

$$\int_{D(p,r)} \|A\|^2 \mathrm{d}\Sigma = 0.$$

Therefore, $\|A\|^2 = 0$ and Σ must be totally geodesic. From (13), this implies that $\Theta = \Theta_0 \leq -1$ is constant on Σ, and then (16) implies that, when $K_M > 0$ somewhere in M, it must be $\Theta_0 = -1$. Finally, by (9) we conclude that Σ must be a slice.

□

Corollary 3 is a direct consequence of Theorem 1.

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