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REMARKS ON CARRIER EVOLUTION EQUATION

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Dedicated to L. A. Medeiros in ocasion of his 80^{th} birthday

Abstract

In this article we consider the quasi linear evolution Carrier equation when the non-linear term depends on the position at the point x, elapsed time t and of the transverse deformation of the segment of the string on the action of a linear internal damping. The Carrier equation is studied first in a cylindrical domain with objective of establishing the some properties in non cylindrical domain through the penalty method. As results of these investigations are established the existence of global weak solutions as well as the exponential decay of total energy for each problem.

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1 Introduction

As an alternative model for small vibrations of an elastic string with fixed ends α and β in the one dimensional case was introduced in 1945 by Carrier [3]. Namely,

$$u_{tt} - \Big(P_0 + P_1 \int_\alpha^\beta u^2 dx\Big) u_{xx} = 0, \tag{1.1}$$

where $u = u(x, t)$ is the displacement of the string and P_0, P_1 are associated with variations of tensions at each point x of the string. Today this model is known as the Carrier's equation. The Eq.(1.1) it was gotten through a numeric outline as a approach of the very well known Kirchhoff equation.

The $n-$dimensional case, i.e., $n = 2,\, 3,\, \cdots$, of Eq.(1.1) is given by

$$u'' - M(\int_\Omega u^2 dx)\Delta u = 0, \tag{1.2}$$

where M is a real valued function, Ω is an open and bounded set of $\mathbb{R}^n$ and $-\Delta$ is the usual Laplace operator. Assuming a nonlinear internal damping acting in (1.2) a mixed problem with the Dirichlet boundary conditions was studied by Frota at al [5], and with acoustic boundary conditions by Frota & Goldstein [6]. In both works global solvability and stability of the energy were established inside a cylindrical domain. An abstract framework in a Hilbert space for Eq.(1.1) can be write as

$$u'' - M(u^2)Au = 0, \tag{1.3}$$

where A denotes the unbounded operator defined by the triplet $\{V,\, H,\, ((\cdot,\cdot))\}$ with V, H real separable Hilbert spaces and $((\cdot,\cdot))$ is the inner product of V. The local solution for the Cauchy problem of (1.3) was proved by Cousin at al [4]. Recently, Rabello & Vieira [11] studied, inside one dimensional noncylidrical domain

$$Q_T = \bigcup_{0<t<T}]\alpha(t), \beta(t)[\times \{t\},$$

the following Carrier's equation with variable coefficients

$$u_{tt} - M\Big(x,\, t,\, \int_{\alpha(t)}^{\beta(t)} u^2 dx\Big) u_{xx} = 0. \tag{1.4}$$

Local solutions and uniqueness were established through a suitable diffeomorphism, which allows do a variable change and to investigate the equation (1.4) in a cylinder. Finally, we mention the paper Medeiros at al [9] in which contains an excellent survey on the Carrier and Kirchhoff equations.

In this work we consider the initial-boundary value problem for the n-dimensional approach of Carrier equation with linear damping given by

$$u''(x,t) - M\Big(x, t, \int_\Omega |u(x,t)|^2 dx\Big)\Delta u(x,t) + \delta u'(x,t) = 0. \tag{1.5}$$

where δ is a positive real constant.

Our goals in this article are, initially, in Section 2, to establish the existence of global weak and strong solutions, and the asymptotic behavior of the energy for Eq.(1.5) in a cylinder $Q = \Omega \times]0, T[$ of $\mathbb{R}^{n+1}$, where Ω is an open and bounded set in $\mathbb{R}^n$ with C^1 boundary Γ. The lateral boundary of Q is denoted by Σ. With the results obtained in Section 2 we will present the study in Section 3, for the moving domain $\widehat{Q}$, which will be defined there. The method we will employ to solve the initial-boundary problem for (1.5) in $\widehat{Q}$, consists in transforming it into a cylindrical problem by means of a perturbation of equation (1.5) adding a singular term depending on a parameter $\epsilon > 0$ which is destined to tend to zero. This method was idealized by Lions [7] and is called by him *penalty method*.

2 Carrier equation in a cylindrical domain

§ 2.1. Existence of weak solutions. We use the standard notation for functional spaces, namely, $L^p(\Omega)$, $1 \leq p \leq \infty$ is the Lebesgue space, $H^m(\Omega)$ is the Sobolev space of order m and $D(\Omega)$ is the space of C^∞ functions in Ω with compact support and with the Schwartz notion of convergence. By $H_0^m(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. Besides, we also work with the spaces $L^p(0,T;H^m(\Omega))$ for $1 \leq p \leq \infty$ and $L^p(0,T;L^2(\Omega))$. To complete this exposition on functional spaces, see for instance, Brézis [2] or Lions [8].

The goal here is to study the initial-boundary value problem

$$u''(x,t) - M\Big(x,t,\int_\Omega |u(x,t)|^2 dx\Big)\Delta u(x,t) + \delta u'(x,t) = 0 \text{ in } Q, \tag{2.1}$$

$$u(x,t) = 0 \text{ on } \Sigma, \tag{2.2}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ in } \Omega, \tag{2.3}$$

where $Q = \Omega \times]0,T[$ is a cylindrical domain. With the results obtained in this Section we will conclude the investigation, in Section 3 for the moving domain case. Concerning with the function M of the equation (2.1), assume the hypotheses:

$$\begin{gathered} M(x,t,\lambda) \text{ is } C^1 - \text{real function in the variables } x \in \overline{\Omega},\ t \geq 0,\ \lambda \geq 0, \\ M(x,t,\lambda) \geq m_0 > 0, \\ \big|\nabla M\big| \leq K_1|\lambda|^p, \quad \Big|\frac{\partial M}{\partial t}\Big| \leq K_2|\lambda|^p \quad \text{and} \quad \Big|\frac{\partial M}{\partial \lambda}\Big| \leq K_3|\lambda|^{p-1} \quad \text{for} \quad p \geq 1. \end{gathered} \tag{2.4}$$

Throughout Section 2, we will represent by K_j for $j = 1, 2, 3, \ldots, 19$ real positive constants.

Definition 2.1. *A weak solution of the initial-boundary value problem*

(2.1)-(2.3) is a real valued function $u = u(x,t)$ *defined in* Q *such that*

$$u \in L^\infty(0,T;H_0^1(\Omega)), \quad u' \in L^\infty(0,T;L^2(\Omega)) \text{ for } T > 0, \tag{2.5}$$

$$-\int_0^T \int_\Omega u'(x,t)v(x)\theta'(t)dxdt+$$
$$\int_0^T \int_\Omega M\left(x,t,|u(t)|^2\right)\Big[\nabla u(x,t)\cdot\nabla v(x)\Big]\theta(t)dxdt+$$
$$\int_0^T \int_\Omega \Big[\nabla M\left(x,t,|u(t)|^2\right)\cdot\nabla u(x,t)\Big]v(x)\theta(t)dxdt+ \tag{2.6}$$
$$\delta\int_0^T \int_\Omega u'(x,t)v(x)\lambda(t)dxdt = 0 \text{ for all } v \in H_0^1(\Omega), \ \theta \in \mathcal{D}(0,T).$$

Moreover, u *satisfies the initial conditions*

$$u(x,0) = u_0(x) \text{ and } u'(x,0) = u_1(x).$$

Theorem 2.1. *Suppose* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ *and*

$$K_6[H(0)]^p < \frac{m_0\delta}{8}, \tag{2.7}$$

where the function $H(t)$ *and the constant* K_6 *are defined in (2.20) and (2.22) respectively. Then there exists at least one weak solution of the initial-boundary value problem (2.1)-(2.3), provided that the hypothesis (2.4) holds.*

Proof. We employ the Faedo-Galerkin approximate method with the hilbertian basis $(w_j)_{j\in\mathbb{N}}$, which is formed by vectors of the Sobolev space $H_0^1(\Omega)$. Thus, for each $N \in \mathbb{N}$ we look for a function $u_N(x,t) = \sum_{j=1}^{N} g_{jN}(t)w_j(x)$ in solution to the approximate initial value problem

$$\begin{aligned}
&\int_\Omega u''_N(x,t)w(x,t)dx+\int_\Omega M\left(x,t,|u_N(t)|^2\right)\Big[\nabla u_N(x,t)\cdot\nabla w(x,t)\Big]dx+\\
&\int_\Omega\Big[\nabla M\left(x,t,|u_N(t)|^2\right)\cdot\nabla u_N(x,t)\Big]w(x,t)dx+\delta\int_\Omega u'_N(x,t)w(x,t)dx=0,\\
&u_N(x,0)=u_{0N}(x)\longrightarrow u_0(x)\ \text{ in }\ H_0^1(\Omega),\\
&u'_N(x,0)=u_{1N}(x)\longrightarrow u_1(x)\ \text{ in }\ L^2(\Omega),
\end{aligned}\tag{2.8}$$

for all $w\in V_N$. Now, we will get estimates for the solution u_N of (2.8). For a sake of simplicity, henceforth, we will write u instead of u_N.

Estimate I. Setting $w=u'$ in (2.8) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|u'(t)\right|^2+\frac{1}{2}\int_\Omega M\left(x,t,|u(t)|^2\right)\frac{d}{dt}|\nabla u(x,t)|^2dx+\\
&\int_\Omega\Big[\nabla M\left(x,t,|u(t)|^2\right)\cdot\nabla u(x,t)\Big]u'(x,t)dx+\delta\left|u'(t)\right|^2=0.
\end{aligned}\tag{2.9}$$

Analysis of the second and third terms in (2.9) gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega M\left(x,t,|u(t)|^2\right)\frac{d}{dt}|\nabla u(x,t)|^2dx=\\
&\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx\Big]-\\
&\frac{1}{2}\int_\Omega\frac{\partial}{\partial t}M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx-\\
&\Big[\int_\Omega\frac{\partial}{\partial\lambda}M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx\Big]\Big(u'(t),u(t)\Big).
\end{aligned}\tag{2.10}$$

Thanks to the hypothesis (2.4) the second and the third terms on the right-hand side of (2.10) can be upper bounded by using usual inequalities as follows:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\frac{\partial}{\partial t}M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx-\\
&\Big[\int_\Omega\frac{\partial}{\partial\lambda}M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx\Big]\Big(u'(t),u(t)\Big)\le\\
&\frac{1}{2}K_2|u(t)|^{2p}\left|\nabla u(t)\right|^2+K_3|u(t)|^{2p-1}\left|\nabla u(t)\right|^2\left|u'(t)\right|.
\end{aligned}\tag{2.11}$$

Similarly, the third term of (2.9) is upper bounded by

$$\begin{aligned} &\int_\Omega \Big[\nabla M\left(x,t,|u(t)|^2\right)\cdot\nabla u(x,t)\Big]u'(x,t)dx \leq \\ &K_1|u(t)|^{2p}|\nabla u(t)|\,|u'(t)| \leq KK_1|u(t)|^{2p-1}|\nabla u(t)|^2\,|u'(t)|, \end{aligned} \tag{2.12}$$

where we have used in (2.11) and (2.12) the Cauchy-Schwarz and Sobolev inequalities and K is the continuous embedded constant of $H_0^1(\Omega)$ into $L^2(\Omega)$. Taking into account (2.10)-(2.12) into (2.9) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\left|u'(t)\right|^2+\int_\Omega M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2\,dx\Big\}+\delta\left|u'(t)\right|^2 \leq \\ &K_4|\nabla u(t)|^2\Big\{|u(t)|^{2p-1}|u'(t)|+|u(t)|^{2p}\Big\}, \end{aligned} \tag{2.13}$$

where $K_4 = K_3 + KK_1 + \frac{1}{2}K_2$.

Estimate II. Setting $w = u$ in (2.8) we get

$$\begin{aligned} &\frac{d}{dt}\Big(u'(t),u(t)\Big)-\left|u'(t)\right|^2+\int_\Omega M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2dx+ \\ &\int_\Omega\Big[\nabla M\left(x,t,|u(t)|^2\right)\cdot\nabla u(x,t)\Big]u(x,t)dx+\frac{\delta}{2}\frac{d}{dt}\,|u(t)|^2=0. \end{aligned} \tag{2.14}$$

By using (2.4) in the third and the fourth terms of (2.14) we get

$$\frac{1}{2}\int_\Omega M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2dx \geq \frac{m_0}{2}|\nabla u(t)|^2, \tag{2.15}$$

$$\begin{aligned} &\int_\Omega\Big[\nabla M\left(x,t,|u(t)|^2\right)\cdot\nabla u(x,t)\Big]u(x,t)dx \leq \\ &K_1|u(t)|^{2p}|\nabla u(t)||u(t)| \leq KK_1|u(t)|^{2p}|\nabla u(t)|^2. \end{aligned} \tag{2.16}$$

Therefore, from (2.15) and (2.16) we modify (2.14) to obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\Big(u'(t),u(t)\Big)+\frac{\delta}{2}\,|u(t)|^2\Big\}-\left|u'(t)\right|^2+\frac{m_0}{2}|\nabla u(t)|^2+ \\ &\frac{1}{2}\int_\Omega M\left(x,t,|u(t)|^2\right)|\nabla u(x,t)|^2dx \leq \\ &KK_1|u(t)|^{2p}|\nabla u(t)|^2. \end{aligned} \tag{2.17}$$

If we multiply (2.17) by $\dfrac{\delta}{4}$ and the resulting expression we add to (2.13) yields

$$
\begin{gathered}
\frac{1}{2}\frac{d}{dt}\Big\{ \left|u'(t)\right|^2 + \frac{\delta}{2}\Big(u'(t), u(t)\Big) + \\
\int_\Omega M\left(x, t, |u(t)|^2\right) |\nabla u(x,t)|^2\, dx + \frac{\delta^2}{4}\left|u(t)\right|^2 \Big\} + \frac{3\delta}{4}|u'(t)|^2 + \\
\frac{\delta}{8}\int_\Omega M\left(x, t, |u(t)|^2\right) |\nabla u(x,t)|^2 dx + \\
|\nabla u(t)|^2 \left\{ \frac{m_0\delta}{8} - K_4\left|u(t)\right|^{2p-1}\left|u'(t)\right| - K_5\left|u(t)\right|^{2p} \right\} \leq 0,
\end{gathered}
\tag{2.18}
$$

where $K_5 = K_4 + \dfrac{KK_1\delta}{4}$. By using (2.18) we denote by $\gamma(t)$ and $H(t)$, respectively, the functions

$$
\gamma(t) = K_4\left|u(t)\right|^{2p-1}\left|u'(t)\right| + K_5\left|u(t)\right|^{2p}, \tag{2.19}
$$

$$
\begin{gathered}
\underset{.}{H}(t) = |u'(t)|^2 + \tfrac{\delta}{2}\Big(u'(t), u(t)\Big) + \tfrac{\delta^2}{4}|u(t)|^2 + \\
\int_\Omega M\left(x, t, |u(t)|^2\right) |\nabla u(x,t)|^2 dx.
\end{gathered}
\tag{2.20}
$$

As

$$
\left|\frac{\delta}{2}(u'(t), u(t))\right|_{\mathbb{R}} \leq \frac{1}{2}|u'(t)|^2 + \frac{\delta^2}{8}|u(t)|^2,
$$

then we obtain from definition of H that

$$
H(t) \geq \frac{1}{2}\left|u'(t)\right|^2 + \frac{\delta^2}{8}|u(t)|^2 + \int_\Omega M\left(x, t, |u(t)|^2\right) |\nabla u(x,t)|^2 dx. \tag{2.21}
$$

From (2.21) we get

$$
\left|u'(t)\right| \leq [2H(t)]^{1/2} \quad \text{and} \quad |u(t)| \leq \frac{[8H(t)]^{1/2}}{\delta}.
$$

From this and (2.19) there exists a positive real constant K_6 such that

$$
\gamma(t) \leq K_6[H(t)]^p, \quad \text{with} \quad K_6 = \frac{8^p}{2\delta^{2p}}\Big(K_4\delta + 2K_5\Big). \tag{2.22}
$$

Next, we will prove that $\gamma(t) < m_0\delta/8$ for all $t \geq 0$. In fact, suppose it is not true. As $\gamma(0) \leq K_6[H(0)]^p < m_0\delta/8$, then by continuity of γ there exists t^* such that

$$\gamma(t) < \frac{m_0\delta}{8} \text{ for all } 0 \leq t < t^* \text{ and } \gamma(t^*) = \frac{m_0\delta}{8}.$$

Integrating (2.18) from 0 to t^* yields $H(t^*) \leq H(0)$. From this, $\gamma(t^*) \leq K_6[H(t^*)]^p$
$< m_0\delta/8$. This contradicts $\gamma(t^*) = m_0\delta/8$. Thus, $\gamma(t) < m_0\delta/8$ for all $0 \leq t$. Therefore, we get $H(t) \leq H(0)$ for all $t \geq 0$. From this, inequality (2.21), hypothesis (2.4) and returning the notation u_N we can write

$$\begin{aligned} (u_N) \quad &\text{is bounded in} \quad L^\infty\left(0, T; H_0^1(\Omega)\right), \\ (u_N') \quad &\text{is bounded in} \quad L^\infty\left(0, T; L^2(\Omega)\right). \end{aligned} \tag{2.23}$$

The estimate (2.23) is sufficient to take the limit in the approximate system (2.8). In fact, we will need strong convergence because of the nonlinear term $M(x,t,\lambda)$. As the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is continuous and compact, then (2.23) enables us to apply a compactness result, see for example Aubin [1] or Lions [8], and thus we can extract a subsequence $(u_\mu)_{\mu\in\mathbb{N}}$, of the sequence $(u_N)_{N\in\mathbb{N}}$, such that

$$u_\mu \longrightarrow u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{2.24}$$

Now, setting $N = \mu$ in the approximate system (2.8), we will analyze the limit $\mu \to \infty$. In fact, initially by $(2.23)_2$ we have

$$-\int_0^T\int_\Omega u_\mu'(t)v(x)\theta'(t)dxdt \to -\int_0^T\int_\Omega u'(t)v(x)\theta'(t)dxdt \text{ as } \mu\to\infty \tag{2.25}$$

for all $v \in H_0^1(\Omega))$, $\theta \in \mathcal{D}(\Omega))$, $T > 0$ and for the case $T \to \infty$.

Next consider the convergence in the nonlinear terms of (2.8). First, it will be shown that

$$\begin{aligned} &\int_0^T\int_\Omega M(x,t,|u_\mu(t)|^2)\big[\nabla u_\mu(t)\cdot\nabla v(x)\big]\theta(t)dxdt \to \\ &\qquad \int_0^T\int_\Omega M(x,t,|u(t)|^2)\big[\nabla u(t)\cdot\nabla v(x)\big]\theta(t)dxdt \end{aligned} \tag{2.26}$$

for all $v \in H_0^1(\Omega))$, $\theta \in \mathcal{D}(\Omega))$, $T > 0$ and for the case $T \to \infty$. In fact, by $(2.23)_1$ we have (∇u_μ) is bounded in $L^2(0,T;L^2(\Omega))$ for all $T > 0$. Thus, we can extract a subsequence (∇u_μ) of (∇u_N) such that $\nabla u_\mu \rightharpoonup \zeta$ weakly in $L^2(0,T;L^2(\Omega))$. Besides, we also have $u_\mu \rightharpoonup u$ weakly in $L^2(0,T;L^2(\Omega))$. Thus, $u_\mu \rightharpoonup u$ in $\mathcal{D}'(\Omega\times(0,T))$. Therefore, $\nabla u_\mu \rightharpoonup \nabla u$ in $\mathcal{D}'(\Omega\times(0,T))$, which implies $\zeta = \nabla u$ in $\mathcal{D}'(\Omega\times(0,T))$. From this

$$\nabla u_\mu \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)) \ \text{ as } \ \mu \to \infty.$$

Also from $(2.23)_1$ we have

$$\int_0^T |u_\mu(t) - u(t)|^2\, dt \to 0 \ \text{ as } \ \mu \to \infty. \tag{2.27}$$

From hypothesis (2.4) and estimate $(2.23)_1$ we have

$$\left|M\left(x,t,|u_\mu(t)|^2\right) - M\left(x,t,|u(t)|^2\right)\right|_{\mathbb{R}} = \left|\int_{|u(t)|^2}^{|u_\mu(t)|^2} \frac{\partial}{\partial\lambda} M(x,t,\lambda)d\lambda\right|_{\mathbb{R}} \le$$

$$K_3\left|\int_{|u(t)|^2}^{|u_\mu(t)|^2} |\lambda|^{p-1}\, d\lambda\right|_{\mathbb{R}} \le K_3K_7\left||u_\mu(t)|^2 - |u(t)|^2\right|_{\mathbb{R}} \le CK_3K_7\left|u_\mu(t) - u(t)\right|.$$

From this and denoting by $\widetilde{C}$ the constant $\widetilde{C} = CK_3K_7$, which is independent of μ and $t \ge 0$ even as $t \to \infty$, we obtain

$$\left|M\left(x,t,|u_\mu(t)|^2\right) - M\left(x,t,|u(t)|^2\right)\right|_{\mathbb{R}} \le \widetilde{C}\left|u_\mu(t) - u(t)\right|. \tag{2.28}$$

Now, observe that

$$\int_0^T\!\!\int_\Omega\Big\{M(x,t,|u_\mu(t)|^2)\big[\nabla u_\mu(t)\cdot\nabla v(x)\big] -$$
$$M(x,t,|u(t)|^2)\big[\nabla u(t)\cdot\nabla v(x)\big]\Big\}\theta(t)dx\, dt =$$
$$\int_0^T\int_\Omega\big[M(x,t,|u_\mu(t)|^2) - M(x,t,|u(t)|^2)\big]\big[\nabla u_\mu(t)\cdot\nabla v(x)\big]\theta(t)dx\, dt +$$
$$\int_0^T\int_\Omega M\left(x,t,|u(t)|^2\right)\left\{\big[\nabla u_\mu(t) - \nabla u(t)\big]\cdot\nabla v(x)\right\}\theta(t)dx\, dt.$$

From this and (2.28) we get

$$\int_0^T \int_\Omega \Big\{ \Big[M(x,t,|u_\mu(t)|^2) - M(x,t,|u(t)|^2) \Big] \big[\nabla u(t) \cdot \nabla v(x) \big] \theta(t) \Big\} dx\, dt \leq$$
$$\widetilde{C} \int_0^T |u_\mu(t) - u(t)|\, |\nabla u_\mu(t)|\, |\nabla v(x)| |\theta(t)| dt \leq$$
$$\widetilde{C}\, |\nabla u_\mu(t)|_{L^\infty(0,T;L^2(\Omega))} \Big(\int_0^T |u_\mu(t) - u(t)|^2\, dt \Big)^{1/2} \Big(\int_0^T |\theta(t) \nabla v(t)|^2 dt \Big)^{1/2}.$$

By using $(2.23)_1$, (2.27) in the last inequality and observing that

$$M(x,t,|u(t)|^2) \nabla v(x) \theta(t) \in L^2(0,T;L^2(\Omega)),$$

we obtain

$$\Big| \int_0^T \int_\Omega M(x,t,|u_\mu(t)|^2) \big[\nabla u_\mu(t) \cdot \nabla v(x) \big] \theta(t) dx dt -$$
$$\int_0^T \int_\Omega M(x,t,|u(t)|^2) \big[\nabla u(t) \cdot \nabla v(x) \big] \theta(t) dx dt \Big|_{\mathbb{R}} \to 0 \text{ as } \mu \to \infty.$$

Thus, we get (2.26). The third term in (2.8) is also obtained by using the hypotheses (2.4) and arguing as in the precedent case. The term with linear damping is similar to the case (2.29). That is,

$$\int_0^T (u'_\mu(t), v(x)) \theta(t) dt \to \int_0^T (u'(t), v(x)) \theta(t) dt \text{ as } \mu \to \infty$$

for all $v \in H_0^1(\Omega))$, $\theta \in \mathcal{D}(\Omega))$ for all $T > 0$ and for the case $T \to \infty$.

Taking into account all these convergence into (2.8) we obtain a solution of problem (2.1)-(2.3) in the sense of the Definition 2.1

□

§ 2.2 Exponential decay. We initially will show the exponential decay estimate for the energy associate with the approximate solutions u_N. The estimates obtained in (2.23) allows us to conclude the same result for the solution u.

Theorem 2.2. *Assuming all hypotheses of Theorem 2.1, the energy $E(t)$ of system (2.1)-(2.3) satisfies*

$$E(t) \leq K_{11} \exp\{-K_{10}\, t\} \quad \textit{for all } t \geq 0, \tag{2.29}$$

where

$$E(t)=\frac{1}{2}\int_{\Omega}\Big(\left|u'(x,t)\right|^{2}+M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}\Big)dx, \tag{2.30}$$

K_{10} and K_{11} are positives real constants defined in (2.37) and (2.38) respectively.

Proof. Note that $0\le H(t)\le H(0)$. Thus, from (2.7) and (2.22) we obtain

$$\gamma(t)\le\frac{m_0\delta}{8}\quad\text{for all}\quad t\ge 0. \tag{2.31}$$

From (2.31), (2.18) and (2.20) we have

$$\frac{d}{dt}H(t)+K_8\int_{\Omega}\Big(\left|u'(x,t)\right|^{2}+M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}\Big)dx\le 0, \tag{2.32}$$

where $K_8=\delta/4$. Again, from (2.20) we can write

$$\begin{aligned}H(t)\le\int_{\Omega}\Big[\Big(1+\frac{\delta}{2}\Big)\left|u'(x,t)\right|^{2}+\frac{K\delta}{8}\Big(1+\delta\Big)|\nabla u(x,t)|^{2}+\\ M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}\Big]dx.\end{aligned} \tag{2.33}$$

From $(2.4)_2$ we have

$$\int_{\Omega}M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}dx\ge m_0\int_{\Omega}|\nabla u(x,t)|^{2}dx. \tag{2.34}$$

Taking into account (2.34) in (2.33) yields

$$\begin{aligned}H(t)\le\int_{\Omega}\Big\{\Big(1+\frac{\delta}{2}\Big)\left|u'(x,t)\right|^{2}+\\ \Big[\frac{K\delta}{8m_0}\Big(1+\delta\Big)+1\Big]M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}\Big\}dx.\end{aligned} \tag{2.35}$$

From this and denoting by $K_9=\max\Big\{1+\frac{\delta}{2},\ \frac{K\delta}{8m_0}\Big(1+\delta\Big)+1\Big\}$ we get

$$\frac{1}{K_9}H(t)\le\int_{\Omega}\Big(\left|u'(x,t)\right|^{2}+M\left(x,t,|u(t)|^{2}\right)|\nabla u(x,t)|^{2}\Big)dx. \tag{2.36}$$

Thus, from (2.32) and (2.36) we can write

$$\frac{d}{dt}H(t) + K_{10}H(t) \leq 0 \;\text{ for all }\; t \geq 0 \;\text{ where }\; K_{10} = \frac{K_8}{K_9}. \tag{2.37}$$

As $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then from (2.37) we have

$$H(t) \leq K_{11} \exp\{-K_{10}t\} \;\text{ for all }\; t \geq 0, \tag{2.38}$$

where K_{11} depends on of the initial data.

Finally, from (2.38), (2.21), (2.30) and Banach-Steinhauss theorem we obtain (2.29). This way we have the desired proof of Theorem 2.2

□

§ 2.3 Existence of strong solutions. To show the existence of a strong solution for the problem (2.1)-(2.3), we assume the hypotheses:

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad u_1 \in H_0^1(\Omega), \tag{2.39}$$

$$M(x,t,\lambda) \leq a + b|\lambda|^p, \tag{2.40}$$

with a and b being positive real constants.

Definition 2.2. *A strong solution for initial-boundary value problem (2.1)-(2.3) is a real function $u = u(x,t)$ defined in Q which belongs to the class*

$$\begin{gathered} u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \;\; u' \in L^2(0,T; H_0^1(\Omega)), \\ u'' \in L^\infty(0,T; L^2(\Omega)), \end{gathered} \tag{2.41}$$

for $T > 0$ and satisfies the equation (2.1) a. e. in Q.

Proposition 2.1. *Assuming the hypotheses (2.4), (2.39) and (2.40), then the problem (2.1)-(2.3) has a unique solution in the sense of Definition 2.2.*

Proof. In this paragraph V_N is the subspace of $H_0^1(\Omega)$ spanned by the N first vectors of the hilbertian basis $(w_j)_{j\in\mathbb{N}}$, where w_j is defined as solution of the eigenvalue problem

$$(-\Delta w_j, v) = \lambda_j\,(w_j, v) \quad \text{for all } v \in H_0^1(\Omega) \text{ and } j \in \mathbb{N}.$$

Thus, the approximate problem (2.8) is now defined with such functions w_j. In these conditions we can get the following estimates:

Estimate III. Setting $w = -\Delta u'$ in (2.8) and proceeding as in the estimates I and II, we can write

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{ \left|\nabla u'(t)\right|^2 + \int_\Omega M\left(x,t,|u(t)|^2\right)|\Delta u(x,t)|^2\,dx\Big\} + \delta\left|\nabla u'(t)\right|^2 = \\
&\qquad -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t} M\left(x,t,|u(t)|^2\right)|\Delta u(x,t)|^2\,dx - \\
&\qquad \Big[\int_\Omega \frac{\partial}{\partial \lambda} M\left(x,t,|u(t)|^2\right)|\Delta u(x,t)|^2\,dx\Big]\Big(u'(t), u(t)\Big) \leq \\
&\qquad K_{12}|\Delta u(t)|^2\Big\{|u(t)|^{2p} + |u(t)|^{2p-1}|u'(t)|\Big\} \leq K_{13}|\Delta u(t)|^2.
\end{aligned} \tag{2.42}$$

Integrating (2.42) from 0 to $t \leq T$, using (2.4) and (2.39) we get

$$\left|\nabla u'(t)\right|^2 + m_0\,|\Delta u(t)|^2 + \int_0^T \delta\left|\nabla u'(t)\right|^2 dt \leq K_{14}T. \tag{2.43}$$

Estimate IV. Setting $w = u''$ into (2.8) and by using (2.40) yields

$$\begin{aligned}
\left|u''(t)\right|^2 &= \int_\Omega M\left(x,t,|u(t)|^2\right)\left(\Delta u(x,t), u''(x,t)\right)dx - \delta\left(u'(t), u''(t)\right) \\
&\leq |u''(t)|\Big[\Big(a + b|u(t)|^{2p}\Big)|\Delta u(t)| + \delta|u'(t)|\Big] \\
&\leq K_{15}|u''(t)|,
\end{aligned}$$

where we have used in the last inequality the previous estimates. Thus, we have

$$|u''(t)| \leq K_{16}. \tag{2.44}$$

Therefore, the estimates (2.43) and (2.44) are sufficient to get solutions for the problem (2.1)-(2.3) in the sense of Definition 2.2. This way, the proof of Proposition 2.1 is concluded. The uniqueness of strong solutions can be gotten by the usual energy method

□

3 Carrier equation in a non-cylindrical domain

§ 3.1 Existence of weak solutions. Let Ω_t be a time-dependent non empty, open and bounded subsets of $\mathbb{R}^n$. We consider a non-cylindrical domain

$$\widehat{Q} = \bigcup_{0 \leq t < \infty} \{\Omega_t \times \{t\}\},$$

such that $\widehat{Q} \subset \mathbb{R}^n \times [0, \infty[$ or $\widehat{Q} \subset \mathbb{R}^n \times [0, T[$ for $T > 0$. We also assume $\widehat{Q} \subset Q = \Omega \times]0, T[$, where Q is a cylinder as in Section 2. By Ω_s, for $0 \leq s \leq T$, we represent the sections of $\widehat{Q} \cap \{t = s\}$, Γ_s is the boundary of Ω_s and the lateral boundary of $\widehat{Q}$ is given by $\Sigma_s = \bigcup_{0<s<T} \Gamma_s$.

In the above conditions we look for a real function $u = u(x, t)$ defined for all $(x, t) \in \widehat{Q}$ solving the mixed problem of Carrier type

$$u''(x,t) - \widehat{M}\Big(x, t, \int_{\Omega_t} |u(x,t)|^2 dx\Big) \Delta u(x,t) + \delta u'(x,t) = 0 \text{ in } \widehat{Q}, \tag{3.1}$$

$$u(x,t) = 0 \text{ on } \widehat{\Sigma}, \tag{3.2}$$

$$u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x) \text{ in } \Omega_0, \tag{3.3}$$

where $\widehat{M}(x, t, \lambda)$ is the restriction of $M(x, t, \lambda)$ to $(x, t) \in \widehat{Q}$ with $\lambda \geq 0$ and $M(x, t, \lambda)$ is the function of Section 2 defined on the cylinder $Q = \Omega \times]0, T[$.

To investigate the solutions of the mixed problem (3.1)-(3.3) it is necessary to assume some hypotheses on $\widehat{Q}$, as was made by Lions [8] in the case of nonlinear wave equation of the type $u'' - \Delta u + |u|^\rho u = 0$. In fact, one condition is about the geometry of $\widehat{Q}$ and the other one is on the regularity of its lateral boundary $\widehat{\Sigma}$. Therefore, we assume:

$$\text{The sections } \Omega_t = \widehat{Q} \cap \{t = s\} \text{ are increasing with } s. \tag{3.4}$$

This condition means that, if $s_1 \leq s_2$ the projections of Ω_{s_1} and Ω_{s_2} on the hyperplane $t = 0$ are increasing, that is,

$$\text{proj}\Big|_{t\,=\,0} \Omega_{s_1} \subseteq \text{proj}\Big|_{t\,=\,0} \Omega_{s_2} \quad \text{if} \quad s_1 \leq s_2.$$

We assumed previously $\widehat{Q} \subset Q = \Omega \times]0, T[$ and we need of the following regularity condition:

$$\text{If } v \in H_0^1(\Omega) \text{ and } v = 0 \text{ a. e. in } \Omega - \Omega_t, \text{ then } v \in H_0^1(\Omega_t). \tag{3.5}$$

Note that $v = 0$ a. e. in $\Omega - \Omega_t$ is the restriction of v to $\Omega - \Omega_t$. Observe that by trace theorem, if Γ_t is of class C^1, $v \in H_0^1(\Omega)$ and $v = 0$ a. e. in $\Omega - \Omega_t$, then $v \in H_0^1(\Omega_t)$. Due to the characteristics of the penalty method it is only possible to obtain estimates that determine weak solutions for the problem (3.1)-(3.3).

Definition 3.1. *A weak solution for the initial-boundary value problem (3.1)-(3.3) is a real valued function $u = u(x, t)$ defined in $\widehat{Q}$ such that*

$$u \in L^\infty(0, T; H_0^1(\Omega_t)), \quad u' \in L^\infty(0, T; L^2(\Omega_t)) \ \textit{for}\ T > 0,$$

u satisfies the identity integral

$$-\int_0^T \int_{\Omega_t} u'(x, t)\phi'(x, t)dxdt+$$

$$\int_0^T \int_{\Omega_t} M\Big(x, t, \int_{\Omega_t} |u(x, t)|^2 dx\Big)\Big[\nabla u(x, t) \cdot \nabla \phi(x, t)\Big] dxdt+$$

$$\int_0^T \int_{\Omega_t} \Big[\nabla M\Big(x, t, \int_{\Omega_t} |u(x, t)|^2 dx\Big) \cdot \nabla u(x, t)\Big] \phi(x, t)dxdt+$$

$$\delta \int_0^T \int_{\Omega_t} u'(x, t)\phi(x, t)dxdt = 0 \ \textit{for all}\ \phi \ \textit{such that}$$

$$\phi \in L^2(0, T; H_0^1(\Omega_t)), \ \ \phi' \in L^2(0, T; L^2(\Omega_t)) \ \textit{with}\ \phi(x, 0) = \phi(x, T) = 0.$$

Moreover, u satisfies the initial conditions

$$u(x,0)=u_0(x) \ \textit{ and } \ u'(x,0)=u_1(x) \ \textit{ for } \ x\in\Omega_0.$$

Theorem 3.1. *Suppose $u_0\in H_0^1(\Omega_0)$ and $u_1\in L^2(\Omega_0)$ such that*

$$\widetilde{K}_6[H(0)]^p<\frac{m_0\delta\theta}{2}, \tag{3.6}$$

where $H(t)$ and $\widetilde{K}_6$ are defined in (3.13) and (3.15) respectively, and θ is a real constant such that $0<\theta<1$. Then there exists at least one function u which is weak solution of the initial-boundary value problem (3.1)-(3.3), provided the hypothesis (2.4) holds.

Proof. We apply the penalty method to transform the non-cylindrical problem in $\widehat{Q}$ into a cylindrical problem in Q, and then we employ the Faedo-Galerkin method. First, we consider the characteristic function

$$\chi(x,t)=\begin{cases} 1 \ \text{ in } \ \Omega\times]0,T[\backslash\widehat{Q}\cup\{\Omega_0\times\{0\}\}, \\ 0 \ \text{ in } \ \widehat{Q}\cup\{\Omega_0\times\{0\}\}, \end{cases}$$

which belongs to $L^\infty(\Omega\times]0,T[)$. Denoting by $\widetilde{u}_0$ and $\widetilde{u}_1$ the extensions of u_0 and u_1 to Ω defined zero outside of $\Omega-\Omega_0$, we have $\widetilde{u}_0\in H_0^1(\Omega)$ and $\widetilde{u}_1\in L^2(\Omega)$. Thus, we define the penalized problem as follows:

Given $\epsilon>0$ we look for $u_\epsilon=u_\epsilon(x,t)$ for $(x,t)\in Q$ such that

$$u_\epsilon\in L^\infty(0,T;H_0^1(\Omega)) \ \text{ and } \ u'_\epsilon\in L^\infty(0,T;L^2(\Omega)) \ \text{ for } \ T>0,$$

u_ϵ satisfies the integral identity

$$\begin{aligned} -\int_Q u'_\epsilon(x,t)\phi'(x,t)dxdt+\int_Q M\left(x,t,|u_\epsilon(t)|^2\right)\Big[\nabla u_\epsilon(x,t)\cdot\nabla\phi(x,t)\Big]dxdt+ \\ \int_Q\Big[\nabla M\left(x,t,|u_\epsilon(t)|^2\right)\cdot\nabla u_\epsilon(x,t)\Big]\phi(x,t)dxdt+ \\ \delta\int_Q u'_\epsilon(x,t)\phi(x,t)dxdt+\frac{1}{\epsilon}\int_Q\chi(x,t)u'_\epsilon(x,t)\phi(x,t)dxdt=0, \end{aligned} \tag{3.7}$$

for all ϕ such that $\phi \in L^2(0,T;H_0^1(\Omega_t))$, $\phi' \in L^2(0,T;L^2(\Omega_t))$ with $\phi(x,0)=\phi(x,T)=0$. Besides, u satisfies the initial conditions

$$u_\epsilon(x,0)=\widetilde{u}_0(x) \text{ and } u'_\epsilon(x,0)=\widetilde{u}_1(x) \text{ for } x\in\Omega_0. \tag{3.8}$$

Note that (3.7) and (3.8) are the version of Definition 3.1 for a weak solution for the penalized problem which is cylindrical. Thus, we can employ the Faedo-Galerkin's method. In fact, considering a hilbertian basis with vectors $\left(w_j\right)_{j\in\mathbb{N}}$, $w_i\in H_0^1(\Omega)$ such that $w_1=\widetilde{u}_0$, we look for $u_{\epsilon N}=\sum\limits_{j=1}^{N} g_{\epsilon jN}(t)w_j(x)\in V_N$, for $\epsilon>0$ fixed, such that

$$\begin{gathered}\int_\Omega u''_{\epsilon N}(x,t)w(x)dx+\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)\Big[\nabla u_{\epsilon N}(x,t)\cdot\nabla w(x)\Big]dx+\\ \int_\Omega\Big[\nabla M\left(x,t,|u_{\epsilon N}(t)|^2\right)\cdot\nabla u_{\epsilon N}(x,t)\Big]w(x)dx+ \qquad (3.9)\\ \delta\int_\Omega u'_{\epsilon N}(x,t)w(x)dx+\frac{1}{\epsilon}\int_\Omega\chi(x,t)u'_{\epsilon N}(x,t)w(x)dx=0 \text{ for all } w\in V_N,\end{gathered}$$

$$u_{\epsilon N}(x,0)=\widetilde{u}_0 \text{ and } u'_{\epsilon N}(x,0)=u_{1N}\to\widetilde{u}_1 \text{ in } L^2(\Omega). \tag{3.10}$$

As before, we now will obtain estimates for the solutions $u_{\epsilon N}$ of the problem (3.9), (3.10). These estimates are those obtained in the Section 2, § 2.1 . Thus, we set $w=u'_{\epsilon N}(x,t)$ and $w=u_{\epsilon N}(x,t)$ in (3.9) and work as in Section 2, § 2.1 . After that, multiplying the second estimate, which is obtained from substitution of w by $u_{\epsilon N}(x,t)$, by $\delta\theta$, and we add to the first one to obtain

$$\frac{1}{2}\frac{d}{dt}\Big\{ \left|u'_{\epsilon N}(t)\right|^2 + 2\delta\theta\left(u'_{\epsilon N}(t), u_{\epsilon N}(t)\right) +$$
$$\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx + \delta^2\theta\,|u_{\epsilon N}(t)|^2 \Big\} +$$
$$\delta(1-\theta)\left|u'_{\epsilon N}(t)\right|^2 + \frac{\delta\theta}{2}\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx + \tag{3.11}$$
$$|\nabla u_{\epsilon N}(x,t)|^2\Big[\frac{m_0\delta\theta}{2} - K_4\,|u_{\epsilon N}(t)|^{2p-1}\left|u'_{\epsilon N}(t)\right| -$$
$$\left(K_4 + \delta\theta K K_1\,|u_{\epsilon N}(t)|^{2p}\right)\Big] + \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u'_{\epsilon N}(x,t)\right|^2 dx +$$
$$\frac{\delta\theta}{\epsilon}\int_\Omega \chi(x,t)u'_{\epsilon N}(x,t)u_{\epsilon N}(x,t)dx \le 0 \ \text{ for all } \ t \ge 0,$$

We denote by $\gamma(t)$ and $H(t)$ the functions

$$\gamma(t) = K_4\,|u_{\epsilon N}(t)|^{2p-1}\,|u'_{\epsilon N}(t)| + (K_4 + \delta\theta K K_1)\,|u_{\epsilon N}(t)|^{2p}, \tag{3.12}$$

$$H(t) = \left|u'_{\epsilon N}(t)\right|^2 + 2\delta\theta\left(u'_{\epsilon N}(t), u_{\epsilon N}(t)\right) +$$
$$\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx + \delta^2\theta\,|u_{\epsilon N}(t)|^2. \tag{3.13}$$

As

$$\left|2\delta\theta(u'_{\epsilon N}(t), u_{\epsilon N}(t))\right|_{\mathbb{R}} \le \theta|u'_{\epsilon N}(t)|^2 + \delta^2\theta|u_{\epsilon N}(t)|^2,$$

then from (3.13) we get

$$\begin{aligned} H(t) &\ge (1-\theta)\left|u'_{\epsilon N}(t)\right|^2 + \int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx \\ &\ge (1-\theta)\left|u'_{\epsilon N}(t)\right|^2 + \frac{m_0}{2K}|u_{\epsilon N}(t)|^2 \\ &+ \frac{1}{2}\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx. \end{aligned} \tag{3.14}$$

From this we get

$$\left|u'_{\epsilon N}(t)\right| \le \left[\frac{1}{(1-\theta)}H(t)\right]^{1/2} \ \text{ and } \ |u_{\epsilon N}(t)| \le \left[\frac{2K}{m_0}H(t)\right]^{1/2}.$$

By using these two inequalities in (3.12) we obtain

$$\gamma(t) \le \widetilde{K}_6[H(t)]^p,$$

$$\widetilde{K}_6 = K_4\Big(\frac{2K}{m_0}\Big)^{p-1/2}\frac{1}{(1-\theta)^{1/2}} + (K_4+\delta\theta K K_1)\left(\frac{2K}{m_0}\right)^{p}. \tag{3.15}$$

From (3.11)-(3.13) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}H(t) + |\nabla u_{\epsilon N}(x,t)|^2\left[\frac{m_0\delta\theta}{2}-\gamma(t)\right] + \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u'_{\epsilon N}(x,t)\right|^2 dx +\\ \frac{\delta\theta}{\epsilon}\int_\Omega \chi(x,t)u'_{\epsilon N}(x,t)u_{\epsilon N}(x,t)dx \le 0 \ \text{ for all } \ t\ge 0.\end{aligned} \tag{3.16}$$

From this, hypothesis (3.6) and as

$$\int_0^t\int_\Omega \chi(x,t)u'_{\epsilon N}(x,t)u_{\epsilon N}(x,t)dxds \ge 0,$$

for this last inequality, see for instance Nakao-Narazaki [10], we obtain

$$H(t)\le H(0) \ \text{ for all } \ t>0 \ \text{ and } \ \epsilon>0. \tag{3.17}$$

Therefore, we obtain from (3.17) estimates independent of N for each $\epsilon>0$ that permit the passing limit as $N\to\infty$ in the approximate equations (3.9). Thus, u_ϵ is a weak solution of (3.7) and (3.8) for each $0<\epsilon<1$.

The next step is to obtain estimates on u_ϵ in order to take the limit $\epsilon\to 0^+$ and conclude the proof of Theorem 3.1. In fact, u_ϵ is defined by the following convergence as $N\to\infty$:

$$\begin{aligned}&u_{\epsilon N}\rightharpoonup u_\epsilon \ \text{ weak star in } \ L^\infty\left(0,T;H_0^1(\Omega)\right),\\ &u'_{\epsilon N}\rightharpoonup u'_\epsilon \ \text{ weak star in } \ L^\infty\left(0,T;L^2(\Omega)\right),\\ &u_{\epsilon N}\to u_\epsilon \ \text{ strongly in } \ L^2\left(0,T;L^2(\Omega)\right),\\ &\chi u'_{\epsilon N}\rightharpoonup \chi u'_\epsilon \ \text{ weak star in } \ L^\infty\left(0,T;L^2(\Omega)\right).\end{aligned} \tag{3.18}$$

From convergence $(3.18)_1$, $(3.18)_2$ and the Banach-Steinhauss theorem we obtain a net $(u_\epsilon)_{0<\epsilon<1}$ and a function $\omega: Q\to\mathbb{R}$ satisfying

$$\begin{aligned}&u_\epsilon\rightharpoonup \omega \ \text{ weak star in } \ L^\infty\left(0,T;H_0^1(\Omega)\right) \ \text{ as } \ \epsilon\to 0,\\ &u'_\epsilon\rightharpoonup \omega' \ \text{ weak star in } \ L^\infty\left(0,T;L^2(\Omega)\right) \ \text{ as } \ \epsilon\to 0,\\ &u_\epsilon\to\omega \ \text{ strongly in } \ L^2\left(0,T;L^2(\Omega)\right) \ \text{ as } \ \epsilon\to 0 \ \text{ and a. e. in } \ Q.\end{aligned} \tag{3.19}$$

From (3.16) we get

$$\int_{\Omega\times]0,T[} \chi \left|u'_{\epsilon N}(x,t)\right|^2 dxdt \leq \epsilon K_{16},$$

where K_{16} is a positive real constant independent of ϵ and N. Thus, from $(3.18)_4$ and the Banach-Steinhauss theorem we have $\|\chi u'_\epsilon\|_{L^2(0,T;L^2(\Omega))} < \epsilon K_{16}$. Then, we affirm that $\chi u'_\epsilon$ converges strong to zero in $L^2\left(0,T;L^2(\Omega)\right)$. Therefore, from $(3.19)_2$ we get

$$\int_0^T \int_\Omega \chi(x,t)|\omega'(x,t)|^2 dxdt = 0. \tag{3.20}$$

From (3.20) we have $\chi(x,t)\omega'(x,t) = 0$ a. e. in $\Omega\times]0,T[= Q$. This implies $\omega'(x,t) = 0$ for all $(x,t) \in Q - \widehat{Q} \cup \Omega_0 \times \{0\}$. Since $\widehat{Q}$ is increasing, we have

$$\int_0^t \omega'(x,s)ds = 0 \text{ for all } 0 < t < T \text{ and } x \in \Omega - \Omega_0.$$

As $w(x,0) = \widetilde{u}_0(x) = 0$ in $\Omega - \Omega_0$, because ω is a solution in Q, then

$$\omega(x,t) = 0 \text{ a. e. in } \Omega - \Omega_t \text{ for } 0 < t < T. \tag{3.21}$$

From (3.21) and since u is the restriction of ω to $\widehat{Q}$, then we get

$$u \text{ belongs to } L^\infty\left(0,T;H_0^1(\Omega_t)\right), \tag{3.22}$$

where we have used to obtain (3.22) the hypothesis of regularity on $\widehat{Q}$ established in (3.5), and also $(3.19)_1$. Arguing as before, if u' is the restriction of ω' to $\widehat{Q}$ then we get by using $(3.19)_2$

$$u' \text{ belongs to } L^\infty\left(0,T;L^2(\Omega_t)\right). \tag{3.23}$$

Thus, by restriction to $\widehat{Q}$ of the penalized problem (3.7) and (3.8) we obtain

$$
\begin{gathered}
-\int_Q u'_\epsilon(x,t)\phi'(x,t)dxdt+ \\
\int_Q M\left(x,t,|u_\epsilon(t)|^2\right)\Big[\nabla u_\epsilon(x,t)\cdot\nabla\phi(x,t)\Big]dxdt+ \\
\int_Q \Big[\nabla M\left(x,t,|u_\epsilon(t)|^2\right)\cdot\nabla u_\epsilon(x,t)\Big]\phi(x,t)dxdt+ \\
\delta\int_Q u'_\epsilon(x,t)\phi(x,t)dxdt=0, \\
u_\epsilon(x,0)=\widetilde{u}_0(x),\;\; u'_\epsilon(x,0)=\widetilde{u}_1(x)\;\text{ for }\; x\in\Omega_0,
\end{gathered}
\tag{3.24}
$$

for all $\phi\in L^2(0,T;H^1_0(\Omega_t))$, $\phi'\in L^2(0,T;L^2(\Omega_t))$ with $\phi(x,0)=\phi(x,T)=0$. Therefore, $\widehat{M}(x,t,\lambda)$ is the restriction of $M(x,t,\lambda)$ to $\widehat{Q}$, for $\lambda>0$, and u the restriction of ω to $\widehat{Q}$. From the convergence $(3.19)_1$-$(3.19)_3$ and making $\epsilon\to 0$ in (3.2) it implies that u is a solution of (3.1)-(3.3) in the sense of Definition 3.1. Thus the proof of Theorem 3.1 is concluded

□

§ 3.2 Exponential decay. The exponential decay for the energy of system (3.1)-(3.3), i. e., inside of the noncylindrical domain, is obtained arguing as in the Section 2, § 2.2, however, we have to make the following hypothesis on δ and θ:

$$\delta=\frac{1}{\theta}>1. \tag{3.25}$$

Thus, we can state the following result

Theorem 3.2. *Assuming all the hypotheses of Theorem 3.1 and (3.25), then the energy E associated with the weak solutions of (3.1)-(3.3) satisfies, for suitable α_0, $\alpha_1>0$,*

$$E(t)\le\alpha_0\exp\{-\alpha_1 t\}\;\; \textit{for all}\;\; t\ge 0, \tag{3.26}$$

where $E(t)$ is defined in (2.30).

Proof. Initially, note that the existence of solutions for the problem (3.1)-(3.3) with the restriction (3.25) can be made as in Theorem 3.1 adding to the penalized problem (3.7) the term

$$\frac{1}{\epsilon}\int_Q \chi(x,t)u_\epsilon(x,t)\phi(x,t)dxdt.$$

Therefore, since $\delta\theta = 1$, we get from (3.11) and (3.14) that

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}H(t) + (\delta-1)\left|u'_{\epsilon N}(t)\right|^2 + \frac{1}{2}\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2\,dx + \\ \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u'_{\epsilon N}(x,t)\right|^2 dx + \frac{2}{\epsilon}\int_\Omega \chi(x,t)u'_{\epsilon N}(x,t)u_{\epsilon N}(x,t)dx + \\ \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u_{\epsilon N}(x,t)\right|^2 dx \le 0 \text{ for all } t \ge 0.\end{gathered} \tag{3.27}$$

Still from (3.13) and (2.4) we can write

$$\begin{aligned} H(t) &\le 2\left|u'_{\epsilon N}(t)\right|^2 + (\delta+1)\left|u_{\epsilon N}(t)\right|^2 + \int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx \\ &\le 2\left|u'_{\epsilon N}(t)\right|^2 + (\delta+1)K\int_\Omega |\nabla u_{\epsilon N}(x,t)|^2\,dx \\ &+ \int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx \qquad (3.28) \\ &\le 2\left|u'_{\epsilon N}(t)\right|^2 + \Big(\frac{(\delta+1)K}{m_0}+1\Big)\int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx \\ &\le K_{17}\Big(\left|u'_{\epsilon N}(t)\right|^2 + \int_\Omega M\left(x,t,|u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx\Big), \end{aligned}$$

$K_{17} = \max\Big\{2,\ \dfrac{(\delta+1)K}{m_0}+1\Big\}$. Besides that,

$$\begin{gathered}\left|\frac{2}{\epsilon}\int_\Omega \chi(x,t)u'_{\epsilon N}(x,t)u_{\epsilon N}(x,t)dx\right|_{\mathbb{R}} \le \\ \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u'_{\epsilon N}(x,t)\right|^2 dx + \frac{1}{\epsilon}\int_\Omega \chi(x,t)\left|u_{\epsilon N}(x,t)\right|^2 dx.\end{gathered}$$

Taking into account this inequality and (3.28) into (3.27) yields

$$\frac{d}{dt}H(t) + K_{18}H(t) \le 0 \text{ for all } t \ge 0,$$

where $K_{18} = \min\{2(\delta - 1),\ \ 1\}/K_{17}$. From this

$$H(t) \leq H(0)\exp\{-K_{18}t\} \ \text{ for all } \ t \geq 0. \tag{3.29}$$

On the other hand,

$$\begin{aligned} E_{\epsilon N}(t) &= \frac{1}{2}\left|u'_{\epsilon N}(t)\right|^2 + \frac{1}{2}\int_{\Omega} M\left(x, t, |u_{\epsilon N}(t)|^2\right)|\nabla u_{\epsilon N}(x,t)|^2 dx \\ &\leq K_{19}H(t) \ \text{ for all } \ t \geq 0, \end{aligned} \tag{3.30}$$

where $K_{19} = 1/\min\{2(1-\theta),\ \ 1\}$. Therefore, from (3.29), (3.30) we have

$$E_{\epsilon N}(t) \leq K_{19}H(0)\exp\{-K_{18}t\} \ \text{ for all } \ t \geq 0.$$

From this, and the Banach-Steinhauss theorem, and denoting $K_{19}H(0)$ and K_{18} by α_0 and α_1, respectively, we get the inequality (3.26)

□

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