Matemática Contemporânea, Vol. 32, 157–178
http://doi.org/10.21711/231766362007/rmc328

GLOBAL AND DECAY OF SOLUTIONS OF A DAMPED KIRCHHOFF-CARRIER EQUATION IN BANACH SPACES

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In Homage to Professor L. A. Medeiros by his Eightieth Birthday

Abstract

This paper is concerned with the study of the existence and the decay of solutions of the following problem:

$$\left| \begin{array}{l} Bu''(t) + M\left(\|u(t)\|_W^{\beta}\right) Au(t) + \delta Bu'(t) = 0, \text{ in } V', t > 0, \\ u(0) = u^0 \ \ , u'(0) = u^1 \ (u^0 \neq 0), \end{array} \right.$$

where A and B are symmetric linear operators from a Hilbert space V into its dual V' satisfying $\langle Bv, v\rangle > 0,\ v \neq 0, \langle Av, v\rangle \geq \gamma \|v\|_V^2, \gamma > 0$; W a Banach space with V continuously embedding in W; β a real number with $\beta \geq 1$, $M(\xi)$ a smooth function with $M(\xi) \geq 0$, and δ a positive real number

AMS Subject Classification: 35L70,35B35.
Key Words and Phrases: Kirchhoff-Carrier Equation, global solutions, decay of solutions.

1 Introduction

Let V be a real separable Hilbert space whose dual is denoted by V' and W a Banach space with V continuously embedding in W. Consider two symmetric linear operators $A, B : V \to V'$ such that

$$\langle Av, v\rangle \geq \gamma \|v\|_V^2\,,\ \forall v \in V\ (\gamma \text{ positive constant});$$

$$\langle Bv, v\rangle > 0,\ \forall v \in V,\ v \neq 0;$$

and a smooth function $M(\xi)$ with

$$M(\xi) \geq 0,\ \forall\, \xi \geq 0.$$

Consider also two real numbers $\beta \geq 1$ and $\delta > 0$. In this conditions we have the following problem:

$$(*)\left|\begin{array}{l} Bu''(t) + M\left(\|u(t)\|_W^\beta\right) Au(t) + \delta Bu'(t) = 0,\ \text{in } V', t > 0,\\ u\,(0) = u^0,\ \ u'(0) = u^1\ (u^0 \neq 0). \end{array}\right.$$

Equation in (*) is a damped abstract version in Banach spaces of the Kirchhoff equation [14] and the Carrier equation [5]. When $B = I, \beta = 2$,W is a Hilbert space and $\delta \geq 0$, there is an extensive literature on this problem (cf. Medeiros, Limaco and Menezes [22]).

The existence of local solutions of problem (*) has been obtained by the Authors in [13].

In this paper we study the existence of global solutions of (*) when $M(\xi) \geq 0$ and the exponential decay of solutions of (*) when $M(\xi) \geq m_0 > 0$. In Section 5, we give some examples.

To obtain global solutions we use the prolongation method and in the decay of solutions, the Lyapunov approach, cf. Komornik and Zuazua [15]. In both cases it is fundamental an appropriate characterization of the derivative of $M(\|u(t)\|_W^\beta)$. We use various results obtained in [13] and in S. S. Souza and the third A. [27].

2 Notations and Main Results

Let V be a real separable Hilbert space whose dual is denoted by V'. Consider two linear operators $A, B : V \to V'$ satisfying

$$(H1) \quad \begin{aligned} &\langle Au, v\rangle = \langle u, Av\rangle, \forall u, v \in V; \\ &\langle Au, u\rangle \geq \gamma \|u\|_V^2, \forall u \in V \ (\gamma \text{ positive constant}); \\ &\langle Bu, v\rangle = \langle u, Bv\rangle, \forall u, v \in V; \\ &\langle Bu, u\rangle > 0, \forall u \in V, u \neq 0. \end{aligned}$$

Here $\langle\ ,\ \rangle$ denotes the duality pairing between V' and V. We have that the scalar product $((u, v)) = \langle Au, v\rangle$ defines a norm $\|u\| = ((u, u))^{1/2}$ in V which is equivalent to the norm $\|u\|_V$. The space V will be equipped with the scalar product $((u, v))$ and norm $\|u\|$.

The bilinear form

$$(u, v) = \langle Bu, v\rangle, \ \forall u, v \in V$$

is a scalar product in V. We denote by H the completed of the space $\{V, (u, v)\}$. The scalar product of the Hilbert space H will be denoted also by (u, v) and its norm by $|u|$. We have that

V is densely and continuously embedding in H.

Consider the coercive self-adjoint operator S of H determined by the triplet $\{V, H, ((u, v))\}$. We have:

$$(Su, v) = ((u, v)) = \langle Au, v\rangle, \ \forall u \in D(S), \ \forall v \in V; \tag{2.1}$$

$$Au = BSu \text{ in } V', \ \forall u \in D(S^{3/2}). \tag{2.2}$$

Identify H with its dual H'. Then expression (2.1) says that A is the extension of S to the space V.

Represent by W a Banach space whose dual

(H2) W' is strictly convex.

Denote by $\theta \geq 0$ a real number and by $(E_\lambda)_{\lambda \in \mathbb{R}}$ the spectral family of S. Then $D(S^\theta)$ is the Hilbert space

$$D\left(S^\theta\right) = \left\{u \in H; \int_0^\infty \lambda^{2\theta} d(E_\lambda u, u) < \infty\right\}$$

equipped with the scalar product

$$(u, v)_{D(S^\theta)} = \left(S^\theta u, S^\theta v\right).$$

Fix $\alpha \geq 0$ a real number. Assume that $D(S^{\alpha+1})$ is continuously embedding in W, that is, there exist a positive number k_0, such that

(H3) $$\|u\|_W \leq k_0 \|u\|_{D(S^{\alpha+1})}, \forall\, u \in D(S^{\alpha+1}).$$

Consider a function $M(\xi)$ and $\beta \geq 1$ a real number satisfying

$$(H4)\left|\begin{array}{l} M \in C^0([0,\infty[),\ M(0) = 0,\ M(\xi) > 0,\ \forall\, \xi > 0; \\ M \in C^1(]0,\infty[); \\ |M'(\xi)|\, \lambda^{1-1/\beta} \leq C_0 M^{1/2}(\xi),\ \forall\, \xi > 0\ (C_0 \text{ positive constant}). \end{array}\right.$$

Under the above considerations, we have the following result:

Theorem 2.1 *Assume hypotheses (H1)-(H4) with $\alpha \geq 0,\ \beta \geq 1$. Consider $\delta > 0$ a real number and*

(H5) $$u^0 \in D(S^{2\alpha+5/2}), u^1 \in D(S^{2\alpha+2}), u^0 \neq 0$$

satisfying

(H6) $$\beta C_0 k_0 \left[\frac{\left|S^{\alpha+1}u^1\right|^2}{M\left(\|u^0\|_W^\beta\right)} + \left|S^{\alpha+3/2}u^0\right|^2\right]^{1/2} < \delta.$$

(a) *Then there exists a function u in the class*

$$\left| \begin{array}{l} u \in L^{\infty}(0,\infty; D(S^{\alpha+3/2})), \\ u' \in L^{\infty}(0,\infty; D(S^{\alpha+1})), \\ u'' \in L^{\infty}(0,\infty; D(S^{\alpha+1/2})) \end{array} \right. \tag{2.3}$$

satisfying

$$(P) \quad \left| \begin{array}{l} u'' + M\left(\|u\|_W^{\beta}\right) Su + \delta u' = 0 \text{ in } L^{\infty}(0,\infty; D(S^{\alpha+1/2})), \\ u(0) = u^0 \ \ , u'(0) = u^1. \end{array} \right.$$

(b) *Let $\mathcal{M}$ be the set constituted by the real numbers $T > 0$ such that there exists a unique function u in the class (2.3) with u solution of (P) in [0, T] and $\|u(t)\|_W > 0$ for all $t \in [0,T]$. Let T_{max} be the supremum of the $T \in \mathcal{M}$. Then $\mathcal{M} \neq \emptyset$ and the solution u obtained in* **(a)** *verifies*

$$u \in L^{\infty}_{loc}(0, T_{max}; D(S^{2\alpha+5/2})),$$
$$u' \in L^{\infty}_{loc}(0, T_{max}; D(S^{2\alpha+2})),$$
$$u'' \in L^{\infty}_{loc}(0, T_{max}; D(S^{\alpha+1/2}))$$

and

$$\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^{\beta}\right)} + \left|S^{\alpha+3/2}u(t)\right|^2 \leq \frac{\left|S^{\alpha+1}u^1\right|^2}{M\left(\|u^0\|_W^{\beta}\right)} + \left|S^{\alpha+3/2}u^0\right|^2, \ 0 \leq t < T_{max}.$$

And if T_{max} is finite,

$$u(t) = 0, \text{ for } \ t \geq T_{max}.$$

In order to obtain the decay of solutions of problem (P), we make the following considerations.

Consider a function $M(\xi)$ and $\beta > 1$ a real number satisfying

$$(H7) \quad \left| \begin{array}{l} M \in C^1([0,\infty[); \\ M(\xi) \geq m_0 > 0, \ \forall\, \xi \geq 0 \ (m_0 \text{ constant}); \\ M'(\xi) \geq 0, \ \forall\, \xi \geq 0; \\ |M'(\xi)|\, \lambda^{1-1/\beta} \leq C_1 M(\xi), \ \forall\, \xi \geq 0 \ (C_1 \text{ positive constant}). \end{array} \right.$$

As V is continuously embedding in H, we have:

$$(Su,u) = \|u\|^2 \geq C_*^2 \, |u|^2 \, , \; \forall \; u \in D(S) \; (C_* \text{ positive constant}).$$

This implies

$$\left|S^{\alpha+1}u\right|^2 \leq \frac{1}{C_*^2} \left|S^{\alpha+3/2}u\right|^2, \; \forall \; u \in D(S^{\alpha+3/2}). \tag{2.4}$$

We introduce the constant $k_1 > 0$ verifying

$$\|u\|_W \leq k_1 \left|S^{\alpha+3/2}u\right|^2, \; \forall \; u \in D(S^{\alpha+3/2}). \tag{2.5}$$

Let $\varphi(t)$ be the function

$$\varphi(t) = \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} + \left|S^{\alpha+3/2}u(t)\right|^2, \; t \geq 0. \tag{2.6}$$

Under the above considerations, we obtain :

Theorem 2.2 *Assume hypotheses (H1) - (H3), (H7) with* $\alpha \geq 0$ *and* $\beta > 1$. *Consider a real number* $\delta > 0$ *and*

(H8) $u^0 \in D(S^{\alpha+3/2}), u^1 \in D(S^{\alpha+1})$;

(H9) $\beta C_1 k_0 M^{1/2}\left(k_1^\beta \varphi^{\beta/2}(0)\right)\varphi^{1/2}(0) < \delta$;

where

$$\varphi(0) = \frac{\left|S^{\alpha+1}u^1\right|^2}{M\left(\|u^0\|_W^\beta\right)} + \left|S^{\alpha+3/2}u^0\right|^2.$$

Then there exists a unique function u in the class (2.3) such that u is solution of Problem (P). Furthermore if

(H10) $$\frac{\left|S^{\alpha+1}u^1\right|^2}{M\left(\|u^0\|_W^\beta\right)} + \left|S^{\alpha+3/2}u^0\right|^2$$

$$< \min\left[\frac{\delta^2}{4\beta^2 C_1^2 k_0^2 M(k_1^\beta \varphi^{\beta/2}(0)}, \frac{m_0 C_*^4}{4\beta^2 C_1^2 k_0^2}, \frac{m_0 C_*^4}{4\delta^2 \beta^2 C_1^2 k_0^2}\right]$$

where C_* were defined in (2.4), we have

$$\varphi(t) \leq 3\varphi(0)e^{-\frac{\tau_0}{3}t}, \quad t \geq 0, \tag{2.7}$$

where

$$\tau_0 = min(\delta, \epsilon_0), \ \epsilon_0 = min(\frac{1}{2P_0}, \frac{\delta}{4}, 1), \ P_0 = \frac{1}{C_* m_0^{1/2}} + \frac{\delta}{2C_* m_0} \tag{2.8}$$

and $\varphi(t)$ was defined in (2.6).

Corollary 2.2 *If $K = \sup_{0\leq t<\infty} M\left(\|u(t)\|_W^\beta\right)$ then (2.7) implies*

$$E(t) \leq 3K\varphi(0)e^{-\frac{\tau_0}{3}t}, \quad t \geq 0,$$

where

$$E(t) = \left|S^{\alpha+1}u'(t)\right|^2 + M\left(\|u(t)\|_W^\beta\right)\left|S^{\alpha+3/2}u(t)\right|^2, \ t \geq 0.$$

Remark 2.1 *By property (2.2), we have that the equations*

$$Bu''(t) + M\left(\|u(t)\|_W^\beta\right)Au(t) + \delta Bu'(t) = 0 \ in \ V', \ t > 0$$

and

$$u''(t) + M\left(\|u(t)\|_W^\beta\right)Su(t) + \delta u'(t) = 0 \ in \ D(S^{3/2}), \ t > 0$$

are equivalents.

3 Proof of Theorem 2.1

We need of the following result, obtained in [13]:

Proposition 3.1 *Let $M : [0,\infty[\to \mathbb{R}$ be a function of class C^1 and*

$$u \in C^1\left([0,\infty[;W\right), \ u(t) \neq 0, \ \forall t \in [0,\infty[.$$

Consider β a real number. Then the Leibniz derivative of $M\left(\|u(t)\|_W^\beta\right)$ is given by

$$\frac{d}{dt}\left\{M\left(\|u(t)\|_W^\beta\right)\right\} = \beta M'(\|u(t)\|_W^\beta)\|u(t)\|_W^{\beta-1}\left\langle \frac{Ju(t)}{\|u(t)\|_W}, u'(t)\right\rangle_{W'\times W},$$
$$t \geq 0,$$

where J is the duality application $J : W \to W'$ defined by

$$\langle Jv, v\rangle_{W'\times W} = \|v\|_W^2, \quad \|Jv\|_{W'} = \|v\|_W, \quad \forall v \in W.$$

By [13] we have also that there exists $T_0 > 0$ and a unique function u in the class

$$\begin{aligned} u &\in L^\infty(0,T_0; D(S^{2\alpha+5/2})), \\ u' &\in L^\infty(0,T_0; D(S^{2\alpha+2})), \\ u'' &\in L^\infty(0,T_0; D(S^{2\alpha+3/2})) \end{aligned} \tag{3.1}$$

such that

$$(LP)\quad \left|\begin{array}{l} u'' + M\left(\|u\|_W^\beta\right) Su + \delta u' = 0 \text{ in } L^\infty(0,T_0; D(S^{2\alpha+3/2})), \\ u(0) = u^0, \;\; u'(0) = u^1 \end{array}\right.$$

and

$$\|u(t)\|_W > 0, \;\; \forall\, t \in [0, T_0]. \tag{3.2}$$

So $\mathcal{M} \neq \emptyset$.

Next we obtain estimates for the solutions u given in $\mathcal{M}$. Note that if u given in $\mathcal{M}$ by the uniqueness of solutions (P) in [0, T], we have that u belongs to class (3.1), u is solution of (LP) in [0, T] and u satisfies (3.2) in [0, T] (see[13]). Consider $0 < t_0 < T_{max}$. Taking the scalar product of H in both sides of equation $(LP)_1$ by $2S^{2\alpha+2}u'$, we obtain:

$$\frac{d}{dt}\left[\left|S^{\alpha+1}u'(t)\right|^2\right] + M\left(\|u(t)\|_W^\beta\right)\frac{d}{dt}\left[\left|S^{\alpha+3/2}u(t)\right|^2\right]$$
$$+ 2\delta\left|S^{\alpha+1}u'(t)\right|^2 = 0, \; t \in [0, t_0],$$

that is,

$$\frac{\frac{d}{dt}\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}+\frac{d}{dt}\left[\left|S^{\alpha+3/2}u(t)\right|^2\right]=-2\delta\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}. \tag{3.3}$$

Introduce the function

$$\varphi(t)=\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}+\left|S^{\alpha+3/2}u(t)\right|^2,\ \ t\in[0,t_0]. \tag{3.4}$$

Our goal is to show that $\varphi(t)$ is not increasing. By Proposition 3.1 and (3.3), we have:

$$\varphi'(t)=-\frac{1}{M\left(\|u(t)\|_W^\beta\right)}\beta M'\left(\|u(t)\|_W^\beta\right)\|u(t)\|_W^{\beta-1}$$

$$\left\langle\frac{Ju(t)}{\|u(t)\|_W},u'(t)\right\rangle_{W\times W'}\left|S^{\alpha+1}u'(t)\right|^2-\frac{2\delta\left|S^{\alpha+1)}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)},\ \ t\in[0,t_0].$$

This gives

$$\varphi'(t)\le\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}\left[\frac{\beta\left|M'\left(\|u(t)\|_W^\beta\right)\right|\|u(t)\|_W^{\beta-1}}{M^{1/2}(\|u(t)\|_W^\beta)}\frac{\|u'(t)\|_W}{M^{1/2}(\|u(t)|_W^\beta)}-2\delta\right].$$

Then hypothesis $(H4)_2$ and embedding (H3) give:

$$\varphi'(t)\le\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}\left[\beta C_0k_0\frac{\left|S^{\alpha+1)}u'(t)\right|^2}{M^{1/2}(\|u(t)|_W^\beta)}-2\delta\right],\ \ t\in[0,t_0]. \tag{3.5}$$

Introduce the function

$$\psi(t)=\beta C_0k_0\frac{\left|S^{\alpha+1)}u'(t)\right|}{M^{1/2}(\|u(t)|_W^\beta)},\ \ t\in[0,t_0].$$

We have

$$\psi(t)\le\beta C_0k_0\varphi^{1/2}(t),\ \ \forall\, t\in[0,t_0]. \tag{3.6}$$

We affirm that

$$\psi(t) < \delta, \quad \forall\, t \in [0, t_0]. \tag{3.7}$$

In fact, suppose that there exists $t_1 \in [0, t_0]$ such that $\psi(t) \geq \delta$. By hypothesis (H6), we have $\psi(0) < \delta$. Consider

$$t^* = inf\,\{t \in [0, t_0]; \psi(t) = \delta\} > 0$$

As $\psi(t)$ is continuous in $[0, t_0]$, we have that $\psi(t^*) = \delta$, which implies by (3.5) that $\varphi(t)$ is not increasing on $[0, t^*]$. Then by hypothesis (H6) and (3.6), we obtain:

$$\psi(t) \leq \beta C_0 k_0 \varphi^{1/2}(0) < \delta, \quad \forall\, t \in [0, t^*],$$

which is a contradiction since $\psi(t^*) = \delta$. So (3.7) holds.

If follows from (3.7), (3.5) and noting that $0 < t_0 < T_{max}$ was arbitrary that

$$\varphi'(t) \leq -\delta \frac{\left|S^{\alpha+1} u'(t)\right|^2}{M(\|u(t)|_W^\beta)}, \quad \forall\, t \in [0, T_{max}[. \tag{3.8}$$

In particular

$$\frac{\left|S^{\alpha+1} u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} + \left|S^{\alpha+3/2} u(t)\right|^2 \leq \frac{\left|S^{\alpha+1} u^1\right|^2}{M\left(\|u^0\|_W^\beta\right)} + \left|S^{\alpha+3/2} u^0\right|^2 <$$

$$< \frac{\delta^2}{\beta^2 C_0^2 k_0^2} = N_0^2, \quad \forall\, t \in [0, T_{max}[. \tag{3.9}$$

Note that if T_{max} is infinite then (3.9) give the theorem. Suppose that T_{max} is finite. Then (3.9) implies

$$\left|S^{\alpha+3/2} u(t)\right|^2 \leq N_0^2, \quad \forall\, t \in [0, T_{max}[. \tag{3.10}$$

As $\|u(t)\|_W \leq k_1 \|u(t)\|_{D(S^{\alpha+3/2})}$ for all $t \in [0, T_{max}[$, we have by (3.9) and (3.10) that

$$\left|S^{\alpha+1} u'(t)\right|^2 \leq N_1^2, \quad \forall\, t \in [0, T_{max}[. \tag{3.11}$$

Consider a sequence of real number (t_ν) such that $0 < t_\nu < T_{max}$ and $t_\nu \to T_{max}$. By (3.10) and (3.11) we have that there exist $\zeta \in D(S^{\alpha+3/2})$ and $\chi \in D(S^{\alpha+1})$ such that

$$u(t_\nu) \to \zeta \text{ weak in } D(S^{\alpha+3/2}), \tag{3.12}$$

$$u'(t_\nu) \to \chi \text{ weak in } D(S^{\alpha+1}). \tag{3.13}$$

We affirm that

$$\zeta = \chi = 0. \tag{3.14}$$

In fact, if $\zeta \neq 0$ with ζ and χ we determine the local solution w of the problem

$$\left| \begin{array}{l} w'' + M\left(\|w\|_W^\beta\right) Sw + \delta w' = 0 \text{ in } L^\infty(0, T_0; D(S^{\alpha+1/2})), \\ w(0) = \zeta \ \ , w'(0) = \chi \end{array} \right.$$

(see [13]). Then the function

$$\widetilde{u}(t) = \left| \begin{array}{l} w(t), \ \ 0 \leq t < T_{max} \\ w(t - T_{max}), \ \ T_{max} \leq t < T_0 + T_{max} \end{array} \right.$$

is a solution of Problem (P) in $[0, T_0 + T_{max}]$, with $\|\widetilde{u}(t)\|_W > 0$ for all $t \in [0, T_0 + T_{max}]$. This gives a contradiction with the definition of T_{max}. So $\zeta = 0$.

Also by (3.11),

$$\|u(t_\nu) - u(t_\mu)\|_{D(S^{\alpha+1})} \leq \int_{t_\mu}^{t_\nu} \|u'(s)\|_{D(S^{\alpha+1})} ds \leq N_1 |t_\nu - t_\mu|,$$

that is, $(u(t_\nu))$ is a Cauchy sequence in $D(S^{\alpha+1})$. As $\zeta = 0$, (3.12) implies then

$$u(t_\nu) \to 0 \ \text{ in } D(S^{\alpha+1}).$$

In particular

$$u(t_\nu) \to 0 \ \text{ in } W.$$

By estimate (3.9), we have:

$$\left|S^{\alpha+1}u'(t_\nu)\right|^2 \leq N_0^2 M\left(\|u(t_\nu)\|_W\right).$$

As $M(0) = 0$ it follows from this inequality and convergence (3.13) that $\chi = 0$. So the affirmation (3.14) is correct.

Also by equation $(P)_1$ we have that

$$u''(t_\nu) \to 0 \ \text{ in } D(S^{\alpha+1/2}).$$

Thus if T_{max} is finite we define $u(t) = 0$ for $t \geq T_{max}$. This extension is a solution of Problem (P) in $[0, \infty[$.

□

4 Proof of Theorem 2.2

We begin with a previous result.

Lemma 4.1 *Let $\beta > 1$ a real number, $M : [0, \infty[\to \mathbb{R}$ a function of class C^1 and $u \in C^1([0, \infty[; W)$. Then if $u(t_0) = 0$, we have that the Leibniz derivative $\dfrac{d}{dt}M\left(\|u(t_0)\|_W^\beta\right)$ is equal to zero.*

Proof: Consider $t_0 > 0$ and $u(t_0) = 0$. Then

$$u(t_0 + h) = u(t_0 + h) - u(t_0) = h\int_0^1 u'(t_0 + \tau h)d\tau,$$

which implies for $0 < |h| < min\{1, t_0/2\}$,

$$\|u(t_0 + h)\|_W^\beta \leq |h|^\beta \left(\int_0^1 \|u'(t_0 + \tau h)\|_W d\tau\right)^\beta \leq |h|^\beta C^\beta,$$

where

$$C = max\left\{\|u'(s)\|_W; t_0/2 \leq s \leq t_0 + 1\right\}.$$

The last inequality gives the result since $\beta > 1$. Analogous arguments give the result when $t_0 = 0$ and $u(0) = 0$. □

Let u_j^0 and u_j^1 be two sequences of vectors of $D(S^{2\alpha+5/2})$ and $D(S^{2\alpha+2})$, respectively, such that

$$u_j^0 \to u^0 \text{ in } D(S^{\alpha+3/2}), \quad u_j^0 \to u^1 \text{ in } D(S^{\alpha+1}).$$

By these convergence and (H9), we have:

$$\beta C_1 k_0 M^{1/2}(k_1^\beta \varphi_j^{\beta/2}(0))\varphi_j^{1/2}(0) < \delta, \;\; j \geq j_0,$$

where

$$\varphi_j(0) = \frac{\left|S^{\alpha+1}u_j^1\right|^2}{M(\left\|u_j^0\right\|_W^\beta)} + \left|S^{\alpha+3/2}u_j^0\right|^2$$

Consider the problem ($j \geq j_0$)

$$(P_j) \quad \left| \begin{array}{l} u_j'' + M\left(\|u_j\|_W^\beta\right) Su_j + \delta u_j' = 0 \text{ in } L^\infty(0,\infty; D(S^{2\alpha+3/2})), \\ u_j(0) = u_j^0, \;\; u_j'(0) = u_j^1. \end{array} \right.$$

By applying the Galerkin method, the successive approximations technique and the spectral theory of the operator S, Arzela-Ascoli Theorem, Proposition (3.1), Lemma (4.1), we obtain a local solution u_j of (P_j), that is, we find a real number $T_0 > 0$ and a function u_j in the class

$$\begin{aligned} u_j \in\; & L^\infty(0,T_0; D(S^{2\alpha+5/2})), \\ u_j' \in\; & L^\infty(0,T_0; D(S^{2\alpha+2})), \\ u_j'' \in\; & L^\infty(0,T_0; D(S^{2\alpha+3/2})) \end{aligned} \tag{4.1}$$

such that, u_j is the solution of Problem (P_j) in $L^\infty(0,T_0; D(S^{2\alpha+3/2}))$ (see the methodology of this approach in S. S. Souza and the third A. [27]). The same arguments allow us to obtain a real number $T_0 > 0$ and a solution u in the class

$$\begin{aligned} u \in\; & L^\infty(0,T_0; D(S^{2\alpha+3/2})), \\ u' \in\; & L^\infty(0,T_0; D(S^{2\alpha+1})), \\ u'' \in\; & L^\infty(0,T_0; D(S^{2\alpha+1/2})) \end{aligned}$$

such that, u is the unique solution of the problem

$$(P') \quad \left| \begin{array}{l} u'' + M\left(\|u\|_W^\beta\right) Su_j + \delta u' = 0 \text{ in } L^\infty(0,T_0;D(S^{2\alpha+1/2})), \\ u(0) = \zeta \ \ , u'(0) = \eta, \end{array} \right.$$

where $\zeta \in D(S^{\alpha+3/2})$ and $\eta \in D(S^{2\alpha+1})$ are arbitrary.

Let $\mathcal{M}_j$ be the set of real numbers $T > 0$ such that there exists a unique function u_j in class (4.1) with T instead T_0 and u_j is the solution of (P_j) in $L^\infty(0,T;D(S^{2\alpha+3/2}))$. Then by the result of local existence above, we have that $\mathcal{M}_j \neq \emptyset$. Denote by $T_{max,j}$ the supremum of $T \in \mathcal{M}$.

Let $\varphi_j(t), j \geq j_0$, be the function

$$\varphi_j(t) = \frac{\left|S^{\alpha+1}u_j'(t)\right|^2}{M\left(\|u_j(t)\|_W^\beta\right)} + \left|S^{\alpha+3/2}u_j(t)\right|^2, \ \ t \in [0,T_{max,j}[. \tag{4.2}$$

Then as in the proof of Theorem (2.1), we obtain:

$$\varphi_j'(t) \leq \frac{\left|S^{\alpha+1}u_j'(t)\right|^2}{M\left(\|u_j(t)\|_W^\beta\right)} \left[\beta C_1 k_0 \left|S^{\alpha+1}u_j'(t)\right| - 2\delta\right], \ \ t \in [0,T_{max,j}[. \tag{4.3}$$

Consider the function

$$\psi_1(t) = \beta C_1 k_0 \left|S^{\alpha+1}u_j'(t)\right|, \ \ t \in [0,T_{max,j}[.$$

By (H9) and noting that $M(\xi)$ is increasing, we have:

$$\begin{aligned} \psi_1(0) = \ \ & \beta C_1 k_0 M^{1/2}\left(\|u_j^0\|_W^\beta\right) \frac{\left|S^{\alpha+1}u_j^1\right|}{M^{1/2}\left(\|u_j^0\|_W^\beta\right)} \\ & \leq \beta C_1 k_0 M^{1/2}\left(k_1^\beta \varphi_j^{\beta/2}(0)\right) \varphi_j^{1/2}(0) < \delta. \end{aligned}$$

By similar arguments used in the proof Theorem 2.1, we obtain:

$$\psi_1(t) < \delta, \ \ t \in [0,T_{max,j}[. \tag{4.4}$$

This result and (4.3) imply

$$\varphi_j'(t) \leq -\delta \frac{\left|S^{\alpha+1)} u_j'(t)\right|^2}{M\left(\|u_j(t)\|_W^\beta\right)}, \quad t \in [0, T_{max,j}[\tag{4.5}$$

and

$$\frac{\left|S^{\alpha+1} u_j'(t)\right|^2}{M\left(\|u_j(t)\|_W^\beta\right)} + \left|S^{\alpha+3/2} u_j(t)\right|^2 \leq \frac{\left|S^{\alpha+1} u_j^1\right|^2}{M\left(\|u_j^0\|_W^\beta\right)} + \left|S^{\alpha+3/2} u_j^0\right|^2 <$$

$$< \frac{\delta^2}{\beta^2 C_1^2 k_0^2 M(k_1^\beta \varphi_j^{\beta/2}(0))}, \quad t \in [0, T_{max,j}[. \tag{4.6}$$

By inequality (4.4), local existence of solution of (P'), uniqueness of solution of Problem (P_j) in any $L^\infty(0,T; D(S^{2\alpha+3/2}))$ and by similar argument used in the proof of Theorem 2.1, we obtain that $T_{max,j}$ is infinite for $j \geq j_0$.

Fix $j \geq j_0$ and consider $\epsilon > 0$. Introduce the functions

$$\rho(t) = \frac{\left(S^{\alpha+1} u_j'(t), S^{\alpha+1} u_j(t)\right)}{M\left(\|u_j(t)\|_W^\beta\right)} + \frac{\delta}{2} \frac{\left|S^{\alpha+1} u_j(t)\right|^2}{M\left(\|u_j(t)\|_W^\beta\right)}, \quad t \in [0, \infty[\tag{4.7}$$

and

$$\varphi_\epsilon(t) = \varphi_j(t) + \epsilon \rho(t), \quad t \in [0, \infty[, \tag{4.8}$$

where u_j is the solution of (P_j).

In what follows, to facilitate the notation, we omit the subscript j. We have:

$$|\rho(t)| \leq \frac{\left|S^{\alpha+1} u'(t)\right|}{C_* m_0^{1/2} M\left(\|u(t)\|_W^\beta\right)^{1/2}} \left|S^{\alpha+3/2} u(t)\right| + \frac{\delta}{2} \frac{\left|S^{\alpha+3/2} u(t)\right|^2}{m_0 C_*},$$

where C_* were defined in (2.4), that is,

$$|\rho(t)| \leq \left[\frac{1}{C_* m_0^{1/2}} + \frac{\delta}{2 m_0 C_*}\right] \varphi(t), \quad t \in [0, \infty[.$$

So, this inequality and (4.8) imply

$$|\varphi_\epsilon(t)| \leq (1+\epsilon P_0)\,\varphi(t), \text{ where } P_0 = \frac{1}{C_* m_0^{1/2}} + \frac{\delta}{2m_0 C_*}.$$

Then taking $0 < \epsilon \leq 1/2P_0$, we have:

$$\frac{1}{2}\varphi(t) \leq \varphi_\epsilon(t) \leq \frac{3}{2}\varphi(t), \quad t \in [0,\infty[. \tag{4.9}$$

On the other side, by taking the scalar product of H in both sides of equation $(P_j)_1$ by $S^{2\alpha+2}u$, we obtain:

$$\left(S^{\alpha+1}u''(t), S^{\alpha+1}u(t)\right) + M\left(\|u(t)\|_W^\beta\right)\left|S^{\alpha+3/2}u(t)\right|^2 + \frac{\delta}{2}\frac{d}{dt}\left|S^{\alpha+1}u(t)\right|^2 = 0$$

or

$$\frac{\frac{d}{dt}\left(S^{\alpha+1}u'(t), S^{\alpha+1}u(t)\right)}{M\left(\|u(t)\|_W^\beta\right)} + \frac{\delta}{2}\frac{\frac{d}{dt}\left|S^{\alpha+1}u(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} = \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} - \left|S^{\alpha+3/2}u(t)\right|^2$$

Combining this equality with the definition (4.7) of $\rho(t)$, we deduce, for $u(t) \neq 0$,

$$\rho'(t) = \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} - \left|S^{\alpha+3/2}u(t)\right|^2 -$$

$$\frac{\beta M'\left(\|u(t)\|_W^\beta\right)\|u(t)\|_W^{\beta-1}}{M\left(\|u(t)\|_W^\beta\right)}\left\langle \frac{Ju(t)}{\|u(t)\|_W}, u'(t)\right\rangle$$

$$\left(\frac{S^{\alpha+1}u'(t)}{M^{1/2}\left(\|u(t)\|_W^\beta\right)}, \frac{S^{\alpha+1}u(t)}{M^{1/2}\left(\|u(t)\|_W^\beta\right)}\right) -$$

$$\frac{\delta\beta M'\left(\|u(t)\|_W^\beta\right)\|u(t)\|_W^{\beta-1}}{2M\left(\|u(t)\|_W^\beta\right)}\left\langle \frac{Ju(t)}{\|u(t)\|_W}, u'(t)\right\rangle \frac{\left|S^{\alpha+1}u(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} =$$

$$= \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} - \left|S^{\alpha+3/2}u(t)\right|^2 - L_1 - L_2; \tag{4.10}$$

and for $u(t) = 0$,

$$\rho'(t) = \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} - \left|S^{\alpha+3/2}u(t)\right|^2 \tag{4.11}$$

(see Lemma 4.1). We have, by hypothesis $(H7)_4$, (4.4), and hypothesis $(H10)$:

$$|L_1| \leq \beta C_1 k_0 \left|S^{\alpha+1}u'(t)\right| \frac{\left|S^{\alpha+1}u'(t)\right|}{M^{1/2}\left(\|u(t)\|_W^\beta\right)} \frac{\left|S^{\alpha+1}u(t)\right|}{M^{1/2}\left(\|u(t)\|_W^\beta\right)} \leq$$

$$\leq \beta C_1 k_0 \left|S^{\alpha+1}u'(t)\right| \left(\frac{\left|S^{\alpha+1}u'(t)\right|^2}{2M\left(\|u(t)\|_W^\beta\right)} + \frac{\left|S^{\alpha+1}u(t)\right|^2}{2M\left(\|u(t)\|_W^\beta\right)} \right) \leq$$

$$\leq \frac{\delta}{4} \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} + \frac{\beta C_1 k_0 \left|S^{\alpha+1}u'(t)\right| \left|S^{\alpha+3/2}u(t)\right|^2}{2m_0^{1/2} C_*^2 M\left(\|u(t)\|_W^\beta\right)^{1/2}} \leq$$

$$\leq \frac{\delta}{4} \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} + \frac{1}{4}\left|S^{\alpha+3/2}u(t)\right|^2.$$

Also

$$|L_2| \leq \frac{\delta C_1 k_0}{2m_0^{1/2} C_*^2} \frac{\left|S^{\alpha+1}u'(t)\right|}{M^{1/2}\left(\|u(t)\|_W^\beta\right)} \left|S^{\alpha+3/2}u(t)\right|^2 \leq \frac{1}{4}\left|S^{\alpha+3/2}u(t)\right|^2.$$

Combining (4.10), (4.11) and the last two inequalities, we obtain:

$$\rho'(t) \leq \frac{\left|S^{\alpha+1}u'(t)\right|}{M\left(\|u(t)\|_W^\beta\right)} - \left|S^{\alpha+3/2}u(t)\right|^2 + \frac{\delta}{4} \frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)} + \frac{1}{2}\left|S^{\alpha+3/2}u(t)\right|^2.$$

This inequality, the definition (4.8) of φ_ϵ and inequality (4.3), give:

$$\begin{aligned}\varphi_\epsilon'(t) &\leq -\delta\frac{\left|S^{\alpha+1}u'(t)\right|}{M\left(\|u(t)\|_W^\beta\right)}+\epsilon\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}-\epsilon\left|S^{\alpha+3/2}u(t)\right|^2\\ &+\epsilon\frac{\delta}{4}\frac{\left|S^{\alpha+1}u'(t)\right|^2}{M\left(\|u(t)\|_W^\beta\right)}+\quad\frac{\epsilon}{2}\left|S^{\alpha+3/2}u(t)\right|^2,\end{aligned}$$

for all $u(t), t\in[0,\infty[$. Noting that $\epsilon\leq min\{1,\delta/4\}$, we obtain:

$$\varphi_\epsilon'(t)\leq-\frac{\delta}{2}\frac{\left|S^{\alpha+1}u'(t)\right|}{M\left(\|u(t)\|_W^\beta\right)}-\frac{\epsilon}{2}\left|S^{\alpha+3/2}u(t)\right|^2,\ \ t\in[0,\infty[,$$

that implies by (4.9),

$$\varphi_\epsilon'(t)\leq-\frac{\tau_0}{3}\varphi_\epsilon(t),\ \ t\in[0,\infty[,$$

which gives

$$\varphi_\epsilon(t)\leq\varphi_\epsilon(0)e^{-\frac{\tau_0}{3}t},\ \ t\in[0,\infty[.$$

Therefore, by (4.9),

$$\varphi_j(t)\leq 3\varphi_j(0)e^{-\frac{\tau_0}{3}t},\ \ t\in[0,\infty[,\ \ j\geq j_0.$$

By the methodology used in [27], we obtain the limit u of the solutions u_j is the solution of Problem (P) with u in class (2.3). Also by taking the lim inf in both sides of the last inequality, we deduce inequality (2.7).

□

5 Examples

The result obtained in Theorem 2.1 can be applied to the equation

$$u''(t)+\|u(t)\|_W^2 Su(t)+\delta u'(t)=0,\ \ t>0.$$

Here $M(\xi) = \xi^{2/\beta}$. And the result obtained in Theorem 2.2, to the equation

$$u''(t) + M\left(\|u(t)\|_W^\beta\right) Su(t) + \delta u'(t) = 0, \quad t > 0,$$

where $M(\xi) = \xi^\sigma + m_0$, $\sigma \geq 1/\beta$ and $m_0 > 0$. For the mixed problem associated to these equation, see [13].

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