

# REMARKS ON THE CONTROLLABILITY OF SOME PARABOLIC EQUATIONS AND SYSTEMS

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## Abstract

This paper is devoted to present a review of recent results concerning the null controllability of some parabolic systems. Among others, we will consider the classical heat equation, the Burgers, Navier-Stokes and Ginzburg-Landau equations, etc.

## 1 Introduction. Controllability and observability

Let us first recall some general ideas that can be applied to a large family of (linear and nonlinear) evolution problems.

Suppose that we are considering an abstract *state equation* of the

form

$$\begin{cases} y_t - A(y) = Bv, & t \in (0, T), \\ y(0) = y^0, \end{cases} \quad (1)$$

which governs the behavior of a physical system. It is assumed that

- $A : D(A) \subset H \mapsto H$  is a (generally nonlinear) operator,
- $y : [0, T] \mapsto H$  is the *state*, i.e. the variable that serves to identify the physical properties of the system,
- $v : [0, T] \mapsto U$  is the *control*, i.e. the variable we can choose,
- $B \in \mathcal{L}(U; H)$  and
- $y^0 \in H$  (for simplicity, we assume that  $U$  and  $H$  are Hilbert spaces).

Suppose that the state equation is well-posed in the sense that, for each  $y^0 \in H$  and each  $v \in L^2(0, T; U)$ , (1) possesses exactly one solution. Then the *null controllability* problem for (1) can be stated as follows:

*For each  $y^0 \in H$ , find  $v \in L^2(0, T; U)$  such that the corresponding solution of (1) satisfies  $y(T) = 0$ .*

For each system of the form (1), the null controllability problem leads to several interesting questions. Among them, let us mention the following:

- First, are there controls  $v$  such that  $y(T) = 0$ ?
- Then, if this is the case, which is the *cost* we have to pay to drive  $y$  to zero? In other words, which is the minimal norm of a control  $v \in L^2(0, T; U)$  satisfying this property?

- How can these controls be computed?

The controllability of differential systems is a very relevant area of research and has been the subject of many papers the last years. In particular, in the context of partial differential equations, the null controllability problem was first analyzed in [29, 30, 26, 27, 22, 25]. For semilinear systems of this kind, the first contributions have been given in [31, 7, 15].

In this paper, we will try to recall some known results concerning the controllability of parabolic systems. More precisely, we will be concerned with the classical heat equation, the Burgers equation and the Navier-Stokes and Ginzburg-Landau systems.

## 2 The classical heat equation. Carleman estimates

In this Section, we will consider the controlled heat equation, complemented with initial and Dirichlet boundary conditions:

$$\begin{cases} y_t - \Delta y = v1_\omega, & (x, t) \in \Omega \times (0, T), \\ y(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ y(x, 0) = y^0(x), & x \in \Omega. \end{cases} \quad (2)$$

Here (and also in the following sections),  $\Omega \subset \mathbb{R}^N$  is a nonempty bounded domain,  $\omega \subset\subset \Omega$  is a (small) nonempty open subset ( $1_\omega$  is the characteristic function of  $\omega$ ) and  $y^0 \in L^2(\Omega)$ .

It is well known that, for every  $y^0 \in L^2(\Omega)$  and every  $v \in L^2(\omega \times (0, T))$ , there exists a unique solution  $y$  to (2), with

$$y \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)).$$

In this context, the null controllability problem reads:

For each  $y^0 \in L^2(\Omega)$ , find  $v \in L^2(\omega \times (0, T))$  such that the corresponding solution of (2) satisfies

$$y(x, T) = 0 \quad \text{in } \Omega. \quad (3)$$

Together with (2), for each  $\varphi^1 \in L^2(\Omega)$ , we can introduce the associated adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & (x, t) \in \Omega \times (0, T), \\ \varphi(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \varphi(x, T) = \varphi^1(x), & x \in \Omega. \end{cases} \quad (4)$$

Then, it is well known that the null controllability of (2) is in practice equivalent to the following property:

*There exists  $C > 0$  such that*

$$\|\varphi(\cdot, 0)\|_{L^2}^2 \leq C \iint_{\omega \times (0, T)} |\varphi|^2 dx dt \quad \forall \varphi^1 \in L^2(\Omega). \quad (5)$$

This is called an observability estimate for the solutions of (4). We thus find that, in order to solve the null controllability problem for (2), it suffices to prove (5).

The estimate (5) is implied by the so called global Carleman inequalities. These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov, see [22, 15]. When they are applied to the solutions of the adjoint systems (4), they take the form

$$\iint_{\Omega \times (0, T)} \rho^2 |\varphi|^2 dx dt \leq K \iint_{\omega \times (0, T)} \rho^2 |\varphi|^2 dx dt \quad \forall \varphi^1 \in L^2(\Omega), \quad (6)$$

where  $\rho = \rho(x, t)$  is an appropriate weight, depending on  $\Omega$ ,  $\omega$  and  $T$  and the constant  $K$  only depends on  $\Omega$  and  $\omega$ .<sup>1</sup>

Combining (6) and the dissipativity of the backwards heat equation (4), it is not difficult to deduce (5) for some  $C$  only depending on  $\Omega$ ,  $\omega$  and  $T$ .

As a consequence, we have:

**Theorem 1.** *The linear system (2) is null controllable. In other words, for each  $y^0 \in L^2(\Omega)$ , there exists  $v \in L^2(\omega \times (0, T))$  such that the corresponding solution of (2) satisfies (3).*

There are many generalizations and variants of this result that provide the null controllability of other similar linear state equations:

- Time-space dependent (and sufficiently regular) coefficients can appear in the equation, other boundary conditions can be used, boundary control (instead of distributed control) can be imposed, etc. For a review of recent applications of Carleman inequalities to the controllability of parabolic systems, see [10].
- The controllability of Stokes-like systems can also be analyzed with these techniques. This includes systems of the form

$$y_t - \Delta y + (a \cdot \nabla)y + (y \cdot \nabla)b + \nabla p = v1_\omega, \quad \nabla \cdot y = 0, \quad (7)$$

where  $a$  and  $b$  are regular enough; see for instance [11].

- Other linear parabolic (non-scalar) systems can also be considered, etc.

As mentioned above, an interesting question related to theorem 1 concerns the cost of null controllability. One has the following result from [13]:

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<sup>1</sup>In order to prove (6), we have to use a weight  $\rho$  decreasing to zero, as  $t \rightarrow 0$  and also as  $t \rightarrow T$ , for instance exponentially.

**Theorem 2.** For each  $y^0 \in L^2(\Omega)$ , let us set

$$C(y^0) = \inf\{\|v\|_{L^2(\omega \times (0,T))} : \text{the solution of (2) satisfies } y(x,T) = 0 \text{ in } \Omega\}.$$

Then we have the following estimate

$$C(y^0) \leq \exp\left(C\left(1 + \frac{1}{T}\right)\right) \|y^0\|_{L^2}, \quad (8)$$

where the constant  $C$  only depends on  $\Omega$  and  $\omega$ .

**Remark 1.** Notice that theorem 1 ensures the null controllability of (2) for *any*  $\omega$  and  $T$ . This is a consequence of the fact that, in a parabolic equation, the information is transmitted at infinite speed. For instance, this is not the case for the wave equation. For the latter null controllability does not always hold, but contrarily the couple  $(\omega, T)$  has to satisfy appropriate geometrical assumptions; see [27] and [4] for more details.

### 3 Positive and negative controllability results for the one-dimensional Burgers equation

In this section, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = y(1, t) = 0, & t \in (0, T), \\ y(x, 0) = y^0(x), & x \in (0, 1). \end{cases} \quad (9)$$

Some controllability properties of (9) have been studied in [15] (see Chapter 1, theorems 6.3 and 6.4). There, it is shown that, in

general, a stationary solution of (9) with large  $L^2$ -norm cannot be reached (not even approximately) at any time  $T$ . In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each  $y^0 \in L^2(0, 1)$ , let us introduce

$$T(y^0) = \inf\{T > 0 : (9) \text{ is null controllable at time } T\}.$$

Then, for each  $r > 0$ , let us define the quantity

$$T^*(r) = \sup\{T(y^0) : \|y^0\|_{L^2} \leq r\}.$$

Our main purpose is to show that  $T^*(r) > 0$ , with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (9).

More precisely, let us set  $\phi(r) = (\log \frac{1}{r})^{-1}$ . We have the following result from [8]:

**Theorem 3.** *One has*

$$C_0\phi(r) \leq T^*(r) \leq C_1\phi(r) \quad \text{as } r \rightarrow 0, \quad (10)$$

for some positive constants  $C_0$  and  $C_1$  not depending of  $r$ .

**Remark 2.** The same estimates hold when the control  $v$  acts on system (9) through the boundary *only* at  $x = 1$  (or only at  $x = 0$ ). Indeed, it is easy to transform the boundary controlled system

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = 0, \quad y(1, t) = w(t), & t \in (0, T), \\ y(x, 0) = y^0(x), & x \in (0, 1) \end{cases} \quad (11)$$

into a system of the kind (9). The boundary controllability of the Burgers equation with *two* controls (at  $x = 0$  and  $x = 1$ ) has been

analyzed in [19]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in that paper that exact controllability does not hold for large time.<sup>2</sup>

**Proof of theorem 3:** The proof of the estimate from above in (10) can be obtained by solving the null controllability problem for (9) via a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of  $T$  in these inequalities.

Let us give more details. First we recall that, as long as  $a \in L^\infty(Q)$ , we can find controls  $v$  such that the solution of the linear system

$$\begin{cases} y_t - y_{xx} + a(x, t)y_x = v1_\omega, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = y(1, t) = 0, & t \in (0, T), \\ y(x, 0) = y^0(x), & x \in (0, 1), \end{cases} \quad (12)$$

satisfies

$$y(x, T) = 0 \quad \text{in } \Omega \quad (13)$$

(see [15]). Moreover, the controls  $v$  can be found in  $L^\infty(\omega \times (0, T))$  and such that

$$\|v\|_\infty \leq e^{C^*/T} \|y^0\|_{L^2(0,1)}, \quad (14)$$

for some  $C^* = C^*(\omega, \|a\|_\infty)$ ; see [14].

Let us now show that this provides a local controllability result for the nonlinear system (9):

**Lemma 1.** *Assume that*

$$y^0 \in H_0^1(0, 1), \quad \|y^0\|_\infty \leq \frac{1}{2} \quad \text{and} \quad \|y^0\|_{L^2(\Omega)} \leq \frac{1}{2T} e^{-C_1^*/T}, \quad (15)$$

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<sup>2</sup>Let us remark that the results in [19] do not allow to estimate  $T(r)$ ; in fact, the proofs are based in contradiction arguments.



where  $C_1^*$  corresponds to the constant  $C^*$  in (14) for  $\|a\|_\infty = 1$ . Then there exist controls  $v \in L^\infty(\omega \times (0, T))$  such that the associated solutions to (9) satisfy (13).

**Sketch of the proof:** The proof of this lemma relies on well known arguments, but we present a sketch of it for the sake of completeness.

Let  $s \in (1/2, 1)$  and let us introduce the set-valued mapping  $\mathcal{A} : H^s(Q) \mapsto H^s(Q)$ , given as follows: for each  $z \in H^s(Q)$ , we first denote by  $\mathcal{A}_0(z)$  the set of all controls  $v \in L^\infty(\omega \times (0, T))$  such that (14) is satisfied and the associated solution of

$$\begin{cases} y_t - y_{xx} + z(x, t)y_x = v1_\omega & (x, t) \in Q, \\ y(0, t) = y(1, t) = 0 & t \in (0, T), \\ y(x, 0) = y^0(x) & x \in (0, 1), \end{cases} \quad (16)$$

fulfills (13); then,  $\mathcal{A}(z)$  is by definition the family of these associated solutions.

Let  $K$  be the closed convex set  $K = \{z \in H^s(Q) : \|z\|_\infty \leq 1\}$ . Let us check that the hypotheses of Kakutani's fixed-point theorem are satisfied by  $\mathcal{A}$  in  $K$  (for the statement of this theorem, see for instance [2]):

- First, we note that the solution of (16) belongs to the space

$$X := L^2(0, T; H^2(0, 1)) \cap H^1(0, T; L^2(0, 1)).$$

In particular,  $y \in H^s(Q)$ . Then, an application of the classical maximum principle yields

$$\|y\|_\infty \leq T\|v\|_\infty + \|y^0\|_\infty.$$

Now, from (14) and (15), we deduce that

$$\|y\|_\infty \leq 1.$$

Consequently,  $\mathcal{A}$  maps  $K$  into  $K$ .

- It is not difficult to prove that, for each  $z \in K$ ,  $\mathcal{A}(z)$  is a nonempty compact convex set of  $H^s(Q)$ , in view of the compactness of the embedding  $X \hookrightarrow H^s(Q)$ .

- Furthermore,  $\mathcal{A}$  is upper hemicontinuous in  $H^s(Q)$ , i.e. for each  $\mu \in (H^s(Q))'$ , the single-valued mapping

$$z \mapsto \sup_{y \in \mathcal{A}(z)} \langle \mu, y \rangle$$

is *upper semi-continuous*. Indeed, let us assume that  $z_n \in K$  for all  $n$  and  $z_n \rightarrow z_0$  in  $H^s(Q)$ . For each  $n$ , there exists  $y_n \in \mathcal{A}(z_n)$  such that

$$\sup_{y \in \mathcal{A}(z_n)} \langle \mu, y \rangle = \langle \mu, y_n \rangle.$$

Then, from classical regularity estimates for the linear heat equation (see for instance [24]), we see that, at least for a subsequence, one has

$$y_n \rightarrow y^* \text{ weakly in } L^2(0, T; H^2(0, 1))$$

and

$$y_{n,t} \rightarrow y_t^* \text{ weakly in } L^2(Q),$$

whence

$$y_n \rightarrow y^* \text{ strongly in } L^2(0, T; H_0^1(0, 1)).$$

Consequently,  $z_n y_{n,x}$  converges weakly in  $L^2(Q)$  to  $z_0 y_x^*$  and the limit function  $y^*$  satisfies (16) and  $y^* \in \mathcal{A}(z_0)$ . This shows that

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathcal{A}(z_n)} \langle \mu, y \rangle \leq \sup_{y \in \mathcal{A}(z_0)} \langle \mu, y \rangle,$$

as desired.

In view of Kakutani's theorem, there exists  $\hat{y} \in K$  such that  $\hat{y} \in \mathcal{A}(\hat{y})$ . This ends the proof of lemma 1.

Let us finish the proof of the right inequality in (10).

Assume that  $y^0 \in L^2(0, 1)$  and  $\|y^0\|_{L^2(0,1)} \leq r$ . In a first step, we take  $v(x, t) \equiv 0$ . Then, from classical parabolic regularity results, we know that  $y(\cdot, t) \in H_0^1(0, 1)$  for any  $t > 0$  and there exist constants  $\tau$  and  $M$  such that the solution of (9) satisfies

$$\|y(\cdot, t)\|_\infty \leq Mt^{-1/4}\|y^0\|_{L^2(0,1)} \quad \forall t \in (0, \tau).$$

We leave the solution evolve freely, until it reaches a set of the form

$$\left\{ w \in L^\infty(0, 1) : \|w\|_\infty \leq \frac{1}{2}, \|w\|_{L^2(0,1)} \leq r \right\}.$$

More precisely, we take  $v(x, t) \equiv 0$  for  $t \in (0, T_0)$ , where  $T_0 = (2M)^4 r^4$ .

Let us set  $\bar{y}^0 = y^0(\cdot, T_0)$ . Then  $\bar{y}^0 \in H_0^1(0, 1)$ ,

$$\|\bar{y}^0\|_\infty \leq \frac{1}{2} \quad \text{and} \quad \|\bar{y}^0\|_{L^2(0,1)} \leq r.$$

Let us now consider the system

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x, t) \in (0, 1) \times (T_0, T_0 + T_1), \\ y(0, t) = y(1, t) = 0, & t \in (T_0, T_0 + T_1), \\ y(x, T_0) = \bar{y}^0(x), & x \in (0, 1), \end{cases} \quad (17)$$

where

$$T_1 = \frac{C_1^*}{\log \frac{1}{r}}.$$

Since one can assume that  $\frac{1}{2C^*} \log \frac{1}{r} \geq 1$ , in view of lemma 1 there exist controls  $\hat{v} \in L^\infty(\omega \times (T_0, T_0 + T_1))$  such that the associated solutions of (17) satisfy

$$y(x, T_0 + T_1) = 0 \quad \text{in} \quad (0, 1).$$

We then set  $v(x, t) \equiv \hat{v}(x, t)$  for  $t \in (T_0, T_0 + T_1)$ .

In this way, we have shown that we can drive the solution of (9) exactly to zero in a time interval of length

$$T_0 + T_1 = (2M)^4 r^4 + \frac{C_1^*}{\log \frac{1}{r}}.$$

Hence, the second inequality in (10) is proved.

Let us now give the proof of the estimate from below in (10). This is inspired by the arguments in [1].

We will prove that there exist positive constants  $C_0$  and  $C'_0$  such that, for any sufficiently small  $r > 0$ , we can find initial data  $y^0$  satisfying  $\|y^0\|_{L^2} \leq r$  with the following property: for any state  $y$  associated to  $y^0$ , one has

$$|y(x, t)| \geq C'_0 r \text{ for some } x \in (0, 1) \text{ and any } t : 0 < t < C_0 \phi(r).$$

Thus, let us set  $T = \phi(r)$  and let  $\rho_0 \in (0, 1)$  be such that  $(0, \rho_0) \cap \omega = \emptyset$ . Notice that this is not restrictive, since it is always possible to work in a suitable open subset  $\tilde{\omega} \subset \omega$ .

We can suppose that  $0 < r < \rho_0$ . Let us choose  $y^0 \in L^2(0, 1)$  such that  $y^0(x) = -r$  for all  $x \in (0, \rho_0)$  and let us denote by  $y$  an associated solution of (9).

Let us introduce the function  $Z = Z(x, t)$ , with

$$Z(x, t) = \exp \left\{ -\frac{2}{t} \left( 1 - e^{-\rho_0^2(\rho_0-x)^3/(\rho_0/2-x)^2} \right) + \frac{1}{\rho_0 - x} \right\}. \tag{18}$$

Then one has  $Z_t - Z_{xx} + ZZ_x \geq 0$ .

Let us now set  $w(x, t) = Z(x, t) - y(x, t)$ . It is immediate that

$$\begin{cases} w_t - w_{xx} + ZZ_x - yy_x \geq 0, & (x, t) \in (0, \rho_0) \times (0, T), \\ w(0, t) \geq 0, \quad w(\rho_0, t) = +\infty, & t \in (0, T), \\ w(x, 0) = r, & x \in (0, \rho_0). \end{cases} \tag{19}$$

Consequently,  $w^-(x, t) \equiv 0$ . Indeed, let us multiply the differential equation in (19) by  $-w^-$  and let us integrate in  $(0, \rho_0)$ . Since  $w^-$  vanishes at  $x = 0$  and  $x = \rho_0$ , after some manipulation we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^{\rho_0} |w^-|^2 dx + \int_0^{\rho_0} |w_x^-|^2 dx \\ = \int_0^{\rho_0} w^- (ZZ_x - yy_x) dx \leq C \int_0^{\rho_0} |w^-|^2 dx. \end{aligned} \quad (20)$$

Hence,

$$y \leq Z \quad \text{in } (0, \rho_0) \times (0, T). \quad (21)$$

Let us set  $\rho_1 = \rho_0/2$  and let  $\tilde{r}$  be a regular function satisfying the following:  $\tilde{r}(0) = \tilde{r}(\rho_1) = 0$ ;  $\tilde{r}(x) = r$  for all  $x \in (\delta\rho_1, (1-\delta)\rho_1)$  and some  $\delta \in (0, 1/4)$ ;  $-r \leq -\tilde{r}(x) \leq 0$ ;

$$|\tilde{r}_x| \leq Cr \quad \text{and} \quad |\tilde{r}_{xx}| \leq C \quad \text{in } (0, \rho_1), \quad (22)$$

where  $C = C(\rho_1)$  is independent of  $r$ .

Let us introduce the solution  $u$  of the auxiliary system

$$\begin{cases} u_t - u_{xx} + uu_x = 0, & (x, t) \in (0, \rho_1) \times (0, T), \\ u(0, t) = Z(\rho_1, t), \quad u(\rho_1, t) = Z(\rho_1, t), & t \in (0, T), \\ u(x, 0) = -\tilde{r}(x), & x \in (0, \rho_1). \end{cases} \quad (23)$$

We will need the following lemma, whose proof can be found in [8]:

**Lemma 2.** *One has*

$$|u| \leq Cr \quad \text{and} \quad |u_x| \leq Cr^{1/2} \quad \text{in } (0, \rho_1) \times (0, \phi(r)), \quad (24)$$

where  $C$  is independent of  $r$ .

Taking into account (21) and that  $u_x, y \in L^\infty((0, \rho_1) \times (0, T))$  (see lemma 2 below), a standard application of Gronwall's lemma shows that

$$y \leq u \quad \text{in } (0, \rho_1) \times (0, T). \quad (25)$$

On the other hand, we see from (24) that  $u_t - u_{xx} \leq C^*r^{3/2}$  in  $(0, \rho_1) \times (0, \phi(r))$  for some  $C^* > 0$ . Let us consider the functions  $p$  and  $q$ , given by  $p(t) = C^*r^{3/2}t - r$  and  $q(x, t) = c(e^{-(x-(\rho_1/4))^2/4t} + e^{-(x-3(\rho_1/4))^2/4t})$ . It is then clear that  $b = u - p - q$  satisfies

$$\begin{aligned} b_t - b_{xx} &\leq 0 \quad \text{in } (\rho_1/4, 3\rho_1/4) \times (0, \phi(r)), \\ b(\rho_1/4, t) &\leq Z(\rho_1, t) - C^*r^{3/2}t + r - c(1 + e^{-\rho_1^2/(16t)}) \quad \text{for } t \in (0, \phi(r)), \\ b(3\rho_1/4, t) &\leq Z(\rho_1, t) - C^*r^{3/2}t + r - c(1 + e^{-\rho_1^2/(16t)}) \quad \text{for } t \in (0, \phi(r)), \\ b(x, 0) &= 0 \quad \text{for } x \in (\rho_1/4, 3\rho_1/4). \end{aligned}$$

Obviously, in the definition of  $q$  the constant  $c$  can be chosen large enough to have  $Z(\rho_1, t) - C^*r^{3/2}t + r - c(1 + e^{-\rho_1^2/(16t)}) < 0$  for any  $t \in (0, \phi(r))$ . If this is the case, we get  $u \leq p + q$  and, in particular,

$$u(\rho_1/2, t) \leq (p + q)(\rho_1/2, t) = 2ce^{-\rho_1^2/(64t)} + C^*r^{3/2}t - r.$$

Therefore, we see that there exist  $C_0$  and  $C'_0$  such that  $u(\rho_1/2, t) < -C'_0r$  for any  $t \in (0, C_0\phi(r))$ .

This proves (10) and, consequently, ends the proof of theorem 3.

## 4 Other more realistic nonlinear equations and systems

There are a lot of more realistic nonlinear equations and systems from mechanics that can also be considered in this context. First, we have the well known Navier-Stokes equations:

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = v1_\omega, & \nabla \cdot y = 0, & (x, t) \in Q, \\ y = 0, & & (x, t) \in \Sigma, \\ y(x, 0) = y^0(x), & & x \in \Omega. \end{cases} \tag{26}$$

Here and below,  $Q$  and  $\Sigma$  respectively stand for the sets

$$Q = \Omega \times (0, T) \quad \text{and} \quad \Sigma = \partial\Omega \times (0, T),$$

where  $\Omega \subset \mathbb{R}^N$  is a nonempty bounded domain,  $N = 2$  or  $N = 3$  and (again)  $\omega \subset\subset \Omega$  is a nonempty open set.

To our knowledge, the best results concerning the controllability of this system have been given in [11] and [12].<sup>3</sup> Essentially, these results establish the local exact controllability of the solutions of (26) to uncontrolled trajectories (this is, more or less, the analog of the *positive* controllability result in theorem 3).

Similar results have been given in [18] for the Boussinesq equations

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = \theta k + v1_\omega, & \nabla \cdot y = 0, \\ \theta_t + y \cdot \nabla \theta - \Delta \theta = u1_\omega, \end{cases} \quad (27)$$

complemented with initial and Dirichlet boundary conditions for  $y$  and  $\theta$  (see [12] for a controllability result with a reduced number of scalar controls).

Let us also mention [3, 21], where the controllability of the MHD and other related equations has been analyzed.

Another system is considered in [9]:

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = \nabla \times w + v1_\omega, & \nabla \cdot y = 0, \\ w_t + (y \cdot \nabla)w - \Delta w - \nabla(\nabla \cdot w) = \nabla \times y + u1_\omega. \end{cases} \quad (28)$$

Here,  $N = 3$ . These equations govern the behavior of a micropolar fluid, see [28]. As usual,  $y$  and  $p$  stand for the velocity field and pressure and  $w$  is the microscopic velocity of rotation of the fluid particles. Again, the local exact controllability of the solutions to the trajectories is established.

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<sup>3</sup>The main ideas come from [16, 23]; some additional results will appear soon in [20] and [17].

Notice that this case involves a nontrivial difficulty. The main reason is that  $w$  is a nonscalar variable and the equations satisfied by its components  $w_i$  are coupled through the second-order terms  $\partial_i(\nabla \cdot w)$ . This is a serious inconvenient and an appropriate strategy has to be applied in order to deduce the required Carleman estimates.

For these systems, the proof of the controllability can be achieved arguing as in the first part of the proof of theorem 3. This is the general structure of the argument:

- First, consider a linearized similar problem and the associated adjoint system and rewrite the original controllability problem in terms of a fixed point equation.
- Then, prove a global Carleman inequality and an observability estimate for the adjoint system. This provides a controllability result for the linearized problem.
- Prove appropriate estimates for the control and the state (this needs some kind of *smallness* of the data); prove an appropriate compactness property of the state and deduce that there exists at least one fixed point.

There is an alternative method that relies on the implicit function theorem. It corresponds to another strategy introduced in [15]:

- First, rewrite the original controllability problem as a nonlinear equation in a space of admissible “state-control” pairs.
- Then, prove an appropriate global Carleman inequality and a regularity result and deduce that the linearized equation possesses at least one solution. Again, this provides a controllability result for a related linear problem.



- Check that the hypotheses of a suitable implicit function theorem are satisfied and deduce a local result.

At present, no negative result is known to hold for these nonlinear systems (apart from the one-dimensional Burgers equation).

## 5 Some remarks on the Ginzburg-Landau equation

We end this paper with a brief section devoted to the controllability of the Ginzburg-Landau equation. The system under consideration is the following:

$$\left\{ \begin{array}{ll} m_t - \alpha m \times m_t - \Delta m + \frac{|m|^2 - 1}{\varepsilon} m = v1_\omega, & (x, t) \in Q, \\ \frac{\partial m}{\partial n} = 0, & (x, t) \in \Sigma, \\ m(x, 0) = m^0, & x \in \Omega. \end{array} \right. \quad (29)$$

Here,  $\Omega \subset \mathbb{R}^3$  is a regular bounded open set,  $m = (m_1, m_2, m_3)$  is the *magnetization* field,  $\varepsilon > 0$  is a parameter,  $\alpha \geq 0$  is a physical constant and it is assumed that  $m^0$  is a measurable initial field satisfying  $|m^0(x)| \equiv 1$ . For the motivation of the system satisfied by  $m$ , see for instance [5].

In this framework, an interesting controllability problem is the following:

*Given a stationary solution  $m^* = m^*(x)$  and an initial field  $m^0 = m^0(x)$  with  $|m^*(x)| \equiv |m^0(x)| \equiv 1$ , find a control  $v \in L^2(\omega \times (0, T))^3$  such that the associated solution of (29) satisfies*

$$m(x, T) = m^* \quad \text{in } \Omega.$$

By introducing the new variable  $y$ , with  $m = m^* + y$ , this can be rewritten in terms of a null controllability problem. Indeed, let us consider the system

$$\begin{cases} y_t - \alpha(y + m^*) \times y_t - \Delta y + G_\varepsilon(x, y)y = v1_\omega, & (x, t) \in Q, \\ \frac{\partial y}{\partial n} = 0, & (x, t) \in \Sigma, \\ y(x, 0) = m^0(x) - m^*(x), & x \in \Omega, \end{cases} \quad (30)$$

where

$$G_\varepsilon(x, y)y \equiv \frac{|m^*(x) + y|^2 - 1}{\varepsilon} (m^*(x) + y) - \frac{|m^*(x)|^2 - 1}{\varepsilon} m^*(x).$$

Then the problem is:

*Given a stationary solution  $m^* = m^*(x)$  and an initial field  $m^0 = m^0(x)$  with  $|m^*(x)| \equiv |m^0(x)| \equiv 1$ , find a control  $v \in L^2(\omega \times (0, T))^3$  such that the associated solution of (30) satisfies*

$$y(x, T) = 0 \quad \text{in } \Omega.$$

A partial (positive) answer to this problem is given in [6]. More precisely, it is shown there that there exists  $\kappa = \kappa(\Omega, \omega, T, \alpha, \varepsilon)$  such that, whenever  $\|m^* - m^0\|_{L^2} \leq \kappa$ , the existence of such controls can be ensured.

**Remark 3.** For any fixed  $v$ , the solutions of (29) converge in some sense as  $\varepsilon \rightarrow 0$  to a solution of the so called Landau-Lifshitz equation:

$$\alpha m_t = m \times (\Delta m - m_t + v1_\omega), \quad |m| = 1. \quad (31)$$

Consequently, it would be very interesting to be able to solve the previous problem with controls  $v$  uniformly bounded with respect to  $\varepsilon$ . However, this is apparently a difficult question.

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