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LIFE SPAN OF SOLUTIONS OF A STRONGLY COUPLED PARABOLIC SYSTEM

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Abstract

We consider the parabolic system $\partial_t w_\lambda - \triangle w_\lambda = F(w_\lambda)$ in $\mathbb{R}$, where $\lambda > 0$, $w_\lambda = (u_\lambda, v_\lambda)$, $F(w_\lambda) = ({u_\lambda}^a {v_\lambda}^b, {u_\lambda}^c {v_\lambda}^d)$. It is assumed that $a, b, c, d \geq 1$, $b > d-1$, $c > a-1$, $\max\{a+b, c+d\} \leq 3$ and $w_\lambda(0) = (\lambda^{b+1-d}\varphi_1, \lambda^{c+1-a}\varphi_2)$ for some positive functions $\varphi_1, \varphi_2 \in C_0(\mathbb{R})$. Under these conditions w_λ blows up for all $\lambda > 0$. We study the life span of w_λ for λ small.

1 Introduction

In this work we consider positive solutions of the fully coupled parabolic system

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$$\begin{cases} u_t - \triangle u = u^a v^b & \text{in } (0,T)\times\mathbb{R}, \\ v_t - \triangle v = u^c v^d & \text{in } (0,T)\times\mathbb{R}, \\ u(0) = u_0, & \text{in } \mathbb{R}, \\ v(0) = v_0 & \text{in } \mathbb{R}, \end{cases} \tag{1.1}$$

where $u_0, v_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$, $v_0 \geq 0$ and $a, b, c, d \geq 1$ are such that

$$(b+1-d)(c+1-a) > 0, \tag{1.2}$$

It is well known that (1.1) has a unique classical solution $w(t) = (u(t), v(t))$ defined over a maximal interval $[0, T)$, $T \leq +\infty$. When $T < +\infty$ we say that w blows up at the blowup time T. Blow up phenomena for semilinear parabolic systems in $\mathbb{R}^N$ has been studied by several authors, see for example [1], [3], [5], [6], [7], [10], [12]. In particular, Escobedo and Levine [8] proved the following result for $N = 1$. Suppose that

$$a > 1 \text{ if } a+b \leq c+d, \qquad d > 1 \text{ if } a+b > c+d, \tag{1.3}$$

and that

$$\min\{a+b, c+d\} \leq 3. \tag{1.4}$$

If $u_0 \neq 0$ and $v_0 \neq 0$ then $w(t)$ blows up. Note that $w(t) = (S(t)u_0, 0)$ and $w(t) = (0, S(t)v_0)$, where $S(t)$ is the heat semi-group operator, are global solutions of (1.1). When $b = c = p$, $a = d = 0$, (1.4) reduces to $p \leq 3$, the well known Fujita blowup condition for the semilinear heat equation in one space dimension. For this reason, we will say that (1.1) is subcritical when (1.3), (1.4) hold. Under assumptions (1.2) and (1.4) we have that $bc > (a-1)(d-1)$. We then define

$$\begin{aligned} D &= bc - (a-1)(d-1) & \beta_1 &= \frac{2(b+1-d)}{D}, \\ \beta_2 &= \frac{2(c+1-a)}{D} & \gamma &= \frac{c-a+1}{b-d+1}. \end{aligned} \tag{1.5}$$

We show here that unconditional blowup for positive solutions also occurs when both u_0 and v_0 decay slowly at infinity. More precisely, given $\sigma > 0$ define

$$\begin{aligned} J(\sigma) = \ & \{u_0 \in C_0(\mathbb{R}), u_0 \geq 0, \text{ there exists } C > 0 \\ & \quad \text{such that } \liminf_{|x| \longrightarrow \infty} |x|^\sigma u_0(x) \geq C\}. \end{aligned} \tag{1.6}$$

If $u_0 \in J(\beta_1)$, $v_0 \in J(\beta_2)$, and the constant C appearing in (1.6) is large enough, then w blows up, see Proposition 3.1.

The main purpose of this work is to study the growth of the blowup time of the solutions of (1.1) for small initial data in the following sense. Given $\lambda > 0$ we define $w_\lambda = (u_\lambda, v_\lambda)$ as the solution of

$$\begin{cases} u_t - \triangle u = u^a v^b & \text{in } (0,T) \times \mathbb{R}, \\ v_t - \triangle v = u^c v^d & \text{in } (0,T) \times \mathbb{R}, \\ u(0) = \lambda^{b+1-d} \varphi_1 & \text{in } \mathbb{R}, \\ v(0) = \lambda^{c+1-a} \varphi_2 & \text{in } \mathbb{R}, \end{cases} \tag{1.7}$$

where $\varphi_1, \varphi_2 \in C_0(\mathbb{R})$ are nonnegative functions such that, either $\varphi_i \in L^1(\mathbb{R})$ or $\varphi_i \in J(\sigma_i)$ for $\sigma_i \leq 1$, $i = 1, 2$. We suppose that either (1.4) holds or that $u_0 \in J(\sigma_1)$, $v_0 \in J(\sigma_2)$ for some $\sigma_1 < \beta_1$, $\sigma_2 < \beta_2$. This ensures that w_λ blows up for all $\lambda > 0$. We analyse the growth of the blowup time T_λ of w_λ as $\lambda \longrightarrow 0$. This allows to distinguish in each case if the solution blows up because of its slow decay at infinity or due to the subcriticality of the problem.

Define

$$I(\sigma, l) = \{\varphi \in C_0(\mathbb{R}), \varphi \geq 0, \lim_{x \longrightarrow \infty} |x|^\sigma \varphi(x) = l\}, \tag{1.8}$$

We now present our main results, concerning some sharp estimates on the growth of T_λ. For φ_1, φ_2 having slow decay, we show the following.

Theorem 1. *Assume (1.2), (1.3), (1.4) and let β_1, γ be given by (1.5). Consider $\varphi_1 \in I(\sigma_1, l_1)$, $\varphi_2 \in I(\sigma_2, l_2)$, where $l_1, l_2 > 0$, $\sigma_1 < 1$ and $\sigma_2 = \gamma\sigma_1 < 1$. Then $\rho_1 = \beta_1 - \sigma_1 > 0$ and there exists $L_1 > 0$ such that*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = L_1.$$

We next consider the case where φ_1 has slow decay and $\varphi_2 \in L^1$.

Theorem 2. *Assume (1.2), (1.3), (1.4). Suppose that $\varphi_1 \in I(\sigma_1, l)$, where $\sigma_1 < 1$, $\gamma\sigma_1 = 1$, $l > 0$ and $\varphi_2 \in L^1$, $\varphi_2 \geq 0$. Assume also that $M = \int \varphi_2 > 0$. Then there exists $L_2 > 0$ such that*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = L_2.$$

For $\varphi_1, \varphi_2 \in L^1$, we prove the following.

Theorem 3. *Assume (1.2), (1.3), (1.4) and suppose further that $a+b = c+d < 3$. Let $\varphi_1, \varphi_2 \in L^1$ be nonnegative functions such that $\displaystyle\int \varphi_1 > 0$, $\displaystyle\int \varphi_2 > 0$. Then there exists $L_3 > 0$ satisfying*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = L_3.$$

To treat the case where φ_1, φ_2 decay as $|x|^{-1}$ at infinity, we define $g(\mu) = \mu^{\rho_1} \log \mu$ for $\mu > \overline{\mu} := e^{1/\rho_1}$. The function g is invertible and we call $h = g^{-1}$ its inverse.

Theorem 4. *Assume (1.2), (1.3), (1.4). Let $a+b = c+d$, $\varphi_1 \in I(1, l_1)$ and $\varphi_2 \in I(1, l_2)$ for some $l_1, l_2 > 0$. Then w_λ blows up at a finite time T_λ and there exists $L_4 > 0$ such that*

$$\lim_{\lambda \longrightarrow 0} (h(\lambda^{-(b+1-d)}))^{-2} T_\lambda = L_4,$$

To prove these results we proceed as follows. Consider $\tilde{w}_\mu(x,t) = (\mu^{\beta_1} u(\mu x, \mu^2 t),$
$\mu^{\beta_2} v(\mu x, \mu^2 t))$, where $\mu > 0$. Then $\tilde{w}_\mu$ is also a solution of (1.1) blowing up at $\tilde{T}_\mu = \mu^{-2}\, T_\lambda$. We choose μ depending on λ in such a way that $\tilde{w}_\mu(t)$ converges (in some topology) to a limit solution $z^*(t)$. It turns out that $z^*(0)$ is a singular initial datum. Suppose $z^*(t)$ blows up at a finite time T^*. Continuity of the blowup time with respect to the initial data holds here, so that $\tilde{T}_\mu = \mu^{-2}\, T_\lambda \longrightarrow T^*$ as $\lambda \longrightarrow 0$. We have that $\mu = \mu(\lambda)$ and $z^*(0) = z_0^*(\varphi_1, \varphi_2)$, that is,

$$\mu^{-2}(\lambda)\, T_\lambda \longrightarrow T^*(\varphi_1, \varphi_2) \quad \text{as } \lambda \longrightarrow 0, \tag{1.9}$$

Therefore, the constants L_1, L_2, L_3 and L_4 in the theorems above are related to the blowup time of some limiting problems.

For $i = 1, 2$ set $\sigma_i = +\infty$ whenever $\varphi_i \in L^1(\mathbb{R})$ and define

$$\rho_1 = \beta_1 - \min\{\sigma_1, 1\}, \qquad \rho_2 = \beta_2 - \min\{\sigma_2, 1\}, \tag{1.10}$$

where we used (1.5). The theorems stated above are restricted to the case $\rho_1 = \rho_2$ for the following reason. It turns out that for $\rho_1 > \rho_2$ the limit solution $z^*(t)$ is equal to $(S(t)\varphi, 0)$ and is global. Under this circunstances our argument breaks down. This does not occur when $a = d = 0$. This is why a more complete description of the blowup behaviour of solutions of the weakly coupled system can be provided, see [6].

Our approach leads to discussing the well-posedness of (1.1) for nonregular initial data. This is done in Section 2, where we state the problem in $\mathbb{R}^N$ and we also consider solutions which are not necessarily positive. In Section 3 we prove the blow up of slowly decaying positive solutions of the Cauchy problem in $\mathbb{R}^N$. We also show the continuity of the blowup time with respect to initial data. Finally, the main results are stated and proved in Section 4.

2 The Semilinear Parabolic Equation with Singular Data

In this section we discuss the existence of solutions of the Cauchy problem in $\mathbb{R}^N$, $N \geq 1$, for the fully coupled system with singular initial data. Consider

$$\begin{cases} u_t - \triangle u = |u|^{a-1}u|v|^{b-1}v & \text{in } (0,T)\times\mathbb{R}^N, \\ v_t - \triangle v = |u|^{c-1}u|v|^{d-1}v & \text{in } (0,T)\times\mathbb{R}^N, \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \\ v(0) = v_0 & \text{in } \mathbb{R}^N\,. \end{cases} \tag{2.1}$$

In the sequel, L^r denotes the Lebesgue space $L^r(\mathbb{R}^N)$ and $\|.\|_r$ its usual norm. Consider $E^{r,s} = L^r + L^s$ the Banach space endowed with the standard norm $\|u\|_{r,s} = \inf \|u_r\|_r + \|u_s\|_s$, where $u = u_r + u_s$, $u_r \in L^r$, $u_s \in L^s$. We also denote by $\mathcal{M}$ the space of finite measures in $\mathbb{R}^N$.

Our results are to be compared with those of [6], where the weakly coupled system corresponding to $a = d = 0$ in (2.1) is considered. In [6] it is shown that the problem is well-posed if $u_0 \in E^{r_1,s_1}$, $v_0 \in E^{r_2,s_2}$ and

$$\max\{\frac{b}{r_2} - \frac{1}{r_1}, \frac{c}{r_1} - \frac{1}{r_2}\} < \frac{2}{N}. \tag{2.2}$$

Singular data in measure spaces was also discussed in [6]. If, for example, $u_0 \in E^{r,s}$, $v_0 \in \mathcal{M}$ then (2.2) with $r_2 = 1$ still ensures well-posedness.

We show in this section that for (2.1), the condition (2.2) should be replaced by

$$\max\{\frac{b}{r_2} + \frac{a-1}{r_1}, \frac{c}{r_1} + \frac{d-1}{r_2}\} < \frac{2}{N}. \tag{2.3}$$

Before proving the main results of this section, we present some preliminary lemmas.

Lemma 2.1. *Let $k > 0$, $T > 0$, $0 < \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 < 1$ be such that $1 + \alpha_1 > \beta_1 + \gamma_1$, $1 + \alpha_2 > \beta_2 + \gamma_2$. Let $A(t)$ be a positive continuous function defined in $(0, T)$ and such that $\int_0^t A(s)\, ds < +\infty$ for $t < T$. Consider $\varphi, \psi : (0, T) \longrightarrow \mathbb{R}^+$ nondecreasing positive functions satisfying*

$$\begin{aligned} \varphi(t) &\leq A(t) + kt^{\alpha_1} \int_0^t (t-s)^{-\beta_1} s^{-\gamma_1} (\varphi(s) + \psi(s))\, ds, \\ \psi(t) &\leq A(t) + kt^{\alpha_2} \int_0^t (t-s)^{-\beta_2} s^{-\gamma_2} (\varphi(s) + \psi(s))\, ds \end{aligned} \tag{2.4}$$

in $(0, T)$. Then there exists $C = C(\alpha_1, \alpha_2, \beta_1, \beta_2, k, T) > 0$ such that, for all $t \in (0, T)$,

$$\varphi(t) + \psi(t) \leq C(A(t) + \int_0^t e^{t-s} A(s)\, ds). \tag{2.5}$$

Proof: Take $\tau > 0$ such that $\varphi(t) \leq A(t) + (\varphi(t) + \psi(t))/4$, $\psi(t) \leq A(t) + (\varphi(t) + \psi(t)/4$ for $t \leq \tau$. Hence,

$$\varphi(t) + \psi(t) \leq 4A(t). \tag{2.6}$$

Consider now $t > \tau$ and choose $0 < a < b < 1$ such that

$$\begin{aligned} T^{1+\alpha_1-\beta_1-\gamma_1}(\int_0^a + \int_b^1)(1-s)^{-\beta_1} s^{-\gamma_1}\, ds \leq \frac{1}{4k}, \\ T^{1+\alpha_2-\beta_2-\gamma_2}(\int_0^a + \int_b^1)(1-s)^{-\beta_2} s^{-\gamma_2}\, ds \leq \frac{1}{4k}. \end{aligned}$$

Then,

$$\begin{aligned} \varphi(t) &\leq A(t) + kt^{\alpha_1}(\int_0^{at} + \int_{at}^{bt} + \int_{bt}^t)(t-s)^{-\beta_1} s^{-\gamma_1} (\varphi(s) + \psi(s))\, ds \\ &\leq A(t) + \tfrac{1}{4}(\varphi(t) + \psi(t)) + kT^{1+\alpha_1-\beta_1-\gamma_1}(1-b)^{-\beta_1} a^{-\gamma_1} \\ &\quad \int_0^t (\varphi(s) + \psi(s))\, ds. \end{aligned}$$

Analogously,

$$\psi(t) \leq A(t) + \tfrac{1}{4}(\varphi(t)+\psi(t)) + kT^{1+\alpha_2-\beta_2-\gamma_2}(1-b)^{-\beta_2}a^{-\gamma_2} \int_0^t (\varphi(s)+\psi(s))\,ds.$$

Adding up both equations and using (2.6) we get for all $t > 0$

$$\begin{aligned}\varphi(t)+\psi(t) \leq\ & 4A(t) + 2k(T^{1+\alpha_1-\beta_1-\gamma_1}(1-b)^{-\beta_1}a^{-\gamma_1} \\ & + T^{1+\alpha_2-\beta_2-\gamma_2}(1-b)^{-\beta_2}a^{-\gamma_2}) \int_0^t (\varphi(s)+\psi(s))\,ds.\end{aligned}$$

Applying Gronwall's Lemma we get the desired result.

□

Lemma 2.2. *Consider $a, b, c, d \geq 1$, $r_1, r_2 \geq 1$. Assume that*

$$\frac{a-1}{r_1} + \frac{b}{r_2} < \frac{2}{N}, \qquad \frac{c}{r_1} + \frac{d-1}{r_2} < \frac{2}{N}. \tag{2.7}$$

Then, there exist $\eta > r_1$, $\xi > r_2$ such that

$$\frac{a}{\eta} + \frac{b}{\xi} \leq 1, \qquad \frac{c}{\eta} + \frac{d}{\xi} \leq 1, \tag{2.8}$$

$$\frac{a}{r_1} + \frac{b}{r_2} - \frac{2}{N} < \frac{a}{\eta} + \frac{b}{\xi}, \qquad \frac{c}{r_1} + \frac{d}{r_2} - \frac{2}{N} < \frac{c}{\eta} + \frac{d}{\xi}. \tag{2.9}$$

Proof: Without loss of generality we suppose that $\frac{a}{r_1} + \frac{b}{r_2} \leq \frac{c}{r_1} + \frac{d}{r_2}$. Since $\frac{c}{r_1} + \frac{d}{r_2} < \frac{2}{N} + \frac{1}{r_2} \leq \frac{2}{N} + 1$, there exists $k \in (0,1)$ such that

$$1-(\frac{a}{r_1}+\frac{b}{r_2})^{-1} \leq 1-(\frac{c}{r_1}+\frac{d}{r_2})^{-1} < k < \frac{2}{N}(\frac{c}{r_1}+\frac{d}{r_2})^{-1} \leq \frac{2}{N}(\frac{a}{r_1}+\frac{b}{r_2})^{-1}. \tag{2.10}$$

Define $\eta > r_1$, $\xi > r_2$ by

$$\frac{1}{\eta} = \frac{1-k}{r_1} \qquad \frac{1}{\xi} = \frac{1-k}{r_2}.$$

Then,

$$\frac{a}{\eta}+\frac{b}{\xi}=(1-k)\left(\frac{a}{r_1}+\frac{b}{r_2}\right), \qquad \frac{c}{\eta}+\frac{d}{\xi}=(1-k)\left(\frac{c}{r_1}+\frac{d}{r_2}\right). \quad (2.11)$$

The result follows from (2.10), (2.11).

□

Lemma 2.3. *Let $a, b, c, d \geq 1$ such that $bc > (a-1)(d-1)$. Consider ρ_0, θ_0, $c_1, c_2 > 0$ such that*

$$(a-1)\rho_0 + b\theta_0 = c_1, \quad c\rho_0 + (d-1)\theta_0 = c_2.$$

For $k \in \mathbb{N}$, define ρ_k, θ_k recursively by

$$a\rho_k + b\theta_k - \rho_{k+1} = c_1 + \varepsilon, \quad c\rho_k + d\theta_k - \theta_{k+1} = c_2 + \delta$$

where $\varepsilon, \delta > 0$ satisfy

$$\frac{d-1}{b}\varepsilon < \delta < \frac{c}{a-1}\varepsilon. \quad (2.12)$$

Then, there exists $k > 1$ such that $\rho_k < 0$, $\theta_k < 0$.

Proof: Let (ρ^*, θ^*) satisfy

$$(a-1)\rho^* + b\theta^* = c_1 + \varepsilon, \quad c\rho^* + (d-1)\theta^* = c_2 + \delta.$$

Using (2.12) we get that $\rho_0 < \rho^*$ and $\theta_0 < \theta^*$. It is easy to verify inductively that for all $k \geq 0$ $\rho_k > \rho_{k+1}$ and $\theta_k > \theta_{k+1}$. Suppose one of the sequences is bounded. Then clearly the other sequence is also bounded and $\rho_k \searrow \rho^*$, $\theta_k \searrow \theta^*$ as $k \longrightarrow \infty$. But this contradicts the fact that $\rho_0 < \rho^*$ and $\theta_0 < \theta^*$. This finishes the proof.

□

We will also use the fact that the heat semigroup $S(t)$ is well defined in $E^{r,s}$ and satisfies

$$\sup_{t \leq T} \|S(t)u\|_\eta \leq \max\{1, T^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}\} t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta})} \|u\|_{r,s}, \quad (2.13)$$

if $1 \leq r \leq s \leq \eta$. This is an immediate consequence of the usual $L^p - L^q$ regularity result for the Laplace operator.

We first consider $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$. We set $\|w\|_{r_1,s_1,r_2,s_2} = \|u\|_{r_1,s_1} + \|v\|_{r_2,s_2}$ for $w = (u,v) \in E^{r_1,s_1} \times E^{r_2,s_2}$.

Theorem 2.4. *Let $a, b, c, d > 1$, $r_1 \geq 1$, $r_2 \geq 1$ be such that $bc - (a-1)(d-1) > 0$ and*

$$\frac{a-1}{r_1} + \frac{b}{r_2} < \frac{2}{N}, \qquad \frac{c}{r_1} + \frac{d-1}{r_2} < \frac{2}{N}. \tag{2.14}$$

Let η, ξ be as in Lemma 2.2 and consider s_1, s_2 such that

$$r_1 \leq s_1 < \eta, \qquad r_2 \leq s_2 < \xi. \tag{2.15}$$

Then given $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$, there exist $T > 0$ and a unique function $u \in C([0,T], E^{r_1,s_1} \times E^{r_2,s_2})$ which is a classical solution of (2.1) in $(0,T)$.

In addition, let $\{w_{n,0}\}_{n\in\mathbb{N}} \subset E^{r_1,s_1} \times E^{r_2,s_2}$ and $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$ be such that $w_{n,0} \longrightarrow w_0$ in $E^{r_1,s_1} \times E^{r_2,s_2}$. Then for t small enough $w_n(t) \longrightarrow w(t)$ uniformly.

Proof: The proof is analogous of the one presented in [6] to show Theorem 2.3. It consists in obtaining a local weak solution, to show the regularity of this solution, the uniqueness of the classical solution and the continuous dependence on the initial data.

Existence. We first construct the functional space in which we prove the existence of a solution of (2.1). Set

$$\alpha = \frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta}\right), \qquad \beta = \frac{N}{2}\left(\frac{1}{r_2} - \frac{1}{\xi}\right). \tag{2.16}$$

By (2.9), we get

$$a\alpha + b\beta < 1 \qquad c\alpha + d\beta < 1. \tag{2.17}$$

Let

$$W = L^\infty((0,T); L^\eta \times L^\xi),$$

be the Banach space with norm $\|w\|_W := \sup_{t\in(0,T)}\{t^\alpha \|u(t)\|_\eta + t^\beta \|v(t)\|_\xi\}$. Given $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$, $T < 1$, and $M > 0$ such that $\|w_0\|_{r_1,s_1,r_2,s_2} \leq M$, let K be the closed ball of radius $M+1$ of W. If $(u,v) \in K$, define $\Phi(u,v) = (\Phi_1(u,v), \Phi_2(u,v))$ as

$$\Phi_1(u,v) = S(t)u_0 + \int_0^t S(t-s)|u|^{a-1}u(s)|v|^{b-1}v(s)\,ds,$$

$$\Phi_2(u,v) = S(t)v_0 + \int_0^t S(t-s)|u|^{c-1}u(s)|v|^{d-1}v(s)\,ds.$$

We will show that $\Phi(K) \subset K$ if T is chosen appropriately. First, we have by (2.13) that

$$t^\alpha \|S(t)u_0\|_\eta + t^\beta \|S(t)v_0\|_\xi \leq \|u_0\|_{r_1,s_1} + \|v_0\|_{r_2,s_2} \leq M. \tag{2.18}$$

Note that by (2.14), we get from $\eta > r_1$, $\xi > r_2$ that

$$\frac{N}{2}\left(\frac{a-1}{\eta} + \frac{b}{\xi}\right) < 1.$$

Using this, the smoothing effect of the heat semigroup, Hölder's inequality and (2.17) we get

$$\begin{aligned} t^\alpha \| &\int_0^t S(t-s)|u|^{a-1}u(s)|v|^{b-1}v(s)\,ds\|_\eta \\ &\leq t^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{a-1}{\eta}+\frac{b}{\xi})} \|u(s)\|_\eta^a \|v(s)\|_\xi^b\,ds \\ &\leq (M+1)^{a+b} t^{1-\frac{N}{2}(\frac{a-1}{r_1}+\frac{b}{r_2})} \int_0^1 (1-s)^{-\frac{N}{2}(\frac{a-1}{\eta}+\frac{b}{\xi})} s^{-a\alpha-b\beta}\,ds, \end{aligned}$$

Thus

$$t^\alpha \|\Phi_1(u,v)\|_\eta \leq M+1,$$

if T is small enough, see (2.18). Analogously, taking T eventually smaller, we obtain

$$t^{\beta}||\Phi_2(u,v)||_{\xi} \leq (M+1).$$

This shows that $\Phi(K) \subset K$ for T small. Similar computations, taking a smaller T if necessary, show that Φ is a contraction in K. This gives the existence of a local weak solution $w = (u,v)$ of (2.1).

Regularity. We use the bootstrap argument of [11] to prove that $u(t) \in L^{\gamma_1}(\mathbb{R}^N)$ for $s_1 \leq \gamma_1 \leq \infty$, $v(t) \in L^{\gamma_2}(\mathbb{R}^N))$ for $s_2 \leq \gamma_2 \leq \infty$ and that there exists $C = C(\gamma_1,\gamma_2) > 0$ such that for all $t < T$,

$$t^{\frac{N}{2}(\frac{1}{r_1}-\frac{1}{\gamma_1})}\|u(t)\|_{\gamma_1} \leq C, \qquad t^{\frac{N}{2}(\frac{1}{r_2}-\frac{1}{\gamma_2})}\|v(t)\|_{\gamma_2} \leq C. \tag{2.19}$$

Note that the existence part of the proof ensures that this is valid for $\gamma_1 = \eta$ and $\gamma_2 = \xi$. Consider first $s_1 \leq \gamma_1 < \eta$. Using (2.13) and the fact that $w \in K$, we get

$$\begin{aligned} t^{\frac{N}{2}(\frac{1}{r_1}-\frac{1}{\gamma_1})}\|u(t)\|_{\gamma_1} &\leq C(1 + t^{\frac{N}{2}(\frac{1}{r_1}-\frac{1}{\gamma_1})}\int_0^t (t-s)^{-\frac{N}{2}(\frac{a}{\eta}+\frac{b}{\xi}-\frac{1}{\gamma_1})} s^{-a\alpha-b\beta}\, ds) \\ &\leq C(1 + t^{1-\frac{N}{2}(\frac{a-1}{r_1}+\frac{b}{r_2})}\int_0^1 (1-s)^{-\frac{N}{2}(\frac{a}{\eta}+\frac{b}{\xi}-\frac{1}{\gamma_1})} s^{-a\alpha-b\beta}\, ds). \end{aligned}$$

A corresponding estimate holds for $\|v(t)\|_{\gamma_2}$ when $s_2 \leq \gamma_2 < \xi$. Thus (2.19) is valid for $\gamma_1 \in [s_1,\eta]$, $\gamma_2 \in [s_2,\xi]$. Suppose now that (2.19) holds for some $\gamma_1 \geq \eta$, $\gamma_2 \geq \xi$ and let θ_1, θ_2 be such that

$$\frac{a}{\gamma_1} + \frac{b}{\gamma_2} - \frac{1}{\theta_1} < \frac{2}{N}, \qquad \frac{c}{\gamma_1} + \frac{d}{\gamma_2} - \frac{1}{\theta_2} < \frac{2}{N}.$$

Write

$$u(2t) = S(t)u(t) + \int_0^t S(t-s)|u(t-s)|^{a-1}u(t+s)|v(t-s)|^{b-1}v(t+s)\, ds$$

to obtain

$$
\begin{aligned}
&t^{\frac{N}{2}(\frac{1}{r_1}-\frac{1}{\theta_1})}\|u(2t)\|_{\theta_1} \leq t^{\frac{N}{2}(\frac{1}{r_1}-\frac{1}{\gamma_1})}\|u(t)\|_{\gamma_1} \\
&+CT^{1-\frac{N}{2}(\frac{a-1}{r_1}+\frac{b}{r_2})}\int_0^1 (1-s)^{-\frac{N}{2}(\frac{a}{\gamma_1}+\frac{b}{\gamma_2}-\frac{1}{\theta_1})}(1+s)^{-\frac{N}{2}(a(\frac{1}{r_1}-\frac{1}{\gamma_1})+b(\frac{1}{r_2}-\frac{1}{\gamma_2}))}\,ds \\
&\leq C(1+T^{1-\frac{N}{2}(\frac{a-1}{r_1}+\frac{b}{r_2})}\int_0^1 (1-s)^{-\frac{N}{2}(\frac{a}{\gamma_1}+\frac{b}{\gamma_2}-\frac{1}{\theta_1})}(1+s)^{-\frac{N}{2}(a(\frac{1}{r_1}-\frac{1}{\gamma_1})+b(\frac{1}{r_2}-\frac{1}{\gamma_2}))}\,ds.
\end{aligned}
$$

Writing an analogous estimate for $\|v(2t)\|_{\theta_2}$, we verify (2.19) for θ_1, θ_2. In this way, we can bootstrap starting from η, ξ. Lemma 2.3 ensures that $\theta_1 = \theta_2 = +\infty$ can be reached in a finite number of steps.

Uniqueness. Uniqueness of classical solutions is proved in [2] for the scalar case. Their arguments extend readily to the present case.

Continuous dependence. Consider $w_0 = (u_{w,0}, v_{w,0})$, $z_0 = (u_{z,0}, v_{z,0})$ and let $w(t) = (u_w(t), v_w(t))$, $z(t) = (u_z(t), v_z(t))$ be their corresponding solutions, defined in $[0,T)$. Call $\Delta u_0 = u_{w,0} - u_{z,0}$, $\Delta u = u_w - u_z$, $\Delta v_0 = v_{w,0} - v_{z,0}$, $\Delta v = v_w - v_z$. We have

$$
\left\{
\begin{aligned}
&\Delta u(t) = S(t)\Delta u_0 \\
&\quad + \int_0^t S(t-s)(|u_w|^{a-1}u_w(s)|v_w|^{b-1}v_w(s) - |u_z|^{a-1}u_z(s)|v_z|^{b-1}v_z(s))\,ds, \\
&\Delta v(t) = S(t)\Delta v_0 \\
&\quad + \int_0^t S(t-s)(|u_w|^{c-1}u_w(s)|v_w|^{d-1}v_w(s) - |u_z|^{c-1}u_z(s)|v_z|^{d-1}v_z(s))\,ds.
\end{aligned}
\right.
\tag{2.20}
$$

By (2.8), we may define $p \geq 1$ such that

$$
\frac{1}{p} = \frac{a}{\eta} + \frac{b}{\xi}.
$$

Using that

$$||u_w|^{a-1}u_w - |u_z|^{a-1}u_z| \le 2^{a-1}\max\{|u_w|^{a-1}, |u_z|^{a-1}\}|\Delta u|,$$

and an analogous inequality for v, we get

$$\begin{aligned} &\|(|u_w|^{a-1}u_w(s) - |u_z|^{a-1}u_z(s))|v_w|^b\|_p \\ &\qquad \le (2M+2)^{a+b-1}s^{-(a-1)\alpha-b\beta}\|\Delta u(s)\|_\eta, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\|(|v_w|^{b-1}v_w(s) - |v_z|^{b-1}v_z(s))|u_z|^a\|_p \\ &\qquad \le (2M+2)^{a+b-1}s^{-a\alpha-(b-1)\beta}\|\Delta v(s)\|_\xi, \end{aligned} \tag{2.22}$$

Define $\varphi(t) = \sup_{s\le t} s^\alpha\|\Delta u(s)\|_\eta$, $\psi(t) = \sup_{s\le t} s^\beta\|\Delta v(s)\|_\xi$. We see from (2.20), (2.13), (2.21) and (2.22) that

$$\varphi(t) \le C(\|\Delta u_0\|_{r_1,s_1} + t^\alpha\int_0^t (t-s)^{-\frac{N}{2}(\frac{a-1}{\eta}+\frac{b}{\xi})}s^{-a\alpha-b\beta}(\varphi(s)+\psi(s)\,ds). \tag{2.23}$$

Analogously,

$$\psi(t) \le C(\|\Delta v_0\|_{r_2,s_2} + t^\beta\int_0^t (t-s)^{-\frac{N}{2}(\frac{c}{\eta}+\frac{d-1}{\xi})}s^{-c\alpha-d\beta}(\varphi(s)+\psi(s)\,ds.) \tag{2.24}$$

Note that

$$1+\alpha-\frac{N}{2}\left(\frac{a-1}{\eta}+\frac{b}{\xi}\right)-a\alpha-b\beta = 1-\frac{N}{2}\left(\frac{a-1}{r_1}+\frac{b}{r_2}\right) > 0,$$

$$1+\beta-\frac{N}{2}\left(\frac{c}{\eta}+\frac{d-1}{\xi}\right)-c\alpha-d\beta = 1-\frac{N}{2}\left(\frac{c}{r_1}+\frac{d-1}{r_2}\right) > 0.$$

It then follows from (2.14) and Lemma 2.1 that

$$t^\alpha\|u_w(t)-u_z(t)\|_\eta + t^\beta\|v_w(t)-v_z(t)\|_\xi \le C\|w_0-z_0\|_{r_1,s_1,r_2,s_2}.$$

This shows the continuity of $w_0 \mapsto w(t)$ from $E^{r_1,s_1} \times E^{r_2,s_2}$ to $L^\eta \times L^\xi$. To obtain uniform convergence, we extend (2.7) beyond η and ξ by bootstrapping, as for regularity result.

□

We next consider initial data in measure spaces. Existence results are still valid under (2.3), with the following modifications. If $u_0 \in \mathcal{M}$ ($v_0 \in \mathcal{M}$) set $r_1 = 1$ ($r_2 = 1$). Note that, as a consequence, $N = 1$ is the only case of interest. We refer to [4] and [6] for the corresponding scalar case and the weakly coupled system, respectively.

Theorem 2.5. *Let $N = 1$, $1 \le r \le s$, $a, b, c, d \ge 1$ satisfy $bc - (a-1)(d-1) > 0$ and*

$$\max\{\frac{c}{r} + d - 1, b + \frac{a-1}{r}\} < 2. \tag{2.25}$$

Endow $E^{r,s}$ with the strong topology, $\mathcal{M}$ with the weak- topology and $E^{r,s} \times \mathcal{M}$ with the product topology. Given $w_0 \in E^{r,s} \times \mathcal{M}$ there exists $T > 0$ and a unique (classical for $t > 0$) solution $C((0,T]; E^{r,s} \times L^1)$ of (2.1) such that $u(t) \longrightarrow u_0$ in $E^{r,s}$, $v(t) \rightharpoonup v_0$ weak-* in $\mathcal{M}$ as $t \longrightarrow 0$.*

In addition, let $\{w_{n,0}\}_{n \in \mathbb{N}} \subset E^{r,s} \times \mathcal{M}$ and $w_0 \in E^{r,s} \times \mathcal{M}$ be such that $u_{n,0} \longrightarrow u_0$ in $E^{r,s}$ and $v_{n,0} \rightharpoonup v_0$ weak- in $\mathcal{M}$. Then for t small enough $w_n(t) \longrightarrow w(t)$ uniformly.*

Proof: We proceed as in the proof of Theorem 2.4. Taking $N = 1$, $r_2 = 1$ we define η, ξ by (2.7), (2.8) and α, β by (2.16). A similar fixed point argument in $L^\infty((0,T); L^\eta \times L^\xi)$ yields the existence of a weak solution of the problem. Regularity also follows as before. To obtain the continuous dependence on the initial data, consider $w = (u_w, v_w)$ and $z = (u_z, v_z)$ two solutions and define $\varphi(t) = \sup_{s \le t} s^\alpha \|u_w(s) - u_z(s)\|_\eta$, $\psi(t) = \sup_{s \le t} s^\beta \|v_w(s) - v_z(s)\|_\xi$. Then (2.20) holds and, consequentely, so does (2.23). We further set $B(t) = t^{\beta}\|S(t)\Delta v_0\|_{\xi}$. Now, we replace (2.24) by

$$\psi(t) \leq B(t)+Ct^{\beta}\int_0^t (t-s)^{-\frac{N}{2}(\frac{c}{\eta}+\frac{d-1}{\xi})} s^{-c\alpha-d\beta}(\varphi(s)+\psi(s))\,ds. \quad (2.26)$$

We apply Lemma 2.1 for (2.23), (2.27). This is possible because of (2.26). We obtain

$$\begin{aligned} t^{\alpha}\|\Delta u(t)\|_{\eta} \quad & +t^{\beta}\|\Delta v(t)\|_{\xi} \leq C(\|\Delta u_0\|_{r_1,s_1} + t^{\beta}\|S(t)\Delta v_0\|_{\xi} \\ & + \int_0^t s^{\beta}\|S(s)\Delta v_0\|_{\xi}\,ds. \end{aligned} \quad (2.27)$$

Consider now $u_{n,0} \longrightarrow u_0$ in $E^{r,s}$, $v_{n,0} \rightharpoonup v_0$ weak-* in $\mathcal{M}$ as $n \longrightarrow \infty$. Using (2.25), it follows from the Lebesgue Dominated Convergence Lemma that $w_n(t) \longrightarrow w(t)$ in $L^{\eta} \times L^{\xi}$ for t small enough. Uniform convergence is obtained by bootstrapping, as for regularity result.

□

We omit the proof of our next result, which can be done as in Theorem 2.5.

Theorem 2.6. *Let $N = 1$, $a, b, c, d \geq 1$ satisfy $bc-(a-1)(d-1) > 0$ and $\max\{b+a, c+d\} < 3$. Given $w_0 \in \mathcal{M}^2$ there exists $T > 0$ and a unique (classical for $t > 0$) solution $C((0,T];L^1 \times L^1)$ of (2.1) such that $w(t) \longrightarrow w_0$ as $t \rightharpoonup 0$ in the weak-* topology of $\mathcal{M}^2$.*

In addition, let $\{w_{n,0}\}_{n\in\mathbb{N}} \subset \mathcal{M}^2$ and $w_0 \in \mathcal{M}^2$ be such that $w_{n,0} \rightharpoonup w_0$ weak- in $\mathcal{M}^2$. Then for t small enough $w_n(t) \longrightarrow w(t)$ uniformly.*

3 Further Results

In this section we present some further results concerning blowing up positive solutions of (2.1). Below, we assume that $u_0, v_0 \in$

$C_0(\mathbb{R}^N)$ and that w is the solution corresponding to $w(0) = (u_0, v_0)$. We have the following.

Proposition 3.1. *Let a, b, c, d be nonnegative real numbers such that β_1, β_2 given by (1.5) are positive. There exists $C > 0$ such that if*

$$\liminf_{|x| \longrightarrow \infty} |x|^{\beta_1} u_0(x) \geq C, \qquad \liminf_{|x| \longrightarrow \infty} |x|^{\beta_2} v_0(x) \geq C. \tag{3.1}$$

then the corresponding solution w of (2.1) blows up.

Proof: We construct blowing up subsolutions of (2.1) as in [5], [10]. We may assume that $\beta_2 \geq \beta_1$. Let $R > 0$ be such that $|x|^{\beta_1} u_0(x) \geq C$ and $|x|^{\beta_2} v_0(x) \geq C$ if $|x| > R$. Define $\varphi(x)$ such that $\varphi(x) = 0$ if $|x| \leq R$ and $\varphi(x) = |x|^{-\beta_1}$ if $|x| > R$. Given $c > 0$ define

$$\underline{u} = \left((S(t)c\varphi)^{-2/\beta_1} - \frac{2t}{\beta_2} \right)^{-\beta_1/2}, \qquad \underline{v} = \left((S(t)c\varphi)^{-2/\beta_1} - \frac{2t}{\beta_2} \right)^{-\beta_2/2}.$$

$$\begin{aligned} \underline{u}_t - \triangle \underline{u} = & \ \tfrac{\beta_1}{\beta_2} \underline{u}^{1+2/\beta_1} - \tfrac{4+2\beta_1}{\beta_1 \beta_2} (S(t)c\varphi)^{-2(1+\beta_1)/\beta_1} \underline{u}^{1+4/\beta_1} |\nabla S(t)c\varphi|^2 t \\ & \leq \underline{u}^{1+2/\beta_1} = \underline{u}^a \underline{v}^b \end{aligned}$$

and

$$\begin{aligned} \underline{v}_t - \triangle \underline{v} = & \ \underline{v}^{1+2/\beta_2} \\ & - \tfrac{1}{\beta_1^2} (S(t)c\varphi)^{-2(1+\beta_1)/\beta_1} \underline{v}^{1+4/\beta_2} |\nabla S(t)c\varphi|^2 (\beta_2(\beta_2 - \beta_1)(S(t)c\varphi)^{-2/\beta} \\ & + (4 + 2\beta_1)t) \leq \underline{v}^{1+2/\beta_2} = \underline{u}^c \underline{v}^d. \end{aligned}$$

Therefore, $(\underline{u}, \underline{v})$ is a subsolution as long as it is defined. We get from (3.1) that $S(t)\varphi \approx t^{-\beta_1/2}$ for t large. Hence, we may find c large enough such that $\|S(t')c\varphi\|_\infty^{-2/\beta_1} = \frac{2t'}{\beta_2}$ for some $t' > 0$. This shows that $\underline{u}$ blows up. If C in (3.1) is large, then $\underline{u}(0) = c\varphi_1 \leq u_0$ and $\underline{v}(0) = (c\varphi_1)^{\beta_2/\beta_1} \leq v_0$. We this choice, $\underline{u} \leq u$, finishing the proof.

$\square$

The following result establishes an upper bound on the growth of $w(t)$ near the blowup time.

Proposition 3.2. *Consider $N = 1$ and assume (1.2), (1.3), (1.4) to hold. Then, given $R > 0$ there exists $C > 0$ such that for all positive solutions $w = (u, v)$ of (2.1) satisfying $\|u(0)\|_\infty + \|v(0)\|_\infty \leq R$. We have*

$$u(x,t) \leq C(T-t)^{-\beta_1}, \qquad v(x,t) \leq C(T-t)^{-\beta_2}, \tag{3.2}$$

in $\mathbb{R} \times [0, T)$, where $T < \infty$ is the blowup time of w.

Proof: The proof follows from a careful analysis of the arguments employed by Chlebík and Fila [3] to show a similar result for the weakly coupled system crresponding to $a = d = 0$. In [3], the authors studied the Cauchy problem in $\mathbb{R}^N$ associated to the weakly couple system $(a = d = 0)$. They assumed that

$$\max\{\beta_1, \beta_2\} \geq N \tag{3.3}$$

to ensure that all positive solutions blow up in finite time. It follows from the results of [8] discussed above that (3.3) should be replaced here by (1.4) (that is why we consider $N = 1$). One may then verify that their arguments may be applied in the present context with no further modifications.

The result announced in [3] concerns a single solution. However, one can easily check in the proof that C in (3.2) can be taken independently of $w(0)$ in a ball of $L^\infty \times L^\infty$.

$\square$

It is shown in [6] that (3.2) ensures the continuity of the blowup time.

Proposition 3.3. *Let* $\mathcal{P} = \{w_0 = (u_0, v_0) \in L^\infty \times L^\infty, u_0 > 0, v_0 > 0\}$ *and define* $T : \mathcal{P} \longrightarrow \mathbb{R}^+$ *such that* $T(w_0)$ *is the blowup time of the corresponding solution* $w = (u, v)$. *Under the assumptions of Proposition 3.2,* T *is continuous.*

For the proof of Proposition 3.3 see Proposition 3.3 of [6].

Remark 3.4. Proposition 3.3 may be applied to singular initial data in $E^{r,s}$ or $\mathcal{M}$ (endowed with the weak-* topology). This is a straightforward consequence of the results discussed in Section 2.

4 Proof of the Main Results

We will now prove our main results concerning the growth as $\lambda \longrightarrow 0$ of the blowup time T_λ of the solution w_λ of (1.1). We recall that we restrict ourselves to the onedimensional case $N = 1$.

It follows from (1.2), (1.4) that $bc - (a-1)(d-1) > 0$. We set

$$D = bc - (a-1)(d-1) \qquad \beta_1 = \frac{2(b+1-d)}{D},$$

$$\beta_2 = \frac{2(c+1-a)}{D} \qquad \gamma = \frac{c-a+1}{b-d+1}.$$

Note that $b > d-1$, $c > a-1$ and $D > 0$ are equivalent assertions. Thus β_1, β_2 and γ are positive numbers. We also have the following.

Lemma 4.1. *If (1.2), (1.4) hold then* $\max\{\beta_1, \beta_2\} > 1$. *As a consequence, if* $\sigma_1 \leq 1$ *is such that* $\gamma\sigma_1 \leq 1$ *then* $\rho_1 = \beta_1 - \sigma_1 > 0$.

Proof: To show that $\max\{\beta_1, \beta_2\} > 1$ we may assume that $a+b \leq c+d$. Then $a+b \leq 3$ implies that $a \leq 2$. Thus $c+1-a \geq c-1 \geq 0$. Using (1.2), we get that $b+1-d > 0$ and $D > 0$. Let $\delta = c+1-a \geq$

$b+1-d$. Using (1.4),

$$\begin{aligned} D &= bc-(a-1)(d-1) \\ &= (a-1)(b+1-d)+b\delta \leq \delta(a+b-1) < 2\delta = 2(c+1-a). \end{aligned}$$

This shows that $\beta_2 > 1$.

Assume now $\sigma_1 \leq 1$, $\gamma\sigma_1 \leq 1$. Then $\beta_1 = \beta_2/\gamma > 1/\gamma \geq \sigma_1$, so that $\rho_1 > 0$.

□

In our proofs we will systematically use the following dilation invariance of (1.1). Given $w_\lambda = (u_\lambda, v_\lambda)$ and $\mu > 1$, the rescaling $\tilde{w}_\mu = (\tilde{u}_\mu, \tilde{v}_\mu)$ defined by

$$\tilde{u}_\mu(x,t) = \mu^{\beta_1} u_\lambda(\mu x, \mu^2 t) \quad \tilde{v}_\mu(x,t) = \mu^{\beta_2} v_\lambda(\mu x, \mu^2 t) \tag{4.1}$$

is also a solution of (1.1). We write $\tilde{w}_{0,\mu}(x) = (\tilde{u}_{0,\mu}(x), \tilde{v}_{0,\mu}(x))$, where

$$\tilde{u}_{0,\mu}(x,t) = \mu^{\beta_1}\lambda^{b-d+1}\varphi_1(\mu x) \quad \tilde{v}_{0,\mu}(x,t) = \mu^{\beta_2}\lambda^{c-a+1}\varphi_2(\mu x). \tag{4.2}$$

Clearly, if w_λ blows up at T_λ then $\tilde{w}_\mu$ blows up at $\tilde{T}_\mu$, where

$$T_\lambda = \mu^2 \tilde{T}_\mu. \tag{4.3}$$

We next consider $\varphi_1 \in I(\sigma_1, l)$, see (1.8).

4.1. The Case $\varphi_1 \in I(\sigma_1, l)$, $\sigma_1 < 1$, $\gamma\sigma_1 \leq 1$

We consider without loss of generality $\varphi_1 \in I(\sigma_1, 1)$.

Theorem 4.2. *Assume (1.2), (1.3), (1.4) and consider* $\varphi_1 \in I(\sigma_1, 1)$, $\varphi_2 \in I(\sigma_2, l)$, *where* $l > 0$, $\sigma_1 < 1$ *and* $\sigma_2 = \gamma\sigma_1 < 1$. *Then* $\rho_1 = \beta_1 - \sigma_1 > 0$ *by Lemma 4.1. We have*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = T(\sigma_1, l\sigma_2),$$

where $T(\sigma_1, l\sigma_2)$ is the blowup time of $w(\sigma_1, l\sigma_2)$, the solution of (1.1) for $w_0 = (|x|^{-\sigma_1}, l|x|^{-\sigma_2})$ given by Theorem 2.4 for $N = 1$.

Proof: We first verify that $w(\sigma_1, l\sigma_2) = (u(\sigma_1, l\sigma_2), v(\sigma_1, l\sigma_2))$ is well defined. Choose $\varepsilon > 0$ small enough so that r_1, r_2, s_1, s_2, defined by $\frac{1}{r_1} = \sigma_1 + \varepsilon$, $\frac{1}{r_2} = \sigma_2 + \varepsilon$, $s_1 = \frac{1}{\sigma_1} + \varepsilon$, $s_2 = \frac{1}{\sigma_2} + \varepsilon$, satisfy $r_1 > 1$, $r_2 > 1$. Since $r_1\sigma_1 < 1 < s_1\sigma_1$, $r_2\sigma_2 < 1 < s_2\sigma_2$, we see that $|x|^{-\sigma_1} \in E^{r_1,s_1}$, $l|x|^{-\sigma_2} \in E^{r_2,s_2}$. We have that

$$\frac{b}{r_2} + \frac{a-1}{r_1} = \frac{2\sigma_1}{\beta_1} + \varepsilon(b + a - 1),$$

$$\frac{c}{r_1} + \frac{d-1}{r_2} = \frac{2\sigma_1}{\beta_1} + \varepsilon(c + d - 1).$$

Using that $\beta_1 > \sigma_1$, we take $\varepsilon > 0$ small enough so that (2.14) takes place. Choosing ε eventually smaller, s_1, s_2 may be taken so that Theorem 2.4 applies. This shows the existence of $w(\sigma_1, l\sigma_2)$. It follows from (1.3), (1.4) that $w(\sigma_1, l\sigma_2)$ blows up at a finite time $T(\sigma_1, l\sigma_2)$.

To study the behaviour of T_λ, consider μ such that $\lambda^{b+1-d}\mu^{\beta_1-\sigma_1} = 1$ and $\tilde{w}_\mu = (\tilde{u}_\mu, \tilde{v}_\mu)$ defined by (4.1). Thus $\tilde{u}_{0,\mu}(x) = \mu^{\sigma_1}\varphi_1(\mu x)$, $\tilde{v}_{0,\mu}(x) = \mu^{\sigma_2}\varphi_2(\mu x)$, see (4.2). Note that $\mu \longrightarrow \infty$ as $\lambda \longrightarrow 0$. Let B be the unitary ball of $\mathbb{R}$ and set $D = \mathbb{R} \setminus B$. It follows from dominated convergence that, as $\mu \longrightarrow \infty$, $\mu^{\sigma_1}\varphi_1(\mu x)I_B \longrightarrow |x|^{-\sigma_1}I_B$ in L^{r_1}, $\mu^{\sigma_1}\varphi_1(\mu x)I_D \longrightarrow |x|^{-\sigma_1}I_D$ in L^{s_1}, where I_Ω denotes the characteristic function of $\Omega \subset \mathbb{R}$. As a consequence, $\tilde{u}_{0,\mu}(x) \longrightarrow |x|^{-\sigma_1}$ in E^{r_1,s_1}. Analogously, $\tilde{v}_{0,\mu}(x) \longrightarrow l|x|^{-\sigma_2}$ in E^{r_2,s_2}. Thus $\tilde{w}_{0,\mu}(x) \longrightarrow (|x|^{-\sigma_1}, l|x|^{-\sigma_2})$ in $E^{r_1,s_1} \times E^{r_2,s_2}$. Using Proposition 3.3 and Remark 3.4, we get that $\tilde{T}_\mu \longrightarrow T(\sigma_1, l\sigma_2)$ as $\mu \longrightarrow \infty$. This finishes the proof, since $\tilde{T}_\mu = \mu^{-2}\, T_\lambda = \lambda^{\frac{2(b+1-d)}{\rho_1}}\, T_\lambda$, see (4.3).

□

Theorem 4.3. *Assume (1.2), (1.3), (1.4). Suppose* $\varphi_1 \in I(\sigma_1, 1)$, *where* $\sigma_1 < 1$ *and* $\gamma\sigma_1 = 1$, $\varphi_2 \in L^1$, $\varphi_2 \geq 0$ *such that* $M = \displaystyle\int \varphi_2 > 0$. *Then,*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = T(\sigma_1, M\delta_0),$$

where $T(\sigma_1, M\delta_0)$ *is the blowup time of* $w(\sigma_1, M\delta_0)$, *the solution of (1.1) for* $w_0 = (|x|^{-\sigma_1}, M\delta_0)$ *given by Theorem 2.5 for* $N = 1$.

Proof: Using that $\gamma > 1$ it follows from Lemma 4.1 that $\beta_2 > 1$. Note that $b = (1-a)/\gamma + 2/\beta_2$ and $c/\gamma = 2/\beta_2 + 1 - d$, so there exist $1 < r < \gamma$, $0 < \theta < 1 < \theta'$ such that

$$b + \frac{a-1}{r} = \frac{2\theta}{\beta_2}, \quad \frac{c}{r} + d - 1 = \frac{2\theta'}{\beta_2}. \tag{4.5}$$

We may choose θ and θ' close enough to 1 so that (2.26) holds. Taking $s > \gamma$, Theorem 2.5 ensures the existence of $w(\sigma_1, M\delta_0) \in C((0,T); E^{r,s} \times \mathcal{M})$. In addition, (1.3), (1.4) yield that $w(\sigma_1, M\delta_0)$ blows up at a finite time $T(\sigma_1, M\delta_0)$.

Consider again $\tilde{w}_\mu = (\tilde{u}_\mu, \tilde{v}_\mu)$, where $\lambda^{b+1-d}\mu^{\beta_1-\sigma_1} = 1$. As before, $\tilde{u}_{0,\mu}(x) = \mu^{\sigma_1}\varphi_1(\mu x) \longrightarrow |x|^{-\sigma_1}$ in $E^{r,s}$. Moreover, $\tilde{v}_{0,\mu}(x) = \mu\varphi_2(\mu x) \rightharpoonup M\delta_0$ weak-* in $\mathcal{M}$. It then follows from Theorem 2.5 that $\tilde{w}_\mu(t) \longrightarrow w(\sigma_1, M\delta_0)(t)$ uniformly for all $t \in (0, T]$. Using the continuity of the blowup time, we get the result.

□

4.2. The Case $\varphi_1 \in L^1$

We set now $\rho_1 = \beta_1 - 1$. Note that by Lemma 4.1, if $a + b = c + d$ then $\rho_1 > 0$. Below, we assume without loss of generality that $\displaystyle\int \varphi_1 = 1$.

Theorem 4.4. *Assume (1.2), (1.3), (1.4) and suppose further that* $a+b=c+d<3$. *Let* $\varphi_1, \varphi_2 \in L^1$ *be nonnegative functions such that* $\int \varphi_1 = 1$, $M = \int \varphi_2 > 0$. *Then*

$$\lim_{\lambda \longrightarrow 0} \lambda^{\frac{2(b+1-d)}{\rho_1}} T_\lambda = T(\delta_0, M\delta_0),$$

where $T(\delta_0, M\delta_0)$ *is the blowup time of* $w(\delta_0, M\delta_0)$, *the solution of (1.1) for* $w_0 = (\delta_0, M\delta_0)$ *given by Theorem 2.6 for* $N = 1$.

Proof: Using that $a+b=c+d<3$, we may obtain $w_{\delta_0, M\delta_0}$ from Theorem 2.6. It is also clear that $w_{\delta_0, M\delta_0}$ blows up.

Define $\tilde{w}_\mu = (\tilde{u}_\mu, \tilde{v}_\mu)$ by (4.1) and set $\lambda^{b+1-d}\mu^{\beta_1 - 1} = 1$. Then $\tilde{u}_{0,\mu}(x) = \mu\varphi_1 \rightharpoonup \delta_0$, $\tilde{v}_{0,\mu}(x) = \mu\varphi_2(\mu x) \rightharpoonup M\delta_0$. Using again Theorem 2.6 and the continuity of the blowup time we get the result.

$\square$

4.3. The Case $\varphi_1 \in I(1,l)$

We suppose $c+d \le a+b$. Then $\rho_1 = \beta_1 - 1 > 0$, see Lemma 4.1. We also consider without loss of generality $\varphi_1 \in I(1,1)$. Define $g(\mu) = \mu^{\rho_1}\log\mu$ for $\mu > \overline{\mu} := e^{1/\rho_1}$. The function g is invertible and we call $h = g^{-1}$ its inverse.

Theorem 4.5. *Assume (1.2), (1.3), (1.4). Let* $a+b=c+d$, $\varphi_1 \in I(1,1)$ *and* $\varphi_2 \in I(1,l)$ *for some* $l > 0$. *Then* w_λ *blows up at a finite time* T_λ *such that*

$$\lim_{\lambda \longrightarrow 0} (h(\lambda^{-(b+1-d)}))^{-2} T_\lambda = T(\delta_0, l\delta_0),$$

where $T(\delta_0, l\delta_0)$ *is the blowup time of* $w(\delta_0, l\delta_0)$, *the solution of (1.1) for* $w_0 = (\delta_0, l\delta_0)$ *given by Theorem 2.6 for* $N = 1$.

Proof: Using that $a+b=c+d<3$ and Theorem 2.6 we conclude that $w(\delta_0,\delta_0)$ is well defined. Consider $\tilde{w}_\mu=(\tilde{u}_\mu,\tilde{v}_\mu)$, where $\mu=h(\lambda^{-(b+1-d)})$. As shown in Theorem 1.5 of [4], we may decompose $\tilde{u}_\mu=\psi_{1,\mu}+\psi_{2,\mu}$ such that $\psi_{1,\mu}\in L^s$ for any $s>1$, $\psi_{2,\mu}\in L^1$ and as $\mu\longrightarrow+\infty$ $\|\varphi_{1,\mu}\|_s\longrightarrow 0$, $\varphi_{2,\mu}\longrightarrow\delta_0$ weak-* in L^1. A similar decomposition holds for $\tilde{v}_\mu$. The rest of the argument follows as in the previous cases.

□

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