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UPPER SEMICONTINUITY OF ATTRACTORS FOR THE DISCRETIZATION OF STRONGLY DAMPED WAVE EQUATIONS

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Abstract

In this paper we prove the upper semicontinuity of attractors for the discretization of damped hyperbolic problems of the form

$$u_{tt} + 2\eta\Lambda^{\frac{1}{2}}u_t + 2au_t + \Lambda u = f(u)$$

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with $D(\Lambda) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}$, $\Lambda : D(\Lambda) \subset X \to X$, $\Lambda u = -u_{xx} + \delta u$, $\delta > 0$, $a > 0$, $\eta \geq 0$ as the discretization step goes to zero.

1 Introduction

For each $\eta > 0$, we consider the strongly damped wave equation

$$\begin{aligned} &u_{tt} + 2\eta\,\Lambda^{1/2} u_t + 2a\,u_t = -\Lambda u + f(u), \quad 0 < x < 1,\ t > 0 \\ &u_x(0) = u_x(1) = 0,\ t > 0, \end{aligned} \tag{1.1}$$

and its discretization given by

$$\ddot{U} + 2\eta\,\Lambda_n^{1/2}\dot{U} + 2a\,\dot{U} = -\Lambda_n U + \mathrm{f}(U) \tag{1.2}$$

where $a > 0$, $\Lambda u = -u_{xx} + \frac{\delta}{2}u$, Λ_n is a $n \times n$ matrix, $\Lambda_n = \Delta_n + \frac{\delta}{2}I$, $\delta > 0$ and Δ_n is the discretization of the Laplacian with Neumann boundary conditions given by

$$\Delta_n = n^2 \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \tag{1.3}$$

$f : \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the dissipative condition

$$\limsup_{|u|\to+\infty} \frac{f(u)}{u} \leq -\delta, \tag{1.4}$$

$\mathrm{f}(U) = (f(u_1), \cdots, f(u_n))^{\top}$ and $U = (u_1, \cdots, u_n)^{\top}$.

In this paper we study how the dynamics of the continuous equation (1.1)can be approximated by the dynamics of the discretization (1.2). More precisely, we prove that the family of global attractors of the discretization (1.2) is upper semicontinuous to the global attractor of the continuous problem (1.1), as n goes to ∞.

We study the problem (1.1) in an abstract form (in the sense of Henry [8]). Let's denote by Λ, the operator $\Lambda : D(\Lambda) \subset X^0 \to X^0$ given by $\Lambda u = -u_{xx} + \frac{\delta}{2}u$, $X = L^2 = X^0$ and $D(\Lambda) = \{u \in H^2(0,1); u'(0) = u'(1) = 0\} = X^1$. So we can write (1.1) as

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = A_\eta \begin{bmatrix} u \\ v \end{bmatrix} + h(\begin{bmatrix} u \\ v \end{bmatrix}) \tag{1.5}$$

where $D(A_\eta) = X^1 \times X^{\frac{1}{2}} = Y^1$,

$$A_\eta = \begin{bmatrix} 0 & I \\ -\Lambda & -2(\eta\Lambda^{1/2} + a) \end{bmatrix} \text{ and } h(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}.$$

For $\eta > 0$, $-A_\eta$ is a sectorial operator and generates an analytic semigroup of contractions (see [6, 7]). For $\eta \geq 0$, the equation (1.5) generates a C^1-semigroup T_η on $Y^0 = H^1 \times L^2$. $T_\eta(t)$, $t \geq 0$, is a gradient system asymptotically smooth. Furthermore, as proved in [5] to a more general case, $T_\eta(t)$ admits a global attractor $\mathcal{A}_\eta$. By using regularity results we have $\mathcal{A}_\eta \subset Y^1$.

In order to keep the similarity, we rewrite (1.2) in a matrix form

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = A_{\eta n} \begin{bmatrix} U \\ V \end{bmatrix} + H(\begin{bmatrix} U \\ V \end{bmatrix}) \tag{1.6}$$

where

$$A_{\eta n} = \begin{bmatrix} 0 & I_n \\ -A_n & -2(\eta A_n^{1/2} + a) \end{bmatrix} \text{ and } \quad H(\begin{bmatrix} U \\ V \end{bmatrix}) = \begin{bmatrix} 0 \\ \mathrm{f}(U) \end{bmatrix}$$

For (1.6), we have a global attractor $\mathcal{A}_{\eta n}$.

Considering
$$\delta_Y(A,B) = \sup_{x\in A}\inf_{y\in B} d_Y(x,y) \tag{1.7}$$
we can define the continuity of a family of sets $B_\eta \subset Y$ in the following form: a family B_η is continuous in η_0 if it is upper semicontinuous at η_0, that is, $\delta_Y(B_\eta, B_{\eta_0}) \to 0$ as $\eta \to \eta_0$; and it is lower semicontinuous at η_0, that is, $\delta_Y(B_{\eta_0}, B_\eta) \to 0$ as $\eta \to \eta_0$.

In most problems, the ideal situation is having the asymptotic dynamics of one equation the same of the asymptotic dynamics of its discretization. Nevertheless, studying the linear wave equation, we noted that the spectrum of the discretization and the spectrum of its continuous counterpart are far away from each other, no matter how fine the discretization is. That also happens to some parabolic equations but in this set of problems the nonconvergent part is controlled by the fact that the real part of the eigenvalues is negative and very large in absolute values (the corresponding modes do not interfere in the asymptotics). The spectrum of A_η with $\eta = 0$ do not have this property. That is restrictive to the hyperbolic equation, i.e. $\eta = 0$.

In order to overcome this problem we propose to approach the semilinear damped wave equation with $\eta = 0$ (hyperbolic case) by a "parabolic equation" strongly damped ($\eta > 0$) and then to make the approximation of this equation by its discretization.

In [3], they proved that the family of global attractors $\mathcal{A}_\eta$, $\eta \geq 0$ is continuous (lower and upper) in $\eta = 0$. Note that $\eta = 0$ in (1.5) give us the damped wave equation.

In order to compare the problems (1.5) and (1.2) it was necessary to consider the space $\mathbb{R}^n \times \mathbb{R}^n$ embedded in the phase space of the continuous problem. We also consider two norms in $\mathbb{R}^n \times \mathbb{R}^n$ which are the discretization of norms in the continuous space (Y^0 and Y^1). Studying the problem, we realize it was not possible to reduce the phase space dimension using a finite dimensional invariant manifold. The reduction to a finite invariant manifold was used to prove the topological equivalence between the dynamics of the discretization and the continuous heat equation, see [4]. The spectrum of A_η do not satisfy the existence of a large gap, since $\lim_{k\to\infty} \mathrm{Re}(\lambda_{\pm(k+1)}) - \mathrm{Re}(\lambda_{\pm k}) = \eta$, where λ_k is the k^{th} eigenvalue of A_η, therefore we could not use this technic.

We workout this problem for a fix $\eta > 0$ as follows. First, we analyze the closeness of the linear semigroups in the norm Y^0. In order to do that, we decompose the semigroups in two parts. One of them, is defined on an infinite space dimension, such that $\mathrm{Re}(\lambda_{\pm k}) \to -\infty$, where $k \to \infty$. It means that the semigroup norm can be set arbitrarily small. So our problem becomes to compare the semigroups in a finite dimension space. We did that using the convergence of the eigenvalues and eigenvectors of the discrete problem to the continuous problem. Then, we compare the nonlinear semigroups and, finally, we prove the upper semicontinuity of the global attractors $\mathcal{A}_{\eta\, n}$. This procedure was used in [1]

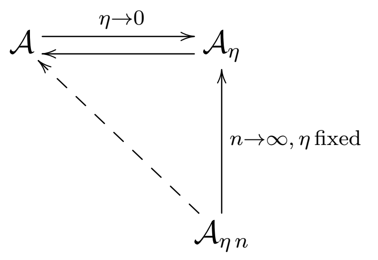
The main result of this paper is

Theorem 1.1. *The family of global attractors $\mathcal{A}_{\eta\, n}$ is upper semi-continuous at $n = \infty$, for any $\eta > 0$.*

Theorem 1.1, and the fact that the family $\mathcal{A}_\eta$ is upper continuous (see [3]), leads to the following important result

Theorem 1.2. *Let $\mathcal{A}$ be the attractor of (1.1) for $\eta = 0$. Then, there exists a sequence (η, n_η) such that $\delta(\mathcal{A}_{\eta\, n_\eta}, \mathcal{A})$ converges to zero when $\eta \to 0$.*

These results can be summarized in the following diagram



where the arrows denotes upper semicontinuity when it points to the limit problem and lower semicontinuity when it points to the family problems.

This paper is organized as follows. Section 2 recalls some spectral properties of A_η and $A_{\eta n}$. We also define the norms and some relations between $\mathbb{R}^n \times \mathbb{R}^n$ and Y^0 and Y^1. In Section 3 we make the comparison of the linear semigroups. The comparison of the nonlinear semigroups is done in Section 4. Finally, the last section proves the upper semicontinuity of attractors $A_{\eta n}$.

2 Spectral properties of A_η and $A_{\eta n}$.

In this section, we recall from [6, 7] some important spectral properties of A_η and $A_{\eta n}$. We also define the norms and some relations between $\mathbb{R}^n \times \mathbb{R}^n$ and Y^0, Y^1.

Let $\nu_k = (k\pi)^2 + \frac{\delta}{2}$ be the eigenvalues of Λ for $k = 0, 1, \cdots$. The eigenvalues, $\lambda_{\pm k}$, of A_η are the solutions of

$$\lambda^2 + (2\eta\nu_k^{\frac{1}{2}} + 2a\nu_k)\lambda + \nu_k = 0$$

and they are given by:

$$\lambda_{\pm k} \;=\; -(\eta\nu_k^{1/2} + a) \pm \sqrt{(\eta\nu_k^{1/2} + a)^2 - \nu_k}$$

For each $\eta > 0$, there exist an $k_0 = k_0(\eta) \geq 0$ such that $\lambda_{\pm k}$ is a real number for $k < k_0$ and $\lambda_{\pm k}$ is a complex number for $k \geq k_0$.

The correspondents eigenfunctions are given by:

$$\phi_{\pm k} = \begin{bmatrix} e_k \\ \lambda_{\pm k} e_k \end{bmatrix} \tag{2.1}$$

where $e_k = \cos(k\pi x)$ is a eigenfunction of Λ with respect the eigenvalue ν_k. If $\lambda_{\pm k}$ is a double eigenvalue then $\psi_k = \begin{bmatrix} 0 \\ e_k \end{bmatrix}$ is a generalized eigenfunction associated with $\lambda_{\pm k}$. If $\lambda_{\pm k}$ is a complex eigenvalue then we consider the following vectors $\psi_{+k} = \mathrm{Re}(\phi_{\pm k})$ and $\psi_{-k} = \mathrm{Im}(\phi_{\pm k})$, in the real eigenspace associated with $\lambda_{\pm k}$.

We have the following properties:

1) the family $(\phi_{+k})_{k=0}^{k_0}, (\psi_{+k})_{k=k_0}^{\infty}$ is orthogonal in Y^0;
2) the family $(\phi_{-k})_{k=0}^{k_0}, (\psi_{-k})_{k=k_0}^{\infty}$ is orthogonal in Y^0;
3) $\langle \phi_{-i}, \phi_{+j} \rangle_{Y^0} = 0$, $\langle \psi_{-i}, \psi_{+j} \rangle_{Y^0} = 0$, $\langle \phi_{-i}, \psi_{+j} \rangle_{Y^0} = 0$, $\langle \psi_{-i}, \phi_{+j} \rangle_{Y^0} = 0$ if $i \neq j$.

Using the same arguments of [3] in section 2, we have that there are $K \geq 1$ and $\gamma > 0$, independent of η, such that $\|e^{A_\eta t}\| \leq Ke^{-\gamma}$, for $\eta > 0$.

Similarly, the eigenvalues of Λ_n are given by $\nu_k^n = 4n^2 \sin^2 \frac{k\pi}{2n} + \frac{\delta}{2}$ and the associated eigenvectors are $e_k^n = (\cos k\pi x_1, \cdots, \cos k\pi x_n)$ for $k = 0, \cdots, n-1$ and $x_i = \frac{2i-1}{2n}$. The eigenvalues, $\lambda_{\pm k}^n$, of $A_{\eta n}$ are the solutions of the equation $\lambda^2 + (2\eta(\nu_k^n)^{\frac{1}{2}} + 2a\nu_k^n)\lambda + \nu_k^n = 0$ and are given by:

$$\lambda_{\pm k}^n = -(\eta(\nu_k^n)^{1/2} + a) \pm \sqrt{(\eta(\nu_k^n)^{1/2} + a)^2 - \nu_k^n}$$

The correspondents eigenvectors are given by:

$$\phi_{\pm k}^n = \begin{bmatrix} e_k^n \\ \lambda_{\pm k}^n e_k^n \end{bmatrix} \tag{2.2}$$

where e_k^n is the eigenvector of Λ_n associated with the eigenvalue ν_k^n. If $\lambda_{\pm k}^n$ is a double eigenvalue then $\psi_k^n = \begin{bmatrix} 0 \\ e_k^n \end{bmatrix}$ is a generalized eigenvector associated with $\lambda_{\pm k}^n$

If $\lambda_{\pm k}^n$ is a complex eigenvalue then we consider the following vectors $\psi_{+k}^n = \mathrm{Re}(\phi_{\pm k}^n)$ and $\psi_{-k}^n = \mathrm{Im}(\phi_{\pm k}^n)$, in the real eigenspace associated with $\lambda_{\pm k}$.

We also get that for each $\eta > 0$ and $n > 0$ exist a $k_0 = k_0(\eta, n) \geq 0$ such that $\lambda_{\pm k}^n$ is a real number for $k < k_0$ and $\lambda_{\pm k}^n$ is a complex number for $k_0 \leq k < n$.

In order to compare the problems (1.5) and (1.6), it is necessary to consider in $\mathbb{R}^n \times \mathbb{R}^n$ a compatible norm with the norm in Y^0. Therefore, we define in $\mathbb{R}^n \times \mathbb{R}^n$ the following inner product:

$$\langle \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} W \\ Z \end{bmatrix} \rangle_0 = \langle \Lambda_n U, W \rangle_{\mathbb{R}^n} + \langle V, Z \rangle_{\mathbb{R}^n} \tag{2.3}$$

where $\langle U, W \rangle_{\mathbb{R}^n} = \sum_{i=1}^n \frac{1}{n} u_i\, w_i$ is the inner product L^2 discretized. We denote for Y_n^0 the space $\mathbb{R}^n \times \mathbb{R}^n$ with the inner product given above.

About the eigenvectors of $A_{\eta n}$ in the space Y_n^0 we have:

1) the family $(\phi_{+k}^n)_{k=0}^{k_0}, (\psi_{+k}^n)_{k=k_0}^n$ is orthogonal in Y_n^0;

2) the family $(\phi_{-k}^n)_{k=0}^{k_0}, (\psi_{-k}^n)_{k=k_0}^n$ is orthogonal in Y_n^0;

3) $\langle \phi_{-i}^n, \phi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \psi_{-i}^n, \psi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \phi_{-i}^n, \psi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \psi_{-i}^n, \phi_{+j}^n \rangle_{Y_n^0} = 0$ if $i \neq j$.

We also need to consider another inner product in $\mathbb{R}^n \times \mathbb{R}^n$ compatible with the inner product in Y^1, that means,

$$\langle \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} W \\ Z \end{bmatrix} \rangle_1 = \langle \Lambda_n U, \Lambda_n W \rangle_{\mathbb{R}^n} + \langle \Lambda_n V, Z \rangle_{\mathbb{R}^n} \tag{2.4}$$

We denote by Y_n^1 the space $\mathbb{R}^n \times \mathbb{R}^n$ with the inner product given above. We make the distinction in the inner products by the index 0 or 1.

With a simple evaluation we get that

$$\langle A_n U, W\rangle_{\mathbb{R}^n} = \sum_{i=1}^{n-1} n(u_{i+1}-u_i)(w_{i+1}-w_i) + \frac{\delta}{2}\sum_{i=1}^{n}\frac{1}{n}u_i w_i.$$

We use the notation Y_n^0 or Y_n^1 to indicate the inner product and the norm considered in $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, we use in $\mathbb{R}^n$, three differents norms given by $\|U\|_{L^2_d} = \langle U, U\rangle_{\mathbb{R}^n}^{1/2}$ which we call L^2-discretized, $\|U\|_{H^1_d} = \langle \Lambda_n U, U\rangle_{\mathbb{R}^n}^{1/2}$ which we call H^1 discretized and $\|U\|_{H^2_d} = \langle \Lambda_n U, \Lambda_n U\rangle_{\mathbb{R}^n}^{1/2}$ which we call H^2 discretized. In order to avoid mistakes, we denote by $\|U'\|_{L^2_d} = \langle \Delta_n U, U\rangle^{\frac{1}{2}}$ the L^2 norm discretized of the discretized derivative.

We also decompose the spaces $\mathbb{R}^n \times \mathbb{R}^n$ and $H^1 \times L^2$.

We write $H^1 \times L^2 = \oplus E_k$ where E_k is the generalized real eigenspace 2-dimensional associated with the eigenvalues $\lambda_{\pm k}$. If $\lambda_{\pm k}$ are real then $E_k = [\phi_{+k}, \phi_{-k}]$. If $\lambda_{\pm k}$ are complex, we consider the vectors $\psi_{+k} = \mathrm{Re}\phi_{+k}$ and $\psi_{-k} = \mathrm{Im}\phi_{+k}$ the base of E_k. We observe that the family E_k is orthogonal.

We denote by $\sphericalangle_{+k}^{-k}$ the angle between ψ_{+k} and ψ_{-k}. We observe that $\cos(\sphericalangle_{+k}^{-k}) < 1-\xi$, for some $\xi > 0$ and for any k. In fact, considering $\|e_k\|_{L^2} = 1$ we get

$$\cos(\sphericalangle_{+k}^{-k}) = \frac{\|e_k\|_{H^1}^2 + \mathrm{Re}(\lambda_{+k})\mathrm{Im}(\lambda_{+k})}{\sqrt{\|e_k\|_{H^1}^4 + (\mathrm{Re}^2(\lambda_{+k}) + \mathrm{Im}^2(\lambda_{+k}))\|e_k\|_{H^1}^2 + (\mathrm{Re}(\lambda_{+k})\mathrm{Im}(\lambda_{+k}))^2}},$$

remembering that $\mathrm{Re}(\lambda_{+k}) = O(-\eta\|e_k\|_{H^1})$ and $\mathrm{Im}(\lambda_{+k}) = O((1-\eta^2)^{\frac{1}{2}}\|e_k\|_{H^1})$ then,

$$\lim_{k\to\infty}\cos^2(\sphericalangle_{+k}^{-k}) \le \frac{(1+\eta(1-\eta^2)^{\frac{1}{2}})^2}{2+\eta(1-\eta^2)^{\frac{1}{2}}} \le 1-\xi$$

for some $\xi > 0$. With this fact, we obtain the equivalence between the sum norm, the max norm and inner product norm in each E_k, with equivalence constants independent of k.

Therefore, for $(u, v) \in H^1 \times L^2$ we write

$$(u, v) = \sum_{k=1}^{\infty} \left((u, v)_k^+ \phi_{+k} + (u, v)_k^- \phi_{-k} \right) = \sum_{k=1}^{\infty} (u, v)_k.$$

Using the orthogonal properties of E_k, we have $\|(u, v)\|_{Y^0} = (\sum_{k=1}^{\infty} \|(u, v)_k\|^2)^{\frac{1}{2}}$ where $(u, v)_k$ is a projection of (u, v) in the space E_k.

Similarly, we write $\mathbb{R}^n \times \mathbb{R}^n = \oplus E_k^n$ where E_k^n is a two dimensional space associated with the eigenvalues $\lambda_{\pm k}^n$. If $\lambda_{\pm k}^n$ are real eigenvalues then $E_k^n = [\phi_{+k}^n, \phi_{-k}^n]$, where $\phi_{\pm k}^n$ is the normalized eigenvector associated with $\lambda_{\pm k}^n$. If $\lambda_{\pm k}^n$ is complex we consider the vectors $\psi_{+k}^n = \mathrm{Re}\phi_{+k}^n$ and $\psi_{-k}^n = \mathrm{Im}\phi_{+k}^n$ a base de E_k^n.

Thus, $(U, V) \in \mathbb{R}^n \times \mathbb{R}^n$ is

$$(U, V) = \sum_{k=1}^{n} \left((U, V)_k^+ \phi_{+k}^n + (U, V)_k^- \phi_{-k}^n \right)$$

and $\|(U, V)\|_{Y_n^0} = (\sum_{k=1}^{n} \|(U, V)_k\|^2)^{1/2}$, where $(U, V)_k$ is a projection of (U, V) in the E_k^n which are orthogonal.

In order to make the comparison proposed, we use a technique of Numerical Analysis which is denominated *Internal Approximation of a Normed Space*, see [9]. We define a family $\{\mathbb{R}^n \times \mathbb{R}^n, P_{2n}, i_{2n}\}$, $n \in \mathbb{N}$ where $P_{2n} : Y^0 \to \mathbb{R}^n \times \mathbb{R}^n$ and $i_{2n} : \mathbb{R}^n \times \mathbb{R}^n \to Y^0$ are denominated projection and inclusion respectively.

Let $(U, V) = (u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) \in \mathbb{R}^n \times \mathbb{R}^n$, the inclusion application, i_{2n}, is defined by $i_{2n}(U, V) = (u(x), v(x))$ where $u(x)$ and $v(x)$ are given by

$$u(x) = u_1\chi_{[0,\frac{1}{2n})} + \sum_{i=1}^{n_1}(u_i + (u_{i+1} - u_i)n(x - x_i))\chi_{[\frac{2i-1}{2n},\frac{2i+1}{2n})} + u_n\chi_{[\frac{2n-1}{2n},1]} \tag{2.5}$$

and

$$v(x) = \sum_{i=1}^{n} v_i\chi_{I_i} \tag{2.6}$$

where I_i is the interval $[\frac{i-1}{n}, \frac{i}{n})$.

We also defined a projection of Y^0 in $\mathbb{R}^n \times \mathbb{R}^n$ in the following way. For each e_k we define $P_n(e_k) = U = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$ where $u_i = e_k(x_i)$, hence, $P_n(e_k) = e_k^n$. We define $P_n' : L^2 \to \mathbb{R}^n$ by $P_n'(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{n} a_k e_k^n$, $P_n'' : H^1 \to \mathbb{R}^n$ by $P_n''(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{\infty} a_k e_k^n$, and $P_{2n} : H^1 \times L^2 \to \mathbb{R}^n \times \mathbb{R}^n$ by $P_{2n}(u,v) = (P_n''(u), P_n'(v))$.

For the inclusion and projection applications we have

Theorem 2.1. *The inclusion application, $i_{2n} : \mathbb{R}^n \times \mathbb{R}^n \to H^1 \times L^2$ is continuous. Furthermore, the continuity is uniform in n.*

Proof: In fact, let $u(x)$ given by (2.5), then

$$\begin{aligned} \|u(x)\|_{L^2}^2 &= \frac{u_1^2}{2n} + \sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}} u^2(x)dx + \frac{u_n^2}{2n} \\ &\leq \frac{u_1^2}{2n} + \sum_{j=1}^{n-1}(\frac{u_{j+1}^2}{2n} + \frac{u_j^2}{2n}) + \frac{u_n^2}{2n} = \sum_{j=1}^{n}\frac{u_j^2}{n} = \|U\|_{L_d^2}^2 \end{aligned}$$

and,

$$\|u(x)\|_{H^1}^2 = \sum_{j=1}^{n-1} n(u_{j+1} - u_j)^2 = \|U\|_{H_d^1}^2$$

and let $v(x)$ given by (2.6), then $\|v(x)\|_{L^2}^2 = \sum_{j=1}^{n}\frac{1}{n}v_i^2 = \|V\|_{L_d^2}^2$. Thus,

$$\|i(U,V)\|_{Y^0} = (\|u(x)\|_{H^1}^2 + \frac{\delta}{2}\|u(x)\|_{L^2}^2 + \|v(x)\|_{L^2}^2)^{\frac{1}{2}} \leq \|(U,V)\|_{Y_n^0}$$

Theorem 2.2. *The projection application, P_{2n}, is continuous. Furthermore, the continuity is uniform in n.*

Proof: In fact, let $U = P_n(\cos(k\pi x)) = (\cos(k\pi x_1), \ldots, \cos(k\pi x_1))$ then we have

$$\|U\|^2_{L^2_d} = \sum_1^n \frac{1}{n} u_i^2 = \sum_1^n \frac{1}{n} \cos^2(k\pi x_i) \leq 1 = 2\|\cos(k\pi x)\|^2_{L^2} \quad (2.7)$$

and

$$\begin{aligned} \|U\|^2_{H^1_d} &= \sum_1^{n-1} n(u_{i+1} - u_i)^2 = \sum_1^{n-1} n(\cos(k\pi x_{i+1}) - \cos(k\pi x_i))^2 \\ &= \sum_1^{n-1} n(k\pi)^2 \operatorname{sen}^2(k\pi \bar{x}_i) \frac{1}{n^2} \leq (k\pi)^2 \leq 2\|k\pi \operatorname{sen}(k\pi x)\|^2_{L^2} \end{aligned}$$

Thus,

$$\begin{aligned} \|P_{2n}(\phi_{\pm k})\|_{Y_n^0} &= \|(P_n(e_k), \lambda_{\pm k} P_n(e_k))\|_{H^1_d \times L^2_d} \\ &= (\|P_n(e_k)\|^2_{H^1_d} + \frac{\delta}{2}\|P_n(e_k)\|^2_{L^2_d} + |\lambda_{\pm k}|\|P_n(e_k)\|^2_{L^2_d})^{\frac{1}{2}} \\ &\leq (2\|e_k\|_{H^1} + \frac{\delta}{2} 2\|e_k\|_{L^2} + |\lambda_{\pm k}| 2\|e_k\|^2_{L^2})^{\frac{1}{2}} \\ &= \sqrt{2}\|\phi_{\pm k}\|_{Y^0} \end{aligned}$$

By using Theorems 2.1 and 2.2 we have that these applications are stable (see [9]).

Another result is

Theorem 2.3. *i) Let $\lambda_{\pm k}$ be the eigenvalues of A_η and $\lambda^n_{\pm k}$ the eigenvalues of $A_{\eta n}$, then for each k fixed we have that $\lambda^n_{\pm k} \to \lambda_{\pm k}$ when $n \to \infty$.*
ii) Let $\phi_{\pm k}$ be the eigenvectors of A_η and $\phi^n_{\pm k}$ the eigenvectors of $A_{\eta n}$, then for each k fixed we have that $i(\phi^n_{\pm k}) \to \phi_{\pm k}$ when $n \to \infty$, and $P_n(e_k) = e^n_n$.

For the inclusion application we use only i and the dimension of the space is omitted.

3 Comparison of Linear Semigroups

Let be $e^{A_\eta t}$ and $e^{A_{\eta n} t}$ the semigroups generated by A_η and $A_{\eta n}$ respectively. We have the following result comparing the semigroups

Theorem 3.1. *For each $\epsilon > 0$, there is a $n_o(\epsilon)$ such that $\forall n \geq n_0$*

$$\|e^{A_\eta t}(u_0, v_0) - i(e^{A_{\eta n} t} P_{2n}(u_0, v_0))\|_{Y^0} \leq M\epsilon t^{-\beta} \|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}, t > 0 \tag{3.1}$$

for all $(u_0, v_0) \in C^{1+\alpha} \times C^\alpha$ and

$$\|e^{A_\eta t} i(U_0, V_0) - i(e^{A_{\eta n} t}(U_0, V_0))\|_{Y^0} \leq M\epsilon t^{-\beta} \|(U_0, V_0)\|_{Y_n^0}, t > 0 \tag{3.2}$$

for all $(U_0, V_0) \in \bigcup_n \mathcal{A}_{\eta n}$.

Proof: We make the proof for the first inequality and when it is necessary we note the changes for the second one.

Let $\epsilon > 0$ be a real parameter. We consider two cases

i) for $0 < t \leq \epsilon$. In this case, when t is small, we use that $e^{-\gamma t}$ is bounded by $K\epsilon^\nu t^{-\beta}$ for $\beta > \nu > 0$. Hence,

$$\begin{aligned} \|e^{A_\eta t}(u_0, v_0) - i(e^{A_{\eta n} t} P_{2n}(u_0, v_0))\|_{Y^0} &\leq K' e^{-\gamma t} \|(u_0, v_0)\|_{Y^0} \\ &\leq M\epsilon^\nu t^{-\beta} \|(u_0, v_0)\|_{Y^0}. \end{aligned}$$

ii) for $t > \epsilon$, we need estimate

$$\|e^{A_\eta t}(u_0, v_0) - i(e^{A_{\eta n} t} P_{2n}(u_0, v_0))\|_{Y^0}.$$

In this case, we decompose Y^0 in two subspaces. In the subspace of finite dimension, we have the uniform convergence of eigenvalues and inclusion of eigenvectors for the eigenvalues and eigenfunctions of the continuous problem. In the subspace of infinite dimension, we have that the real part of eigenvalues goes to $-\infty$.

By using that $\lambda^n_{\pm k} \to \lambda_{\pm k}$ when $n \to \infty$ and $\mathrm{Re}(\lambda_{\pm k}) \to -\infty$, when $k \to \infty$ and considering $\beta \in (0,1)$ a fixed number then there are $K(\epsilon)$ and $N(\epsilon)$ such that

$$e^{Re(\lambda^n_{\pm k})t} \leq \epsilon t^{-\beta}, \quad e^{Re(\lambda_{\pm k})t} \leq \epsilon t^{-\beta} \text{ for all } \ n \geq N(\epsilon) \text{ and } k \geq K(\epsilon). \tag{3.3}$$

Using $K = K(\epsilon)$ given in (3.3), we consider the following subspaces $\oplus E^n_k$, $1 \leq k \leq K$, $\oplus E^n_k$, $K+1 \leq k \leq n$ of $\mathbb{R}^n \times \mathbb{R}^n$ and $\oplus E_k$, $1 \leq k \leq K$, $\oplus E_k$, $K+1 \leq k < \infty$ of Y^0. Then,

$$\begin{aligned}
&\|e^{A_\eta t}(u_0,v_0) - i(e^{A_{\eta n} t}P_{2n}(u_0,v_0))\|_{Y^0} \\
&\qquad \leq \|e^{A_\eta t}\sum_{k=1}^{K}(u_0,v_0)_k - i(e^{A_{\eta n} t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0} \\
&\qquad + \|e^{A_\eta t}\sum_{k=K+1}^{\infty}(u_0,v_0)_k\|_{Y^0} + \|i(e^{A_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0,v_0))_k)\|_{Y^0}
\end{aligned}$$

By the continuity, uniform in n, of the applications inclusion and projection, we have:

$$\begin{aligned}
&\|i(e^{A_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0,v_0))_k)\|_{Y^0} \leq M\|e^{A_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0,v_0))_k\|_{Y^0_n} \\
&\quad = M(\sum_{k=K+1}^{n}\|e^{A_{\eta n} t}(P_{2n}(u_0,v_0))_k\|^2_{Y^0_n})^{1/2} \\
&\quad \leq M(\sum_{k=K+1}^{n}(e^{Re\lambda^n_{\pm} t}\|(P_{2n}(u_0,v_0))_k\|_{Y^0_n})^2)^{1/2} \\
&\quad \leq M\epsilon t^{-\beta}(\sum_{k=K+1}^{n}\|(P_{2n}(u_0,v_0))_k\|^2_{Y^0_n})^{1/2} \leq M\epsilon t^{-\beta}\|P_{2n}(u_0,v_0)\|_{Y^0_n} \\
&\quad \leq M'\epsilon t^{-\beta}\|(u_0,v_0)\|_{Y^0}
\end{aligned}$$

In the similar way, we have

$$\|e^{A_\eta t}\sum_{k=K+1}^{\infty}(u_0,v_0)_k\|_{Y^0}\le M\epsilon t^{-\beta}\|(u_0,v_0)_k\|_{Y^0}$$

We consider another operator B_n, which possess the same eigenvalues of $A_{\eta n}$ but, associated with the eigenvectors of A_η. Thus, we have

$$\begin{aligned}\|e^{A_\eta t}\sum_{k=1}^{K}(u_0,v_0)_k &- i(e^{A_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\\ &\le \|e^{A_\eta t}\sum_{k=1}^{K}(u_0,v_0)_k-e^{B_n t}\sum_{k=1}^{K}(u_0,v_0)_k\|_{Y^0}\\ &+ \|e^{B_n t}\sum_{k=1}^{K}(u_0,v_0)_k-i(e^{A_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\end{aligned}$$

If each $\lambda^n_{\pm k}$, for $1\le k\le K$, is real then

$$\begin{aligned}&\|\sum_{k=1}^{K}e^{A_\eta t}(u_0,v_0)_k-\sum_{k=1}^{K}e^{B_n t}(u_0,v_0)_k\|_{Y^0}\\ &\le(\sum_{k=1}^{K}\|(e^{\lambda_{+k}t}-e^{\lambda^n_{+k}t})(u_0,v_0)_{+k}+(e^{\lambda_{-k}t}-e^{\lambda^n_{-k}t})(u_0,v_0)_{-k}\|^2_{Y^0})^{1/2}\\ &\le 2M\max_{1\le k\le K}\{|e^{\lambda_{+k}t}-e^{\lambda^n_{+k}t}|,|e^{\lambda_{-k}t}-e^{\lambda^n_{-k}t}|\}.(\sum_{k=1}^{K}\|(u_0,v_0)_k\|^2)^{1/2}\\ &\le 2Mt\max_{1\le k\le K}\{|e^{\tilde\lambda^n_{+k}t}||\lambda_{+k}-\lambda^n_{+k}|,|e^{\bar\lambda^n_{-k}t}||\lambda_{-k}-\lambda^n_{-k}|\}\|(u_0,v_0)\|_{Y^0}\\ &\le\epsilon t^{-\beta}\|(u_0,v_0)\|_{Y^0}\end{aligned}$$

for $n\ge n_1\ge n_0$, where $\bar\lambda^n_{-k}$ is between λ_{-k} and λ^n_{-k}; and $\tilde\lambda^n_{+k}$ is between λ_{+k} and λ^n_{+k}.

In the case $\lambda_{\pm k}$ complex, we denote by $(u_0, v_0)_k$ the component of (u_0, v_0) in E_k and $(u_0, v_0)_k = 2a(cos\delta, -sen\delta)$ in the base ψ_{+k}, ψ_{-k}. In this case, we have

$$e^{A_\eta t}(u_0, v_0)_k = 2ae^{\alpha_k t}(\cos(\beta_k t + \delta)\psi_{+k} - \operatorname{sen}(\beta_k t + \delta)\psi_{-k}),$$

$$e^{B_n t}(u_0, v_0)_k = 2ae^{\alpha_k^n t}(\cos(\beta_k^n t + \delta)\psi_{+k} - \operatorname{sen}(\beta_k^n t + \delta)\psi_{-k}),$$

where $\lambda_{\pm k} = \alpha_k \pm \beta_k$ and $\lambda_{\pm k}^n = \alpha_k^n \pm \beta_k^n$, then

$$\begin{aligned}
&\|e^{A_\eta t}(u_0, v_0)_k - e^{B_n t}(u_0, v_0)_k\|_{Y^0} \\
&\quad \leq \|2ae^{\alpha_k t}[(\cos(\beta_k t + \delta) - \cos(\beta_k^n t + \delta))\psi_{+k} - (\operatorname{sen}(\beta_k t + \delta) - \operatorname{sen}(\beta_k^n t + \delta))\psi_{-k}]\| \\
&\qquad + \|2a[\cos(\beta_k^n t + \delta)\psi_{+k} - \operatorname{sen}(\beta_k^n t + \delta)\psi_{-k}]\| |e^{\alpha_{+k} t} - e^{\alpha_{+k}^n t}| \\
&\qquad \leq e^{\alpha_{+k} t} t|\beta_k - \beta_k^n| \| - 2a\operatorname{sen}(\bar{\beta}_k^n + \delta)\psi_{+k} - 2a\cos(\bar{\beta}_k^n t + \delta)\psi_{-k}\| \\
&\qquad + e^{\bar{\alpha}_{+k} t} t|\alpha_k - \alpha_k^n| \|2a\cos(\beta_k^n t + \delta)\psi_{+k} - 2a\operatorname{sen}(\beta_k^n t + \delta)\psi_{-k}\| \\
&\qquad \leq L(e^{\alpha_{+k} t} t|\beta_k - \beta_k^n| + e^{\bar{\alpha}_{+k} t} t|\alpha_k - \alpha_k^n|)\|(u_0, v_0)_k\|
\end{aligned}$$

If $(P_{2n}(u_0, v_0))_k = a_{+k}^n \psi_{+k}^n + a_{-k}^n \psi_{-k}^n$ and $(u_0, v_0)_k = a_{+k}\psi_{+k} + a_{-k}\psi_{-k}$ then

$$\begin{aligned}
&\|e^{B_n t}\sum_{k=1}^K (u_0, v_0)_k - i(e^{A_{\eta n} t}\sum_{k=1}^K (P_{2n}(u_0, v_0))_k)\|_{Y^0} \\
&\quad \leq \|\sum_{k=1}^K e^{B_n t}(u_0, v_0)_k - \sum_{k=1}^K e^{B_n t}(a_{+k}^n \psi_{+k} + a_{-k}^n \psi_{-k})\|_{Y^0} \\
&\quad + \|\sum_{k=1}^K e^{B_n t}(a_{+k}^n \psi_{+k} + a_{-k}^n \psi_{-k}) - \sum_{k=1}^K i(e^{A_{\eta n} t}(P_{2n}(u_0, v_0))_k)\|_{Y^0} \\
&\quad \leq e^{\alpha_1^n t}\sum_{k=1}^K (|a_{+k} - a_{+k}^n| \|\psi_{+k}\| + |a_{-k} - a_{-k}^n| \|\psi_{-k}\|) \\
&\quad + \|\sum_{k=1}^K e^{B_n t}(a_{+k}^n \psi_{+k} + a_{-k}^n \psi_{-k}) - \sum_{k=1}^K i(e^{A_{\eta n} t} a_{+k}^n \psi_{+k}^n + a_{-k}^n \psi_{-k}^n)\|_{Y^0}
\end{aligned}$$

Hence, we need to estimate $|a_{+k} - a_{+k}^n|$ and $|a_{-k} - a_{-k}^n|$. In order to calculate this, we write $a_{+k}^n = \frac{b_{+k}^n}{c_{+k}^n}$ and $a_{+k} = \frac{b_{+k}}{c_{+k}}$ where

$$b_{+k}^n = \langle P_{2n}(u_0, v_0), \psi_{+k}^n \rangle \|\psi_{-k}^n\|^2 - \langle P_{2n}(u_0, v_0), \psi_{-k}^n \rangle \langle \psi_{-k}^n, \psi_{+k}^n \rangle,$$

$$c^n_{+k} = \|\psi^n_{+k}\|^2\|\psi^n_{-k}\|^2 - \langle\psi^n_{+k}, \psi^n_{-k}\rangle^2$$

and

$$b_{+k} = \langle(u_0, v_0), \psi_{+k}\rangle\|\psi_{-k}\|^2 - \langle(u_0, v_0), \psi_{-k}\rangle\langle\psi_{-k}, \psi_{+k}\rangle,$$

$$c_{+k} = \|\psi_{+k}\|^2\|\psi_{-k}\|^2 - \langle\psi_{+k}, \psi_{-k}\rangle^2$$

and

$$|a_{+k} - a^n_{+k}| \leq \frac{|c^n_{+k}||b_{+k} - b^n_{+k}| + |c_{+k} - c^n_{+k}||b^n_{+k}|}{|c_{+k}||c^n_{+k}|}.$$

Thus, it is sufficient estimate $|b_{+k} - b^n_{+k}|$ and $|c_{+k} - c^n_{+k}|$. We consider two cases:

I)$(u_0, v_0) = i(U_0, V_0)$ and $P_{2n}(u_0, v_0) = (U_0, V_0)$ for $(U_0, V_0) \in \mathcal{A}_{\eta\, n}$;

II) (u_0, v_0) in $C^{1+\alpha} \times C^\alpha$.

Since that

i) $\|\psi_{+k}\|_{Y^0} = \|\psi^n_{+k}\|_{Y^0_n} + O(\frac{1}{n})$,

ii) $\langle\psi_{+k}, \psi_{-k}\rangle_{Y^0} = \langle\psi^n_{+k}, \psi^n_{-k}\rangle_0 + O(\frac{1}{n})$

iii)$\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} = \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} n(u_{i+1} - u_i)k\pi \operatorname{sen}(k\pi x)dx$,

iv)$\langle P_n(u_0), P_n(\cos(k\pi x)\rangle_{H^1_d} = \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} n(u_{i+1} - u_i)k\pi \operatorname{sen}(k\pi\bar{x}_i)dx$.

then

$$|\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x)\rangle_{H^1_d}| \leq k^2\pi^2\sum_{i=1}^{n-1}\frac{1}{n}(u_{i+1} - u_i).$$

If $(u_0, v_0) = i(U_0, V_0)$ for some $(U_0, V_0) \in \mathcal{A}_{\eta\, n}$ then, by [2], we have $\bigcup_n \mathcal{A}_{\eta\, n}$ is bounded in $H^2_d \times H^1_d$ and $n|u_{i+1} - u_i| \leq \|U\|_{H^1_d} + \|U\|_{H^2_d} \leq 2K$ for $1 \leq i \leq n-1$, thus

$$|\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x)\rangle_{H^1_d}| \leq \frac{k^2\pi^2}{n}(\|U_0\|_{H^1_d} + \|U_0\|_{H^2_d}).$$

We also have

v)$\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} = \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} v_i\cos(k\pi x)dx$,

iv)$\langle P_n(v_0), P_n(\cos(k\pi x)\rangle_{L^2_d} = \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} v_i\cos(k\pi x_i)dx$.

Hence,

$$|\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} - \langle P_n(v_0), P_n(\cos(k\pi x)\rangle_{L^2_d}| \leq k\pi\sum_{i=1}^{n}\frac{1}{n^2}v_i.$$

We are in the case of $(u_0, v_0) = i(U_0, V_0)$ for some $(U_0, V_0) \in \mathcal{A}_{\eta n}$ then, using that $\bigcup_n \mathcal{A}_{\eta n}$ is bounded in $H^2_d \times H^1_d$ and $|v_i| \le \|V\|_{L^2_d} + \|V\|_{H^1_d} \le 2K$ for $1 \le i \le n$ then,

$$|\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} - \langle P_n(v_0), P_n(\cos(k\pi x)\rangle_{L^2_d}| \le \frac{k^2\pi^2}{n}(\|V_0\|_{L^2_d} + \|V_0\|_{H^1_d}).$$

Therefore,

$$|\langle i(P_{2n}(u_0, v_0)), \psi_{\pm k}\rangle_{Y^0} - \langle P_{2n}(u_0, v_0), \psi^n_{\pm k}\rangle_{Y^0_n}| \le \frac{\tilde{K}}{n}\|(U_0, V_0)\|_{Y^1_n}.$$

If $(u_0, v_0) \neq i(U_0, V_0)$ and $(u_0, v_0) \in C^{1+\alpha} \times C^{\alpha}$ then we have $|u_{i+1} - u_i| \le n^{-1}\|u_0\|_{C^{1+\alpha}}$, $|v_i| \le \|v\|_{C^\alpha}$. Thus

$$\begin{aligned}
&|\langle u_0, \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x)\rangle_{H^1_d}| \\
&\quad \le |\langle u_0, \cos(k\pi x)\rangle_{H^1} - \langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1_d}| \\
&\quad + |\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x)\rangle_{H^1_d}|.
\end{aligned}$$

However,

$$\|(u_0, v_0) - i(P_{2n}(u_0, v_0))\|_{Y^0} \le \frac{1}{n^\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}.$$

Hence,

$$|\langle (u_0, v_0), \psi_{\pm k}\rangle_{Y^0} - \langle P_{2n}(u_0, v_0), \psi^n_{\pm k}\rangle_{Y^0_n}| \le \frac{\tilde{K}}{n^\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}.$$

Therefore, for case I)

$$|b_{+k} - b^n_{+k}| \le Mn^{-1}\|(U_0, V_0)\|_{Y^1_n},$$

for case II),

$$|b_{+k} - b^n_{+k}| \le Mn^{-\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}$$

and in analogous form, for k, $1 \le k \le K$ we get

$$|c_{+k} - c^n_{+k}| \le Mn^{-1}.$$

Analogously, we obtain $|a_{-k} - a^n_{-k}|$.

We came back to estimate $e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a^n_{+k}|\|\psi_{+k}\|+|a_{-k}-a^n_{-k}|\|\psi_{-k}\|)$. In the case I)

$$\begin{aligned}
&e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a^n_{+k}|\|\psi_{+k}\|+|a_{-k}-a^n_{-k}|\|\psi_{-k}\|)\\
&\leq e^{\alpha_1^n t}\tilde{M}n^{-1}\|(U_0,V_0)\|_{Y^1_n}\sum_{k=1}^{K}(\|\psi_{+k}\|+\|\psi_{-k}\|)\\
&\leq \epsilon t^{-\beta}\|(U_0,V_0)\|_{Y^1_n}
\end{aligned}$$

and in the case II)

$$\begin{aligned}
&e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a^n_{+k}|\|\psi_{+k}\|+|a_{-k}-a^n_{-k}|\|\psi_{-k}\|)\\
&\leq e^{\alpha_1^n t}\tilde{M}n^{-\alpha}\|(u_0,v_0)\|_{C^{1+\alpha}\times C^{\alpha}}\sum_{k=1}^{K}(\|\psi_{+k}\|+\|\psi_{-k}\|)\\
&\leq \epsilon t^{-\beta}\|(u_0,v_0)\|_{C^{1+\alpha}\times C^{\alpha}}
\end{aligned}$$

Now we go to estimate

$$\|\sum_{k=1}^{K} e^{B_n t}(a^n_{+k}\psi_{+k}+a^n_{-k}\psi_{-k})-\sum_{k=1}^{K} i(e^{A_{\eta n}t}a^n_{+k}\psi^n_{+k}+a^n_{-k}\psi^n_{-k})\|_{Y^0}$$

In order to do this, we consider a complex inclusion, that means the inclusion of real part and the inclusion of imaginary part. In this case, we are considering complex solutions.

Since that $a^n_{+k}\psi_{+k}+a^n_{-k}\psi_{-k}=d^n_{+k}\phi_{+k}+d^n_{-k}\phi_{-k}$ and $a^n_{+k}\psi^n_{+k}+a^n_{-k}\psi^n_{-k}=d^n_{+k}\phi^n_{+k}+d^n_{-k}\phi^n_{-k}$ where $d_{+k}=1/2(a^n_{+k}-ia^n_{-k})$ and $d_{-k}=1/2(a^n_{+k}+ia^n_{-k})$ then

$$e^{B_n t}(a^n_{+k}\psi_{+k}+a^n_{-k}\psi_{-k})=e^{\lambda^n_{+k}t}d^n_{+k}\phi_{+k}+e^{\lambda^n_{-k}t}d^n_{-k}\phi_{-k}$$

and

$$\begin{aligned}
i(e^{A_{\eta n}t}(a^n_{+k}\psi^n_{+k}+a^n_{-k}\psi^n_{-k})) &= i(e^{\lambda^n_{+k}t}d^n_{+k}\phi^n_{+k}+e^{\lambda^n_{-k}t}d^n_{-k}\phi^n_{-k})\\
&= d^n_{+k}i(e^{\lambda^n_{+k}t}\phi^n_{+k})+d^n_{-k}i(e^{\lambda^n_{-k}t}\phi^n_{-k})
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|e^{B_nt}(a^n_{+k}\psi_{+k}+a^n_{-k}\psi_{-k})-i(e^{A_{\eta n}t}(a^n_{+k}\psi^n_{+k}+a^n_{-k}\psi^n_{-k}))\|\\
&\quad\le |d^n_{+k}|\|e^{\lambda^n_{+k}t}\phi_{+k}-i(e^{\lambda^n_{+k}t}\phi^n_{+k})\|+|d^n_{-k}|\|e^{\lambda^n_{-k}t}\phi_{-k}-i(e^{\lambda^n_{-k}t}\phi^n_{-k})\|\\
&\quad= |d^n_{+k}|\|e^{\lambda^n_{+k}t}\phi_{+k}-e^{\lambda^n_{+k}t}i(\phi^n_{+k})\|+|d^n_{-k}|\|e^{\lambda^n_{-k}t}\phi_{-k}-e^{\lambda^n_{-k}t}i(\phi^n_{-k})\|\\
&\quad\le |d^n_{+k}|e^{\alpha^n_{+k}t}\|\phi_{+k}-i(\phi^n_{+k})\|+|d^n_{-k}|e^{\alpha^n_{-k}t}\|\phi_{-k}-i(\phi^n_{-k})\|\\
&\quad\le |d^n_{+k}|e^{\alpha^n_{+k}t}K/n\|\phi_{+k}\|+|d^n_{-k}|e^{\alpha^n_{-k}t}K/n\|\phi_{-k}\|\\
&\quad\le e^{\alpha^n_{+k}t}\tilde{K}/n\|(u_0,v_0)_k\|\le \epsilon t^{-\beta}\|(u_0,v_0)_k\|
\end{aligned}$$

Hence,

$$\|\sum_{k=1}^{K}e^{B_nt}(a^n_{+k}\psi_{+k}+a^n_{-k}\psi_{-k})-\sum_{k=1}^{K}i(e^{A_{\eta n}t}a^n_{+k}\psi^n_{+k}+a^n_{-k}\psi^n_{-k})\|_{Y^0}\le \epsilon t^{-\beta}\|(u_0,v_0)\|_{Y^0}$$

Finally, in the case I),

$$\|e^{B_nt}\sum_{k=1}^{K}(u_0,v_0)_k-i(e^{A_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\le \epsilon t^{-\beta}\|(u_0,v_0)\|_{Y^0},$$

and in the case II)

$$\|e^{B_nt}\sum_{k=1}^{K}(u_0,v_0)_k-i(e^{A_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\le \epsilon t^{-\beta}\|(u_0,v_0)\|_{C^{1+\alpha}\times C^{\alpha}}.$$

4 Comparison of nonlinear semigroups

About the nonlinear semigroups we have

Theorem 4.1. *Let $T_\eta(t)$ and $T_{\eta n}(t)$ be the nonlinear semigroups generated by (1.5) and (1.6), respectively, then*

$$\|T_\eta(t,i(U_0,V_0))-i(T_{\eta n}(t,(U_0,V_0)))\|_{Y^0}\le M\epsilon K_0t^{-\beta}, \tag{4.1}$$

for $t\in(0,\tau)$, $(U_0,V_0)\in\mathcal{A}_{\eta n}$ and for $n\le n(\epsilon)$

Proof: By the variation of constants formula and for $(U_0, V_0) \in \mathcal{A}_{\eta\, n}$ we have

$$T_{\eta n}(t,(U_0,V_0)) = e^{A_{\eta n}t}(U_0,V_0) + \int_0^t e^{A_{\eta n}(t-s)} H(T_{\eta n}(s,(U_0,V_0)))ds \quad (4.2)$$

$$T_{\eta}(t,i(U_0,V_0)) = e^{A_{\eta}t} i(U_0,V_0) + \int_0^t e^{A_{\eta}(t-s)} h(T_{\eta}(s,i(U_0,V_0)))ds \quad (4.3)$$

Then, for $t \in (0,\tau)$

$$\begin{aligned}
&\|T_\eta(t,i(U_0,V_0)) - i(T_{\eta n}(t,(U_0,V_0)))\|_{Y^0}\\
&\le \|e^{A_\eta t} i(U_0,V_0) - i(e^{A_{\eta n}t}(U_0,V_0))\|_{Y^0}\\
&+\|\int_0^t e^{A_\eta(t-s)}h(T_\eta(s,i(U_0,V_0))) - i(e^{A_{\eta n}(t-s)}P_{2n}h(T_\eta(s,i(U_0,V_0))))ds\|_{Y^0}\\
&+\|\int_0^t i(e^{A_{\eta n}(t-s)}P_{2n}h(T_\eta(s,i(U_0,V_0)))) - i(e^{A_{\eta n}(t-s)}H(T_{\eta n}(s,(U_0,V_0))))ds\|_{Y^0}\\
&\le \epsilon t^{-\beta}\|(U_0,V_0)\|_{Y^0_n}\\
&+\int_0^t \|e^{A_\eta(t-s)}h(T_\eta(s,i(U_0,V_0))) - i(e^{A_{\eta n}(t-s)}P_{2n}h(T_\eta(s,i(U_0,V_0))))\|_{Y^0}ds\\
&+\int_0^t \|i(e^{A_{\eta n}(t-s)}P_{2n}h(T_\eta(s,i(U_0,V_0)))) - i(e^{A_{\eta n}(t-s)}H(T_{\eta n}(s,(U_0,V_0))))\|_{Y^0}ds
\end{aligned}$$

Since that $i(U_0,V_0)$ is bounded in $H^1 \times L^2$ we have $(T_\eta(s,i(U_0,V_0)))_1$ is bounded in C^α, thus $\|h(T_\eta(s,i(U_0,V_0)))\|_{C^{1+\alpha}\times C^\alpha}$ is bounded for all $(U_0,V_0) \in \bigcup_n \mathcal{A}_{\eta\, n}$. We also have $H(T_{\eta n}(s,(U_0,V_0))) = P_{2n}(h(i(T_{\eta n}(s,(U_0,V_0)))))$ then

$$\begin{aligned}
&\|T_\eta(t,i(U_0,V_0)) - i(T_{\eta n}(t,(U_0,V_0)))\|_{Y^0}\\
&\quad\le \epsilon t^{-\beta}\|(U_0,V_0)\|_{Y^0_n} + \epsilon\int_0^t (t-s)^{-\beta}\|h(T_\eta(s,i(U_0,V_0)))\|_{C^{1+\alpha}\times C^\alpha}ds\\
&\quad+\int_0^t \|e^{A_{\eta n}(t-s)}\|\|P_{2n}(h(T_\eta(s,i(U_0,V_0)))) - P_{2n}(h(i(T_{\eta n}(s,(U_0,V_0)))))\|_{Y^0_n}ds\\
&\quad\le \epsilon t^{-\beta}K_0 + \epsilon\tau\frac{t^{-\beta}K_0}{1-\beta} + L\int_0^t \|(T_\eta(s,i(U_0,V_0))) - i(T_{\eta n}(s,(U_0,V_0)))\|_{Y^0}ds
\end{aligned}$$

Hence, by Gronwall Inequality, we have that exists a constant $M(\beta, \tau, L)$ such that

$$\|T_\eta(t, i(U_0, V_0)) - i(T_{\eta n}(t, (U_0, V_0)))\|_{H^1 \times L^2} \leq M\epsilon K_0 t^{-\beta}, \qquad (4.4)$$

for $t \in (0, \tau)$, $(U_0, V_0) \in \mathcal{A}_{\eta\, n}$ and for $n \leq n(\epsilon)$

5 Upper Semicontinuity of global attractors $\mathcal{A}_\eta$ and $i(\mathcal{A}_{\eta\, n})$ in $H^1 \times L^2$

Now we can prove the main result

Theorem 5.1. *The family of global attractors $\mathcal{A}_{\eta\, n}$ is upper semicontinuous at $n = \infty$, for any $\eta > 0$.*

Proof: Since that $\bigcup_n \mathcal{A}_{\eta\, n}$ is bounded in $\mathbb{R}^n \times \mathbb{R}^n$ and also $\|i(U, V)\|_{H^1 \times L^2} \leq \|(U, V)\|_{\mathbb{R}^n \times \mathbb{R}^n}$ then $\|i(\bigcup_n \mathcal{A}_{\eta\, n})\|_{H^1 \times L^2} \leq \| \bigcup_n \mathcal{A}_n\|_{\mathbb{R}^n \times \mathbb{R}^n} \leq K$.

The global attractor $\mathcal{A}_\eta$ attracts bounded of $H^1 \times L^2$ thus, $\forall \delta > 0$, exists $\tau = \tau(\delta)$ such that

$$\delta_{Y_0}(T_\eta(\tau, i(\phi_n)), \mathcal{A}_\eta) \leq \delta/2$$

for all $\phi_n \in \mathcal{A}_{\eta\, n}$ and for all n.
The attractors $\mathcal{A}_{\eta\, n}$ are invariant, thus if $\psi_n \in \mathcal{A}_{\eta\, n}$ then exists $\phi_n \in \mathcal{A}_{\eta\, n}$ such that $T_{\eta n}(\tau, \phi_n) = \psi_n$.
Hence, we choose $n_0(\delta) = n(\epsilon(\delta)) > 0$ such that

$$\|T_\eta(\tau, i(\phi_n)) - i(T_{\eta n}(\tau, \phi_n))\| \leq M\epsilon\tau^{-\beta}\|\phi_n\| \leq \delta/2$$

for $n \geq n_0(\delta)$.
Therefore,

$$\delta_{Y_0}(i(\psi_n), \mathcal{A}_\eta) \leq \delta_{Y_0}(i(\psi_n), T_\eta(\tau, i(\phi_n)) + \delta_{Y_0}(T_\eta(\tau, i(\phi_n)), \mathcal{A}_\eta) \leq \delta$$

for all $\psi_n \in \mathcal{A}_{\eta n}$ and for all $n \geq n_0(\delta)$.

References

[1] Arrieta, J. M.; Carvalho, A. N.; Rodríguez-Bernal, A., *Upper Semicontinuity of Attractors for Parabolic Problems with Localized Large Diffusion and Nonlinear Boundary Conditions*, Journal of Differential Equations, 168 (2001), 33–59.

[2] Bruschi, S. M., *Discretização de problemas semilineares dissipativos parabólicos e hiperbólicos em domínios unidimensionais*, Tese de Doutorado, USP-São Carlos, Julho (2000).

[3] Bruschi, S. M.; Carvalho, A. N.; Cholewa, J.; Dlotko, T., *Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations*, Journal of Dynamics and Differential Equations 18 (3) (2006), 767-814.

[4] Bruschi, S. M.; Carvalho, A. N.; Ruas Filho, J. G., *The dynamics of a one-dimensional parabolic problem versus the dynamics of its discretization*, Journal of Differential Equations, 168-1, Part 1, (2000), 67-92.

[5] Carvalho, A. N.; Cholewa, J.; Dlotko, T., *Attractors for strongly damped wave equation with critical nonlinearities*, Pacific Journal of Mathematics, 207, (2002), 287-310.

[6] Chen, S.; Triggiani, R., *Proof of Extensions of two Conjectures on Structural Damping for Elastic Systems*, Pacific Journal of Mathematics, 136 (1), (1989), 15–55.

[7] Chen, S.; Triggiani, R., *Characterization of Domains of Fractional Powers of Certain Operators Arising in Elastic Systems, and Applications*, Journal of Differential Equations, 88 (1990), 279–293.

[8] Henry, D., *Geometric theory of semilinear parabolic equations,* Lecture Notes in Mathematics 840, Springer-Verlag, Berlin (1981).

[9] Temam, R., *Numerical Analysis*, D. Reidel Publishing Company, (1973).

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