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ON EQUATIONS OF NAVIER-STOKES TYPE IN MOVING DOMAINS

G. M. de Araújo **S. B. de Menezes**

Dedicated to Prof. L. A. Medeiros on occasion of his 80^{th} *birthday*

Abstract

In this work we are concerned with the existence and uniqueness of weak solutions for an initial-boundary value problem associated with equations of Navier-Stokes type in a domain $\widehat{Q}$ with moving boundary. The technique, to show the existence and uniqueness of solutions, consists in transforming $\widehat{Q}$ into a cylinder Q by using a suitable diffeomorphism and to apply in Q the Faedo-Galerkin method and basic result of the theory of monotone operators in the transformed initial-boundary value problem.

1 Introduction

In this article we study evolution equations of Navier-Stokes type, in a domain of $\mathbb{R}^n_x \times \mathbb{R}_t$ whose boundary is moving with respect to t, for $t \in [0, T]$ and $T > 0$. More precisely, we consider an open bounded

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domain $\widehat{Q}$ of $\mathbb{R}_x^n \times \mathbb{R}_t$ which is the union of open bounded sets $\Omega_t \subset \mathbb{R}_x^n$ and Ω_t are deformations of a fixed set Ω of $\mathbb{R}_x^n$ by a diffeomorphism τ_t to be defined as follows. Henceforth, we will write $\mathbb{R}^n$ instead of $\mathbb{R}_x^n$, for $n \in \mathbb{N}$. Thus, let Ω fixed, non-empty, open bounded set of $\mathbb{R}^n$, whose points are represented by $y = (y_1, y_2, ..., y_n)$ with y_i real numbers for $i = 1, 2, ..., n$. Let Ω_t be the diffeomorphic images of Ω by the matrix valued function

$$\begin{array}{ccc} [0,T] & \rightarrow & \mathbb{R}^{n^2} \\ t & \longmapsto & K(t). \end{array}$$

The vectors of Ω_t are represented by $x = (x_1, x_2, ..., x_n)$ where x_i is a real number for each $i = 1, 2, ..., n$. Thus we have

$$x = K(t)y, \ \text{ for } \ i = 1, 2, ..., n.$$

The non cylindrical domain $\widehat{Q}$ of $\mathbb{R}_x^n \times \mathbb{R}_t$ is defined by

$$\widehat{Q} = \bigcup_{0 \le t \le T} \{\Omega_t \times \{t\}\}.$$

If the boundary of Ω_t is Γ_t, then the lateral boundary of $\widehat{Q}$ is

$$\widehat{\Sigma} = \bigcup_{0 \le t \le T} \{\Gamma_t \times \{t\}\}.$$

We represent by Q the cylinder $Q = \Omega \times [0, T[$, with lateral boundary Σ given by $\Sigma = \Gamma \times [0, T[$, where Γ is the boundary of Ω. In these conditions, we have the natural diffeomorphism between Q and $\widehat{Q}$ given by

$$(y, t) \in Q \rightarrow (x, t) \in \widehat{Q} \ \text{ with } \ x = K(t)y \ \text{ and } \ 0 \le t \le T.$$

Finally we propose the non cylindrical initial boundary value problem for a differential equation of Navier-Stokes type:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu_1 \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) - \nu_0 \Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} \\ = f - \nabla \widehat{p} \quad \text{in} \quad \widehat{Q} \\ div\, u = 0 \text{ in } \widehat{Q} \\ u = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x, 0) = u_0(x) \quad \text{in} \quad \Omega_0. \end{cases} \tag{1.1}$$

In (1.1), $u = (u_i)_{1\leq i\leq n}$ is a vector velocity of the fluid , f is the density of forces acting on it, $\widehat{p} = \widehat{p}(x,t)$ is the pressure at point $(x,t) \in \widehat{Q}$, $\Delta u = (\Delta u_1, \Delta u_2, ..., \Delta u_n)$, ν_0, ν_1 are positive constants.

If we have $\nu_1 = 0$, then the problem (1.1) reduces to the classical Navier-Stokes equation in non cylindrical domain. Global existence and uniqueness results for such nonhomogeneous, incompressible Navier-Stokes type equation (1.1) were first obtained by J. L. Lions [3], under standard hypotheses on f and u_0 in the dimension $n \geq 2$, in context of cylindrical domains. Here we are considering the same equations as in [3] however in more general non cylindrical domains.

From a physical point of view, a real fluid is evolutional, so the region filled with a moving fluid usually move along the trajectories of the incompressible fluid motion. Thus, the space-time domain is not a cylindrical one as often treated. So we treat with the case of a non cylindrical space-time domains in this paper. To investigate the existence and uniqueness of solutions for the initial and moving boundary value problem (1.1) we assume the following hypotheses:

$(H1) \quad K(t) = k(t)M$

where $k = k(t)$ is a real function for $0 \leq t \leq T$ continuously derivable with $k(t) \geq k_0 > 0$, k_0 a positive constant, and M is an invertible $n \times n$ matrix whose entries are real constants.

We adopt the notation $K(t) = (\alpha_{ij}(t))$ and $K^{-1}(t) = (\beta_{ij}(t))$. The method we employ to obtain the existence of solutions for the problem (1.1) consists in transforming it in another equivalent problem proposed in the cylinder Q by means of the diffeomorphism $(x,t) = \big(K(t)y, t\big)$ for $x \in \Omega_t$, $y \in \Omega$ and $0 \leq t \leq T$, i.e., for $(x,t) \in \widehat{Q}$ and $(y,t) \in Q$. In fact we set

$$\begin{aligned} u(x,t) &= v(K^{-1}(t)x,t), \quad f(x,t) = g(K^{-1}(t)x,t) \\ p(x,t) &= q(K^{-1}(t)x,t), \quad u_0(x) = v_0(K^{-1}(0)x). \end{aligned} \tag{1.2}$$

Then we transform the system (1.1) to the following problem defined in the cylinder Q :

$$\begin{cases} \dfrac{\partial v}{\partial t} -\nu_1 \dfrac{\partial}{\partial y_r}\left[\left(\displaystyle\sum_{i,k=1}^{n}\left|\sum_{s=1}^{n}\beta_{sk}(t)\frac{\partial v_i}{\partial y_s}\right|^2\right)^{\frac{p-2}{2}}\displaystyle\sum_{l,r=1}^{n} a_{lr}(t)\frac{\partial v}{\partial y_l}\right] \\ -\nu_0\left(\displaystyle\sum_{l,r=1}^{n} a_{lr}(t)\frac{\partial^2 v}{\partial y_l y_r}\right) + \displaystyle\sum_{i,l=1}^{n}\beta_{li}(t)v_i\frac{\partial v}{\partial y_l} \\ + \displaystyle\sum_{j,l,r=1}^{n}\beta'_{lr}(t)\alpha_{rj}(t)y_j\frac{\partial v}{\partial y_l} = g - (\nabla q)K^{-1}(t) \text{ in } Q \\ div(M^{-1}v^T) = 0 \text{ in } Q \\ v = 0 \text{ on } \Sigma \\ v(y,0) = v_0(y) \text{ in } \Omega \end{cases} \tag{1.3}$$

where v^T is the transposed of the row vector $v = (v_1, ..., v_n)$ and

$$a_{lr}(t) = \sum_{j=1}^{n}\beta_{lj}(t)\beta_{rj}(t). \tag{1.4}$$

The obtention of (1.3) is given in Appendix 4, and the equivalence of problems (1.1) and (1.3) is proved in Theorem 2.3.

Remark 1.1. *We note that the particular form of the function $K(t)$ given by hypothesis (H_1) is considered in order to have the equivalence between the conditions $div\, u = 0$ in $\widehat{Q}$ and $div\left(M^{-1}v^T\right) = 0$ in Q.*

In order to formulate problems (1.1) and (1.3) we need some notations about Sobolev spaces. In fact, let us consider the following spaces

$$\mathcal{V}_t = \{\varphi \in (\mathcal{D}(\Omega_t))^n; \quad div\ \varphi = 0\}, \quad V(\Omega_t) = \overline{\mathcal{V}}_t^{(W^{1,p}(\Omega_t))^n},$$

$$V_s(\Omega_t) = \overline{\mathcal{V}}_t^{(H^s(\Omega_t))^n} \quad \text{and} \quad H(\Omega_t) = \overline{\mathcal{V}}_t^{(L^2(\Omega_t))^n}.$$

The norm of the space $V(\Omega_t)$ and the inner product and norm of the space $H(\Omega_t)$ are denoted, respectively, by $\|u\|_{V(\Omega_t)} = \left(\int_{\Omega_t} |\nabla u|^p \, dx\right)^{\frac{1}{p}}$, $(u,z)_{H(\Omega_t)} = \sum_{i=1}^{n} \int_{\Omega_t} u_i(x) z_i(x) \, dx$ and $|u|^2_{H(\Omega_t)} = \sum_{i=1}^{n} \int_{\Omega_t} |u_i(x)|^2 \, dx$.

By analogy, we define the spaces

$\mathcal{V} = \{\psi \in (\mathcal{D}(\Omega))^n \; ; \; div(M^{-1}\psi^T) = 0\}$, $\; V = \overline{\mathcal{V}}^{(W^{1,p}(\Omega))^n}$,

$V_s = \overline{\mathcal{V}}^{(H^s(\Omega))^n}$ and $H = \overline{\mathcal{V}}^{(L^2(\Omega))^n}$.

The norm of the space V and the inner product and norm of the space H are represented, respectively, by $\|v\| = \left(\int_{\Omega} |\nabla v|^p \, dy\right)^{\frac{1}{p}}$, $(v,w) = \sum_{i=1}^{n} \int_{\Omega} v_i(y) w_i(y) \, dy$ and $|v|^2 = \sum_{i=1}^{n} \int_{\Omega} |v_i(y)|^2 \, dy$.

Finally, in the case of non cylindrical domains $\widehat{Q}$, the spaces $L^p(0,T;V(\Omega_t))$, $L^\infty(0,T;V(\Omega_t))$, $L^p(0,T;H(\Omega_t))$ and $L^\infty(0,T;H(\Omega_t))$ are defined like in Lions [3]. By $\langle , \rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X, and by C (sometimes C_1, C_2,...) we denote various positive constants.

Next, to state the variational formulation of problems (1.1) and (1.3), we introduce some bilinear and trilinear forms and some operators.

Concerning the non cylindrical problem, we introduce the notations

$$\widehat{a}(t;u,z) = \sum_{i,j=1}^{n} \int_{\Omega_t} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial z_i}{\partial x_j}(x) \, dx = ((u,z))_{V(\Omega_t)} \tag{1.5}$$

$$\widehat{b}(t;u,z,\xi) = \sum_{i,j=1}^{n} \int_{\Omega_t} u_i(x) \frac{\partial z_j}{\partial x_i}(x) \xi_j(x) \, dx, \quad u,z,\xi \in V(\Omega_t) \tag{1.6}$$

$$\widehat{A}(t) : V(\Omega_t) \longrightarrow V'(\Omega_t), \;\; \widehat{A}(t)u = -\Delta u \tag{1.7}$$

$$\widehat{\mathcal{A}}(t) : V(\Omega_t) \longrightarrow V'(\Omega_t), \quad \widehat{\mathcal{A}}(t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla|^{p-2}\frac{\partial u}{\partial x_i}\right) \tag{1.8}$$

$$\widehat{B}(t) : V(\Omega_t) \longrightarrow V'(\Omega_t), \quad \widehat{B}(t)u = \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} \tag{1.9}$$

and for the cylindrical problem,

$$a(t; v, w) = \sum_{i,l,r=1}^{n} \int_\Omega a_{lr}(t) \frac{\partial v_i}{\partial y_r}(y) \frac{\partial w_i}{\partial y_l}(y)\, dy, \quad v, w \in V \tag{1.10}$$

$$\begin{aligned} b(t; v, w, \psi) = \sum_{i,j,l=1}^{n} \int_\Omega \beta_{li}(t) v_i(y) \frac{\partial w_j}{\partial y_l}(y)\psi_j(y)\, dy,& \\ v, w, \psi \in V& \end{aligned} \tag{1.11}$$

$$\begin{aligned} c(t; v, w) = \sum_{i,j,l,r=1}^{n} \int_\Omega \beta'_{lr}(t)\alpha_{rj}(t)\ y_j\ \frac{\partial v_i}{\partial y_l}(y) w_i(y)\, dy,& \\ v \in V, w \in H& \end{aligned} \tag{1.12}$$

$$A(t) : V \longrightarrow V', \quad A(t)v = -\sum_{l,r=1}^{n} a_{lr}(t) \frac{\partial^2 v}{\partial y_l \partial y_r} \tag{1.13}$$

$$\begin{aligned} &\mathcal{A}(t) : V \longrightarrow V', \\ &\mathcal{A}(t)v = -\frac{\partial}{\partial y_r}\left[\left(\sum_{i,k=1}^{n}\left|\sum_{s=1}^{n}\beta_{sk}(t)\frac{\partial v_i}{\partial y_s}\right|^2\right)^{\frac{p-2}{2}} \sum_{l,r=1}^{n} a_{lr}(t)\frac{\partial v}{\partial y_l}\right] \end{aligned} \tag{1.14}$$

$$B(t) : V \longrightarrow V', \quad B(t)v = \sum_{i,l=1}^{n} \beta_{li}(t) v_i \frac{\partial v}{\partial y_l} \tag{1.15}$$

$$C(t) : V \longrightarrow H, \quad C(t)v = \sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t) y_j \frac{\partial v}{\partial y_l}. \tag{1.16}$$

Remark 1.2. *The mapping $\mathcal{A}$ takes objects of V into V', and bounded sets of V into bounded sets of V'. In fact*

$$
\begin{aligned}
|\langle \mathcal{A}(t)v, w\rangle| \leq & \sum_{j,r=1}^{n} \int_{\Omega} \left| \left(\sum_{i,k=1}^{n} \left| \sum_{s=1}^{n} \beta_{sk} \frac{\partial v_i}{\partial y_s}(y) \right|^2 \right)^{\frac{p-2}{2}} \right| \\
& \times \left| \sum_{\mu,l=1}^{n} \beta_{lj} \frac{\partial v_\mu}{\partial y_l}(y) \frac{\partial w_\mu}{\partial y_r}(y) \right| dy \\
& \leq c \sum_{i,l,\mu,r,s=1}^{n} \int_{\Omega} \left| \frac{\partial v_i}{\partial y_s}(y) \right|^{p-2} \left| \frac{\partial v_\mu}{\partial y_l}(y) \right| \left| \frac{\partial w_\mu}{\partial y_r}(y) \right| dy.
\end{aligned}
\tag{1.17}
$$

On the other hand, using Hölder inequality with $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{p-2}{p} = 1$, we obtain $|\langle \mathcal{A}(t)v, w\rangle| \leq c\|v\|^{p-1}\|w\|$ or $\|\mathcal{A}v\|_{V'} \leq c\|v\|^{p-1}$.

The proof that these operators are well defined is given in Section 3.

2 Solution concept and main results

The solution concept and the main results for the equivalent problems (1.1) and (1.3) are given by

Definition 2.1. *A weak solution for (1.1) is a function $u : \widehat{Q} \to \mathbb{R}$ in the class $u \in L^\infty(0,T; H(\Omega_t)) \cap L^p(0,T; V(\Omega_t))$ for $T > 0$, satisfying the integral identity*

$$
\begin{aligned}
& -\int_0^T (u(t), \xi'(t))_{H(\Omega_t)}\, dt + \nu_0 \int_0^T \widehat{a}(t; u(t), \xi(t))\, dt \\
& +\nu_1 \int_0^T \langle \widehat{\mathcal{A}}(t)u(t), \xi(t)\rangle_{V'(\Omega_t)V(\Omega_t)}\, dt \\
& + \int_0^T \widehat{b}(t; u(t), u(t), \xi(t))\, dt = \int_0^T \langle f(t), \xi(t)\rangle_{V'(\Omega_t)V(\Omega_t)}\, dt,
\end{aligned}
\tag{2.1}
$$

for all $\xi \in L^p(0,T; V(\Omega_t)), \xi' \in L^1(0,T; H(\Omega_t))$, with $\xi(0) = \xi(T) = 0$. Moreover, u verifies the initial condition $u(x,0) = u_0(x)$ in Ω_0.

Theorem 2.1. *If* $n \geq 2$, $p \geq 1 + \dfrac{2n}{n+2}$, $f \in L^{p'}(0,T;V'(\Omega_t))$, $u_0 \in H(\Omega_0)$, *and* (H_1) *hold, then the initial boundary value problem* (1.1) *has a weak solution in the sense of Definition 2.1. Moreover, if* $p \geq \dfrac{n+2}{2}$ *then the initial boundary value problem* (1.1) *has only one weak solution in the sense of Definition 2.1.*

Definition 2.2. *A weak solution for (1.3) is a function* $v : Q \to \mathbb{R}$ *in the class* $v \in L^\infty(0,T;H) \cap L^p(0,T;V)$ *for* $T > 0$, *satisfying the integral identity*

$$\begin{aligned}
&-\int_0^T (v(t), \psi'(t))\,dt + \nu_0 \int_0^T a(t; v(t), \psi(t))\,dt \\
&+\nu_1 \int_0^T \langle \mathcal{A}(t)v(t), \psi(t)\rangle\,dt + \int_0^T b(t; v(t), v(t), \psi(t))\,dt \\
&+\int_0^T c(t; v(t), \psi(t))\,dt = \int_0^T \langle g(t), \psi(t)\rangle\,dt,
\end{aligned} \tag{2.2}$$

for all $\psi \in L^p(0,T;V)$, $\psi' \in L^1(0,T;H)$, *with* $\psi(0) = \psi(T) = 0$. *Besides,* v *satisfies the initial condition* $v(y,0) = v_0(y)$ *in* Ω.

Theorem 2.2. *If* $v_0 \in H$, $g \in L^{p'}(0,T;V')$, $n \geq 2$, $p \geq 1 + \dfrac{2n}{n+2}$ *and hypothesis* (H_1) *hold, then the initial boundary value problem (1.3) has a weak solution in the sense of Definition 2.2. Moreover, if* $p \geq \dfrac{n+2}{2}$ *then the initial boundary value problem* (1.3) *has only one weak solution in the sense of Definition 2.2*

Theorem 2.3. *The problems* (2.1) *and* (2.2) *are equivalents.*

Remark 2.1. *Applying Lemma 3.1 and Lemma 3.6 , we obtain that the weak solution* u *of the problem* (1.1) *satisfies*

$$\left\{\begin{array}{l}
u' + \nu_0 \widehat{A}u + \nu_1 \widehat{\mathcal{A}}u + \widehat{B}u = f \text{ in } L^{p'}(0,T;V'(\Omega_t)) \\
u(x,0) = u_0(x)
\end{array}\right. \tag{2.3}$$

and the weak solution v *of the problem (1.3) satisfies*

$$\left\{\begin{array}{l}
v' + \nu_0 Av + \nu_1 \mathcal{A}v + Bv + Cv = g \text{ in } L^{p'}(0,T;V') \\
v(y,0) = v_0(y).
\end{array}\right. \tag{2.4}$$

Remark 2.2. *Following the ideas of Lions [3] or Temam [6], we deduce from the equation*

$$u' + \nu_0 \widehat{A}u + \nu_1 \widehat{\mathcal{A}}u + \widehat{B}u = f \ \text{in}\ L^{p'}(0,T;V'(\Omega_t)),$$

given in (2.3)*, that there exists* $p \in L^{p'}(0,T;L^2(\Omega_t))$ *such that*

$$u' + \nu_0 \widehat{A}u + \nu_1 \widehat{\mathcal{A}}u + \widehat{B}u = f - \nabla \widehat{p} \ \text{in}\ L^{p'}(0,T;(H^{-1}(\Omega_t))^n).$$

3 Proof of the results

We begin by stating some lemmas that will be used in the proof of the results.

Lemma 3.1. *Concerning to the bilinear form* $a(t;v,w)$ *and the operator* $A(t)$ *defined, respectively, by (1.10) and (1.13), we have:*

i) $\langle A(t)v, w\rangle = a(t;v,w),\ \forall v,w \in V.$

ii) $a(t;v,v) \geq a_0\|v\|^2,\ \forall v \in V$ *(a_0 positive constant).*

iii) $|a(t;v,w)| \leq a_1\|v\|\ \|w\|,\ \forall v,w \in V$ *(a_1 positive constant).*

Lemma 3.2. *If* $s > 1 + \dfrac{n}{2}$ *and* $n \geq 2$*, then* $b(t;v,w,\psi)$*,* $c(t;v,w)$*,* $B(t)$ *and* $C(t)$ *satisfies*

i) $b(t;v,v,w) = -b(t;v,w,v),\ \forall v \in V, w \in V_s.$

ii) For all $t \in [0,T]$*,* $v \in V$ *and* $w \in V_s$*, the linear form* $w \longmapsto b(t;v,v,w)$ *is continuous on* V_s *and verifies*

$$b(t;v,v,w) = \langle B(t)v, w\rangle, \quad \|B(t)v\|_{V_s'} \leq c_1\|v\|^2, \quad \forall v \in V.$$

iii) $|c(t;v,w)| \leq c_2\|v\|\ |w|,\ \forall v \in V$ *and* $w \in H.$

iv) For all $t \in [0,T]$ *and* $v \in V$*, the linear form* $w \longmapsto c(t;v,w)$ *is continuous on* H *and verifies*

$$c(t;v,w) = (C(t)v, w), \quad |C(t)v| \leq c_3\|v\|.$$

The positive constants $c_i,\ i=1,2,3$, are independents of v and w.

Lemma 3.3. *If* $s > 1 + \dfrac{n}{2},\ n \geq 2,\ \widehat{b}(t;u,z,\zeta)$ *and* $\widehat{B}(t)$ *are defined, respectively, by* (1.6) *and (1.9), then for each* $t \in [0,T]$ *and* $u \in V(\Omega_t), z \in V_s(\Omega_t)$, *the linear form* $z \longrightarrow \widehat{b}(t;u,u,z)$ *is continuous on* $V_s(\Omega_t)$ *and* $\widehat{b}(t;u,u,z) = \langle \widehat{B}(t)u, z\rangle$.

Lemma 3.4. *For each fixed* $t \in [0,T]$, *the operator* $\mathcal{A}(t) : V \longmapsto V'$ *defined by*

$$\mathcal{A}(t)(v) = -\frac{\partial}{\partial y_r}\left[\left(\sum_{i,k=1}^{n}\left|\sum_{s=1}^{n}\beta_{sk}(t)\frac{\partial v_i}{\partial y_s}\right|^2\right)^{\frac{p-2}{2}}\sum_{l,r=1}^{n}a_{lr}(t)\frac{\partial v}{\partial y_l}\right],$$

where $a_{lr}(t) = \displaystyle\sum_{j=1}^{n}\beta_{lj}(t)\beta_{rj}(t)$, *is monotone and hemicontinuous.*

Lemma 3.5. *For the operator* $\mathcal{A}$, *defined in Lemma 3.4, we have*

$$\langle \mathcal{A}v, v\rangle \geq c\|v\|^p, \quad \textit{for all } v \in V.$$

Proof. Indeed, we observe that

$$\langle \mathcal{A}v, v\rangle = \int_\Omega \Big[\left(\sum_{i,k=1}^{n}\left|\sum_{s=1}^{n}\beta_{sk}(t)\frac{\partial v_i}{\partial y_s}(y)\right|^2\right)^{\frac{p-2}{2}}$$
$$\times \sum_{j,l,\mu,r}^{n}\beta_{rj}(t)\beta_{lj}(t)\frac{\partial v_\mu}{\partial y_l}(y)\frac{\partial v_\mu}{\partial y_r}(y)\Big]\,dy,$$
$$\left(\sum_{i,k=1}^{n}\left(\sum_{s=1}^{n}\beta_{sk}(t)\frac{\partial v_i}{\partial y_s}(y)\right)^2\right)^{\frac{p-2}{2}} \geq c|\nabla v|^{p-2},$$

where $c > 0$ is a constant independent of v. For $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, we have

$$\begin{aligned}\|\xi K^{-1}(t)\|^n_{\mathbb{R}^n} &= \left\| \left[\sum_{s=1}^n \beta_{s1}(t)\xi_s, \dots \sum_{s=1}^n \beta_{sn}(t)\xi_s \right] \right\|^2_{\mathbb{R}^n} \\ &= \sum_{k=1}^n \left(\sum_{s=1}^n \beta_{sk}(t)\xi_s \right)^2 .\end{aligned} \tag{3.1}$$

Again, by using Cauchy-Schwarz inequality and $\xi K^{-1}(t) = \eta$ in (3.1) yields $\|\eta K(t)\|^2_{\mathbb{R}^n} \leq c\|\eta\|^2_{\mathbb{R}^n}$, for all $t \in [0, T]$. It follows from above results that

$$\begin{aligned}\sum_{k=1}^n \left(\sum_{s=1}^n \beta_{sk}(t)\xi_s \right)^2 &= \|\xi K^{-1}(t)\|^2_{\mathbb{R}^n} = \|\eta\|^2_{\mathbb{R}^n} \\ &\geq c\|\eta K(t)\|^2_{\mathbb{R}^n} = c\|\xi\|^2_{\mathbb{R}^n} = \sum_{s=1}^n (\xi_s)^2.\end{aligned}$$

Taking $\xi_s = \dfrac{\partial v_i}{\partial y_s}$, we obtain

$$\left(\sum_{i,k=1}^n \left(\sum_{s=1}^n \beta_{sk}(t) \frac{\partial v_i}{\partial y_s}(y) \right)^2 \right)^{\frac{p-2}{2}} \geq c|\nabla v|^{p-2}.$$

Using similar argument, we prove that

$$\sum_{j,r,l,\mu=1}^n \beta_{rj}(t)\beta_{lj}(t) \frac{\partial v_\mu}{\partial y_l}(y) \frac{\partial v_\mu}{\partial y_r}(y) \geq c|\nabla v|^2.$$

Lemma 3.6. *Assuming* $p \geq 1 + \dfrac{2n}{n+2}$, *if* $u, v \in L^p(0, T; V) \cap L^\infty(0, T; H)$ *then* $b(u(t), u(t), v(t)) \in L^1(0, T)$.

The proof of Lemma 3.6 can be found in Lions [3], p. 212 to 213. Lemma 3.1 to Lemma 3.3 can be obtained with slight modifications from Lions, loc. cit., and Miranda-Límaco [5]. Lemma 3.4 follow directly.

Proof of Theorem 2.2. We employ Faedo-Galerkin approximate method with a hilbertian basis $(w_\nu)_{\nu \in \mathbb{N}}$ of Sobolev space V_s, cf. Brezis [1], defined as solution of the eigenvalue problem

$$((w_\nu, v))_{V_s} = \lambda(w_\nu, v) \text{ for all } v \in V_s \text{ and } \nu \in \mathbb{N}. \tag{3.2}$$

Identifying H with its dual and assuming that $s > 1 + \frac{n}{2}$, we have the continuous embedding $V_s \hookrightarrow V \hookrightarrow H \hookrightarrow V' \hookrightarrow V_s'$, with immersion of V_s into H compact. It follows that the spectral problem (3.2) has a solution $(w_\nu)_{\nu\in\mathbb{N}}$ and $(\lambda_\nu)_{\nu\in\mathbb{N}}$. If V_m is the subspace spanned by the m first vectors of $\{w_1, w_2, w_3, ...\}$, the approximate problem will consist of determining one function $v_m(y,t) = \sum_{j=1}^{m} h_{jm}(t)w_j$ in V_m solution of the following system of ordinary differential equations

$$\begin{cases} (v_m', w_j) + \nu_0 a(t; v_m, w_j) + \nu_1\langle \mathcal{A}(t)v_m, w_j\rangle + b(t; v_m, v_m, w_j) \\ +c(t; v_m, w_j) = \langle g(t), w_j\rangle, \ j = 1, 2, ..., m \\ v_m(y, 0) = v_{0_m}, \quad v_{0_m} \longrightarrow v_0 \text{ in } H. \end{cases} \tag{3.3}$$

System (3.3) has local solution v_m in $0 \le t < t_m$, see for instance, Coddington-Levinson [2]. The main point is to obtain the necessary a priori estimates in order to extend the local solutions to the whole interval $[0,T]$. They are also needed in the convergence analysis of the approximate solutions to a solution of (1.3) in the sense of Definition 2.2.

First estimate Lemma 3.6 and Remark 1.2 implies that

$$\langle \mathcal{A}(t)v_m(t), v_m(t)\rangle \ge c_1\|v_m(t)\|^p \tag{3.4}$$

and

$$\|\mathcal{A}(t)v_m(t)\|_{V'} \le c_2\|v_m(t)\|^{p-1}, \tag{3.5}$$

where c_1 and c_2 are constants independents of m and $t \in [0,T]$. Substituting w_j by $v_m'(t)$ in (3.3), integrating this result from 0 to t and using Lemma 3.1, Lemma 3.2 and inequality (3.4), we get

$$|v_m(t)|^2 + \int_0^t \|v_m(s)\|^2 ds + \int_0^t \|v_m(s)\|^p ds \le c_3 + c_4\int_0^t |v_m(s)|^2 ds,$$

where c_3 and c_4 are constants independents of m and t. Then Gronwall's inequality implies

$$(v_m) \text{ is bounded in } L^\infty(0,T;H) \tag{3.6}$$

$$(v_m) \text{ is bounded in } L^p(0,T;V) \tag{3.7}$$

$$(v_m) \text{ is bounded in } L^2(0,T;V) \tag{3.8}$$

Second estimate. Let P_m be the orthogonal projection of H on V_m, that is, $P_m\varphi = \sum_{j=1}^{m}(\varphi, w_j)w_j, \quad \varphi \in H$. Since (w_ν) are the solutions of the spectral problem (3.2), we have

$$\|P_m\|_{\mathcal{L}(V,V)} \leq 1 \text{ and } \|P_m^*\|_{\mathcal{L}(V',V')} \leq 1. \tag{3.9}$$

Note that $P_m v'_m = v'_m$. Multiplying both sides of the approximate equation $(3.2)_1$ by $h_{jm}(t)$ and adding from $j = 1$ to $j = m$, we obtain

$$\begin{aligned} v'_m(t) &= -\nu_0 P_m^* A(t)v_m(t) - \nu_1 P_m^* \mathcal{A}(t)v_m(t) \\ &-P_m^* B(t)v_m(t) - P_m^* C(t)v_m(t) + P_m^* g(t). \end{aligned} \tag{3.10}$$

Taking into account (3.6) to (3.9) into (3.10), using Lemma 3.1 to Lemma 3.6 and estimates (3.4) and (3.5), we obtain

$$(v'_m) \text{ is bounded in } L^{p'}(0,T;V'_s). \tag{3.11}$$

Estimates (3.6), (3.7), (3.11) and Aubin-Lion's Compacteness Theorem applied to (3.7) and (3.11), imply that there exists a subsequence from (v_m), still denoted by (v_m), such that

$$v_m \rightharpoonup v \text{ weak star in } L^\infty(0,T;H) \tag{3.12}$$

$$v_m \rightharpoonup v \text{ weak in } L^p(0,T;V) \tag{3.13}$$

$$v'_m \rightharpoonup v' \text{ weak in } L^{p'}(0,T;V'_s) \tag{3.14}$$

$$v_m \longrightarrow v \text{ strong in } L^2(0,T;H) \text{ and a.e in Q} \tag{3.15}$$

$$\mathcal{A}v_m \rightharpoonup \chi \text{ weak in } L^{p'}(0,T;V'). \tag{3.16}$$

Convergence results obtained above allow us to pass to the limit in the approximate equation $(3.2)_1$ to obtain

$$\begin{aligned}&\langle v', w\rangle + \nu_0 a(t;v,w) + \nu_1\langle \chi, v\rangle + b(t;v,v,w) + c(t;v,w)\\ &= \langle g, w\rangle \ \text{ for all } \ w \in V_s.\end{aligned} \tag{3.17}$$

It remains to show $\chi = \mathcal{A}v$. This is proved by a standard monotonicity argument for the operator $\mathcal{A}(t)$, coupled with some technical ideas, cf. Lions [3].

Indeed, for $s_0, s \in]0,T[$, with $s > s_0$, we define $\psi_m : [0,T] \longrightarrow \mathbb{R}$ by

$$\psi_m(t) = \begin{cases} 1 & \text{if } \ s_0 + \dfrac{2}{m} < t < s - \dfrac{2}{m} \\ 0 & \text{if } \ t > s - \dfrac{1}{m} \quad \text{or} \quad t < s_0 + \dfrac{1}{m}. \end{cases}$$

We introduce a regularizing sequence $\rho_n \in \mathcal{D}(\mathbb{R})$, such that

$$\rho_n(t) = \rho_n(-t), \ \int_{-\infty}^{+\infty} \rho_n(t)\,dt = 1, \ \rho_n \text{ with support in } \left[-\frac{1}{n}, \frac{1}{n}\right].$$

By using $w = w(t) = ((\psi_m(t)v(t)) * \rho_n(t) * \rho_n(t))\psi_m(t)$, $n > 2m$, in (3.17), we obtain (see Lions [3], p. 214)

$$\begin{aligned}\int_0^T \langle v', w\rangle\,dt &= \int_0^T \langle \psi_m v', (\psi_m v) * \rho_n * \rho_n\rangle\,dt\\ &= \int_0^T \langle (\psi_m v)' * \rho_n, (\psi_m v) * \rho_n\rangle\,dt\\ &- \int_0^T (\psi_m' v, (\psi_m v) * \rho_n * \rho_n)\,dt\\ &= -\int_0^T (\psi_m' v, (\psi_m v) * \rho_n * \rho_n)\,dt,\end{aligned} \tag{3.18}$$

Here, we have used above that $w(t) \in V_s$ and the following result cf. Brezis [1], p. 128:

$$\int_0^T \langle (\psi_m u)' * \rho_n, (\psi_m u) * \rho_n\rangle\,dt = 0.$$

Since $\psi_m'(t)v(t) \in H$ and $(\psi_m(t)v(t) * \rho_n) \longrightarrow \psi_m(t)v(t)$ in H, as $n \longrightarrow \infty$, we obtain

$$\int_0^T (u', v)\, dt \longrightarrow \int_0^T \psi_m \psi_m' |v|^2 dt \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

Thus applying Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned} &\int_0^T b(v, v, w)\, dt = \int_0^T \psi_m^2 b(v, v, v * \rho_n * \rho_n)\, dt \\ &\longrightarrow \int_0^T \psi_m^2 b(v, v, v)\, dt = 0 \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\int_0^T a(t; v, w)\, dt = \int_0^T \psi_m^2 a(t; v, v * \rho * \rho)\, dt \\ &\longrightarrow \int_0^T \psi_m^2 a(t; v, v)\, dt \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\int_0^T c(t; v, w)\, dt = \int_0^T \psi_m^2 c(t; v, v * \rho * \rho)\, dt \\ &\longrightarrow \int_0^T \psi_m^2 c(t; v, v)\, dt \end{aligned} \tag{3.22}$$

$$\int_0^T \langle \chi, w\rangle\, dt = \int_0^T \psi_m^2 \langle \chi, v * \rho * \rho\rangle\, dt \longrightarrow \int_0^T \psi_m^2 \langle \chi, v\rangle\, dt \tag{3.23}$$

$$\int_0^T \langle g, w\rangle\, dt = \int_0^T \psi_m^2 \langle g, v * \rho * \rho\rangle\, dt \longrightarrow \int_0^T \psi_m^2 \langle g, v\rangle\, dt. \tag{3.24}$$

From (3.19) to (3.24) we obtain

$$\begin{aligned} &\int_0^T (-\psi_m \psi_m')|v|^2 dt + \nu_0 \int_0^T \psi_m^2 a(t; v, v)\, dt \\ &+\nu_1 \int_0^T \psi_m^2 \langle \chi, v\rangle\, dt + \int_0^T \psi_m^2 c(t; v, v)\, dt = \int_0^T \psi_m^2 \langle g, v\rangle\, dt. \end{aligned} \tag{3.25}$$

Since $\dfrac{d}{dt}(\psi_m^2(t)|v(t)|^2) = 2\psi_m(t)\psi_m'(t)|v(t)|^2 + \psi_m^2(t)\dfrac{d}{dt}|v(t)|^2$,

$$\begin{aligned} &\int_0^T -\psi_m(t)\psi_m'(t)|v(t)|^2 dt = -\frac{1}{2}\int_{s_0+\frac{2}{m}}^{s-\frac{2}{m}} \frac{d}{dt}(\psi_m^2(t)|v(t)|^2)\, dt \\ &+\frac{1}{2}\int_{s_0+\frac{2}{m}}^{s-\frac{2}{m}} \psi_m^2(t)\frac{d}{dt}|v(t)|^2 dt. \end{aligned} \tag{3.26}$$

Thus, $\displaystyle\int_0^T -\psi_m(t)\psi_m'(t)|v(t)|^2 dt \longrightarrow \frac{1}{2}|v(s)|^2 - \frac{1}{2}|v(s_o)|^2.$

Consequently for almost every s and s_0

$$\begin{aligned} \frac{1}{2}|v(s)|^2 &+ \nu_0 \int_{s_0}^s a(t;v,v)\,dt + \nu_1 \int_{s_0}^s \langle \chi, v\rangle\,dt + \\ &+ \int_{s_0}^s c(t;v,v)\,dt = \frac{1}{2}|v(s_0)|^2 + \int_{s_0}^s \langle g, v\rangle\,dt \end{aligned} \tag{3.27}$$

Since $v \in L^\infty(0,T;H)$, we can to find a sequence $s_{0n} \longrightarrow 0$ with $v(s_{0n})$ bounded in H and thus, $v(s_{0n}) \rightharpoonup \varphi$ in H weak. From (3.6) and (3.11) we conclude that $v \in C^0([0,T];V_s')$. This implies that $v(s_{0n}) \longrightarrow v(0) = v_0$ in V_s'. Therefore $v(s_{0n}) \rightharpoonup v_0$ weak in H, which implies that

$$|v_0|^2 \leq \liminf |v(s_{0n})|^2. \tag{3.28}$$

Let us consider $s_0 = s_{0n}$ and s fixed. Taking $\liminf$ in (3.27) and using (3.28), we obtain

$$\begin{aligned} \frac{1}{2}|v(s)|^2 &+ \nu_0 \int_0^s a(t;v,v)\,dt + \nu_1 \int_0^s \langle \chi, v\rangle\,dt + \\ &+ \int_0^s c(t;v,v)\,dt \geq \frac{1}{2}|v_0|^2 + \int_0^s \langle g, v\rangle\,dt. \end{aligned} \tag{3.29}$$

We denote by

$$\begin{aligned} \mathcal{Y}_\mu^s &= \nu_1 \int_0^s \langle \mathcal{A}v_\mu - \mathcal{A}\varphi, v_\mu - \varphi\rangle\,dt + \frac{1}{2}|v_\mu(s)|^2 + \\ &+ \nu_0 \int_0^s a(t;v_\mu(t),v_\mu(t))\,dt + \int_0^s c(t;v_\mu(t),v_\mu(t))\,dt, \end{aligned} \tag{3.30}$$

for all $\varphi \in L^p(0,T;V)$. From estimate (3.6), we can see that there exists a subsequence $(v_\mu)_{\mu\in\mathbb{N}}$ such that $v_\mu(s) \rightharpoonup v(s)$ weak in H and this imply that

$$|v(s)|^2 \leq \liminf |v_\mu(s)|^2. \tag{3.31}$$

On the other hand, it follows from Lemma 3.1 that

$$a_0 \int_0^s \|w(t)\|^2 dt \leq \int_0^s a(t;w(t),w(t))\,dt \leq a_1 \int_0^s \|w(t)\|^2 dt,$$

for all $w \in L^2(0,T;V)$. Therefore $\left(\int_0^s a(t;v(t),v(t))\,dt\right)^{\frac{1}{2}}$ is a norm equivalent to the norm $\|w\|_{L^2(0,T;V)}$ in $L^2(0,T;V)$.

Since $v_m \longrightarrow v$ weakly in $L^2(0,T;V)$, we obtain

$$\int_0^s a(t;v(t),v(t))\,dt \leq \liminf \int_0^s a(t;v_\mu(t),v_\mu(t))dt. \tag{3.32}$$

Moreover, we have by (3.7) that $\dfrac{\partial v_{\mu_i}}{\partial y_l} \rightharpoonup \dfrac{\partial v_i}{\partial y_l}$ weakly in $L^2(0,T;L^2(\Omega))$, $i,l = 1,2...,n$, and by (3.15) we conclude, $v_{\mu_i} \longrightarrow v_i$ strongly in $L^2(0,T;L^2(\Omega))$. These two last convergence imply

$$\int_0^s c(t;v_\mu(t),v_\mu(t))\,dt \longrightarrow \int_0^s c(t;v(t),v(t)). \tag{3.33}$$

Besides, since $\mathcal{A}$ is a monotone operator, we obtain

$$\int_0^s \langle \mathcal{A}(t)v_\mu(t) - \mathcal{A}(t)\varphi(t), v_\mu(t) - \varphi(t)\rangle\,dt \geq 0, \tag{3.34}$$

for all $\varphi \in L^p(0,T;V)$. Taking into account (3.31) to (3.34) into (3.30) yields

$$\begin{aligned}&\liminf \mathcal{Y}_\mu^s \geq \frac{1}{2}|v(s)|^2 + \nu_0 \int_0^s a(t;v(t),v(t))\,dt \\ &+ \int_0^s c(t;v(t);v(t))\,dt.\end{aligned} \tag{3.35}$$

The approximate equation $(3.2)_1$ give us

$$\begin{aligned}&\nu_1 \int_0^s \langle \mathcal{A}(t)v_\mu(t), v_\mu(t)\rangle\,dt = \int_0^s \langle g(t), v_\mu(t)\rangle\,dt \\ &-\nu_0 \int_0^s a(t;v_\mu(t),v_\mu(t))\,dt + \frac{1}{2}|v_\mu(0)|^2 - \frac{1}{2}|v_\mu(s)|^2 \\ &- \int_0^s c(t;v_\mu(t),v_\mu(t))\,dt.\end{aligned} \tag{3.36}$$

Observe that

$$\begin{aligned}&\mathcal{Y}_\mu^s = \nu_1 \int_0^s \langle \mathcal{A}(t)v_\mu(t), v_\mu(t)\rangle\,dt - \nu_1 \int_0^s \langle \mathcal{A}(t)v_\mu(t), \varphi(t)\rangle\,dt \\ &-\nu_1 \int_0^s \langle \mathcal{A}(t)\varphi(t), v_\mu(t) - \varphi(t)\rangle + \frac{1}{2}|u_\mu(s)|^2 \\ &+\nu_0 \int_0^s a(t;v_\mu(t),v_\mu(t))\,dt + \int_0^s c(t;v_\mu(t),v_\mu(t))\,dt,\end{aligned} \tag{3.37}$$

for all $\varphi \in L^p(0,T;V)$. Combining (3.36) with (3.37) yields

$$\begin{aligned}\mathcal{Y}_\mu^s = \int_0^s \langle g(t), v_\mu(t)\rangle\, dt + \frac{1}{2}|v_\mu(0)|^2 - \nu_1 \int_0^s \langle \mathcal{A}(t)v_\mu(t), \varphi(t)\rangle \\ -\nu_1 \int_0^s \langle \mathcal{A}(t)\varphi(t), v_\mu(t) - \varphi(t)\rangle\, dt \longrightarrow \mathcal{Y}^s,\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}\mathcal{Y}^s &= \int_0^s \langle g(t), v(t)\rangle\, dt + \frac{1}{2}|u_0|^2 - \nu_1 \int_0^s \langle \chi, \varphi(t)\rangle - \\ &- \nu_1 \int_0^s \langle \mathcal{A}\varphi(t), v(t) - \varphi(t)\rangle\, dt.\end{aligned} \tag{3.39}$$

Hence, by (3.35) and (3.38), we obtain

$$\begin{aligned}&\int_0^s \langle g(t), v(t)\rangle\, dt + \frac{1}{2}|v_0|^2 - \nu_1 \int_0^s \langle \chi, \varphi(t)\rangle \\ &-\nu_1 \int_0^s \langle \mathcal{A}\varphi(t), v(t) - \varphi(t)\rangle\, dt \geq \frac{1}{2}|v(s)|^2 \\ &+\nu_0 \int_0^s a(t; v(t), v(t))\, dt + \int_0^s c(t; v(t); v(t))\, dt.\end{aligned} \tag{3.40}$$

Finally, combining (3.29) with (3.40) yields

$$\nu_1 \int_0^s \langle \chi - \mathcal{A}\varphi(t), v(t) - \varphi(t)\rangle\, dt \geq 0, \tag{3.41}$$

a.e. s, for all $\varphi \in L^p(0,T;V)$. Setting $\varphi(t) = v(t) - \lambda w(t)$, with $\lambda > 0$ and $w \in L^p(0,T;V)$ arbitraries, into (3.41) yields

$$\int_0^s \big\langle \chi - \mathcal{A}(t)[v(t) - \lambda w(t)], w(t)\big\rangle\, dt \geq 0. \tag{3.42}$$

Now, from hemicontinuity of the operator $\mathcal{A}(t)$, we have

$$\int_0^s \langle \chi - \mathcal{A}(t)[v(t) - \lambda w(t)], w(t)\rangle\, dt \longrightarrow \int_0^s \langle \chi - \mathcal{A}(t)v(t), w(t)\rangle\, dt,$$

for all $w \in L^p(0,T;V)$. This convergence and (3.42) give

$$\int_0^s \langle \chi - \mathcal{A}(t)v(t), w(t)\rangle\, dt = 0, \;\; \forall \;\; w \in L^p(0,T;V),$$

and this implies that $\mathcal{A}(t)v = \chi$ in $L^{p'}(0, T; V')$.

Proof of uniqueness of solutions of Theorem 2.2. Analogue to the proof of Theorem 5.2, *p.* 217, of the reference [3] for the case of a cylindrical domain.

Proof of Theorem 2.3. We recall that $K(t) = k(t)M = (\alpha_{ij}(t))$, $K^{-1}(t) = \dfrac{1}{k(t)}M^{-1} = (\beta_{ij}(t))$, $x = K(t)y$, $y = K^{-1}(t)x$, $x_r = \sum_{j=1}^{n} \alpha_{rj}(t)y_j$, $y_l = \sum_{r=1}^{n} \beta_{lr}(t)x_r$. We establish that $u(x,t) = v(K^{-1}(t)x, t)$ and $u_0(x) = v_0(K^{-1}(0)x)$. We shall show that if u is a weak solution of Problem (1.1) then v is a weak solution of Problem (1.3) and reciprocally. Let $\xi(x,t)$ be in the conditions of definition of weak solutions of Problem (1.3).

Consider $\psi(y,t)$ defined by $\xi(x,t) = |\det K^{-1}(t)|\ \psi\big(K^{-1}(t)x, t\big)$. First we prove that

$$\begin{aligned} &-\int_0^T (u(t), \xi'(t))_{H(\Omega_t)}dt = -\int_0^T (v(t), \psi'(t))dt \\ &+\int_0^T c(t; v(t), \psi(t))dt. \end{aligned} \tag{3.43}$$

In fact, since $\dfrac{\partial y_l}{\partial t} = \sum_{j,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t)y_j$, we deduce that

$$\begin{aligned} &\frac{\partial \xi_i}{\partial t}(x,t) = |\det K^{-1}(t)| \left[\sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t)y_j \frac{\partial \psi_i}{\partial y_l}(y,t) \right. \\ &\left. + \frac{\partial \psi_i}{\partial t}(y,t) \right] + |\det K^{-1}(t)|' \psi_i(y,t). \end{aligned} \tag{3.44}$$

Since $\det K^{-1}(t) = \dfrac{1}{k(t)^n} \det M^{-1}$, we obtain

$$
\begin{aligned}
&|\det K^{-1}(t)|' = -n\frac{k'(t)}{k(t)^{n+1}}|\det M^{-1}| = \\
&-n\frac{k'(t)}{k(t)}|\det K^{-1}(t)|.
\end{aligned}
\tag{3.45}
$$

By substituting (3.45) in the second member of (3.44) and integrating on Ω_t, we find

$$
\begin{aligned}
&-\int_{\Omega_t} u_i(x,t)\frac{\partial \xi_i}{\partial t}(x,t)\,dx \\
&= -\left\{\int_\Omega \left[\sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t)y_j v_i(y,t) \quad \times \frac{\partial \psi_i}{\partial y_l}(y,t)\,dy\right]\right\} \\
&-\int_\Omega v_i(y,t)\frac{\partial \psi_i}{\partial t}(y,t)\,dy + \int_\Omega n\frac{k'(t)}{k(t)}v_i(y,t)\psi_i(y,t)\,dy.
\end{aligned}
\tag{3.46}
$$

We observe that

$$
\begin{aligned}
&\sum_{l,r=1}^{n} \beta'_{lr}(t)\alpha_{rl}(t) = tr\left[(K^{-1}(t))'K(t)\right] \\
&= tr\left(-\frac{k'(t)}{k(t)}I\right) = -n\frac{k'(t)}{k(t)},
\end{aligned}
$$

where tr denotes the trace of the $n \times n$ matrix N. Applying in (3.46) the Green theorem, we obtain

$$
\begin{aligned}
&-\int_\Omega \sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t)y_j v_i(y,t)\frac{\partial \psi_i}{\partial y_l}(y,t)\,dy \\
&= \int_\Omega \sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t)y_j\frac{\partial v_i}{\partial y_l}(y,t)(y,t)\psi_i(y,t)\,dy \\
&-\int_\Omega n\frac{k'(t)}{k(t)}v_i(y,t)\psi_i(y,t)\,dy.
\end{aligned}
\tag{3.47}
$$

Combining (3.47) and (3.46) and cancellating similar terms with opposite sign, we find

$$-\int_{\Omega_t} u_i(x,t)\frac{\partial \xi_i}{\partial t}(x,t)\,dx = -\int_{\Omega} v_i(y,t)\frac{\partial \psi_i}{\partial t}(y,t)\,dy +$$

$$+\int_{\Omega}\sum_{j,l,r=1}^{n}\beta'_{lr}(t)\alpha_{rj}y_j\frac{\partial v_i}{\partial y_l}(y,t)\psi_i(y,t)\,dy.$$

By adding both sides of this expression from $i=1$ to $i=n$, integrating on $[0,T]$ and recalling the definition of $c(t;v,w)$ given in (1.12), we obtain the required equality (3.43). We have also

$$\sum_{i,j=1}^{n}\int_{\Omega_t}\frac{\partial u_i}{\partial x_j}(x,t)\frac{\partial \xi_i}{\partial x_j}(x,t)\,dx = \sum_{i,l,r=1}^{n}\int_{\Omega} a_{lr}(t)\frac{\partial v_i}{\partial y_l}(y,t)\frac{\partial \psi_i}{\partial y_r}(y,t)\,dy.$$

Integrating both sides of this expression from 0 to T implies

$$\int_0^T \widehat{a}(t;u(t),\xi(t))\,dt = \int_0^T a(t;v(t),\psi(t))\,dt. \tag{3.48}$$

Similarly, we obtain

$$\int_0^T \langle\widehat{\mathcal{A}}(t)u(t),\xi(t)\rangle_{V'(\Omega_t)\times V(\Omega_t)}dt = \int_0^T \langle\mathcal{A}(t)v(t),\psi(t)\rangle\,dt \tag{3.49}$$

and

$$\int_0^T \widehat{b}(t;u(t),u(t),\xi(t))\,dt = \int_0^T b(t;v(t),v(t),\psi(t))\,dt. \tag{3.50}$$

By the results of Appendix 4, we deduce that

$$\int_0^T \langle f(t),\xi(t)\rangle_{V'(\Omega_t)\times V(\Omega_t)}dt = \int_0^T \langle g(t),\psi(t)\rangle\,dt, \tag{3.51}$$

where ξ and ψ verify respectively, the conditions of (2.1) and (2.2). Clearly $u(x,0)=u_0(x)$ implies $v(y,0)=v_0(y)$ and reciprocally. Results (3.43) and (3.48) to (3.51) permit us to prove the Theorem 2.3.

4 Appendix

Obtention of Problem (1.3). We follow the notation of the proof of Theorem 2.3. In particular $x_r = \sum_{j=1}^{n} \alpha_{rj}(t) y_j$ and $y_l = \sum_{r=1}^{n} \beta_{lr}(t) x_r$. We have $\dfrac{\partial y_l}{\partial t} = \sum_{r,j=1}^{n} \beta' lr(t) \alpha_{rj}(t) y_j$, $\dfrac{\partial y_l}{\partial x_j} = \beta_{lj}(t)$ and therefore

$$\frac{\partial u_i}{\partial t}(x,t) = \sum_{j,l,r=1}^{n} \beta'_{lr}(t)\alpha_{rj}(t) y_j \frac{\partial v_i}{\partial y_l}(y,t) + \frac{\partial v_i}{\partial t}(y,t). \tag{4.1}$$

Since $\dfrac{\partial y_l}{\partial x_j} = \beta_{lj}(t)$,

$$\frac{\partial u_i}{\partial x_j}(x,t) = \sum_{l=1}^{n} \beta_{lj}(t) \frac{\partial v_i}{\partial y_l}(y,t). \tag{4.2}$$

Hence, $\dfrac{\partial^2 u_i}{\partial x_j^2}(x,t) = \sum_{l,r=1}^{n} \beta_{lj}(t)\beta_{rj}(t) \dfrac{\partial^2 v_i}{\partial y_l \partial y_r}(y,t)$. Whence

$$\Delta u_i(x,t) = \sum_{j,l,r=1}^{n} a_{lr}(t) \frac{\partial^2 v_i}{\partial y_l \partial y_r}(y,t), \tag{4.3}$$

where $a_{lr}(t)$ is defined in (1.4). By (4.2) we deduce that

$$\|u(t)\|^2_{V(\Omega_t)} = \sum_{i,j=1}^{n} \int_{\Omega} \left[\sum_{l=1}^{n} \beta_{lj}(t) \frac{\partial v_i}{\partial y_l}(y,t) \right]^2 |\det K(t)| dy. \tag{4.4}$$

From (4.2) it follows that

$$\begin{aligned} &u_i(x,t)\frac{\partial u}{\partial x_i}(x,t) \\ &= \left(v_i(y,t) \sum_{l=1}^{n} \beta_{li}(t) \frac{\partial v_1}{\partial y_l}(y,t), ..., v_i(y,t) \sum_{l=1}^{n} \beta_{li}(t) \frac{\partial v_n}{\partial y_l}(y,t) \right), \end{aligned}$$

that is,

$$u_i(x,t)\frac{\partial u}{\partial x_i}(x,t) = \sum_{l=1}^{n} \beta_{li}(t) v_i(y,t) \frac{\partial v}{\partial y_l}(y,t). \tag{4.5}$$

We have that $\frac{\partial p}{\partial x_i}(x,t) = \sum_{l=1}^{n} \beta_{li}(t)\frac{\partial q}{\partial y_l}(y,t) = \left(\nabla q(y,t)K^{-1}(t)\right)_i$. Thus

$$\nabla p(x,t) = \nabla q(y,t)K^{-1}(t). \tag{4.6}$$

By (4.1), (4.3) to (4.6) we obtain that the first equations of (1.1) and (1.3) are equivalents. On the other side, expression (4.2) gives $div\ u(x,t) = \sum_{i,l=1}^{n} \beta_{li}(t)\frac{\partial v_i}{\partial y_l}(y,t)$. We know that $\beta_{ij}(t) = \frac{1}{k(t)}n_{ij}$ where $M^{-1} = (n_{ij})$. Therefore $div\, u(x,t) = \frac{1}{k(t)}div\ \left(M^{-1}v^T(y,t)\right)$. This shows that the second equations of (1.1) and (1.3) are equivalents. The other two conditions of there problem are clearly equivalents.

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Departamento de Matemática-CCEN
Universidade Federal do Pará
66.075-110 - Belém, Pará, Brazil
E-mail: gera@ufpa.br silvano@ufpa.br