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THE BILLIARD ON AN ELLIPSOID AS AN INTEGRABLE SYSTEM

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Abstract

We show that the diffeomorphism induced by the billiard map on an ellipsoid is the time one map of an evolution equation with remarkable algebraic properties. The description of the evolution contains a non-local operator similar to a Hilbert transform and the associated flow can be explicitly written by making use of the solution of a singular Riemann-Hilbert problem.

The Korteweg-de Vries equation is the archetypical example of an evolution equation for which an underlying algebraic structure gives rise to conserved quantities ([L]). These quantities in turn provide a priori estimates which were employed in the proof of (long time) existence and uniqueness of the Cauchy problem for initial conditions in a variety of spaces ([BS]). Later, researchers in PDE's became interested in the analogous problems for evolution equations which are not necessarily described by differential operators, as in the case of the Benjamin-Ono equation ([I]). There are new evolution equations of a similar nature: some well known diffeomorphisms in numerical analysis, like the QR step in the computation of eigenvalues of symmetric matrices ([P]), have been shown to be the time one maps of evolution equations with an equally rich algebraic structure ([S]),[DNT]), but in this case, usually, the natural phase space is a subset of the set of $n \times n$ matrices (see however [DLT] for a similar evolution in the space of bounded operators in $\ell_2(N)$).

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In this short note, we show that the diffeomorphism induced by the billiard map in the n-dimensional ellipsoid (to be described in detail below) is (essentially) the time one map of an evolution equation in an infinite dimensional functional space with remarkable algebraic properties. The description of the evolution equation contains a non-local operator similar to a Hilbert transform and the associated flow can be explicitly written by making use of the solution of a singular Riemann-Hilbert problem (corresponding to a Bruhat-type group factorization). Details and similar evolutions associated in particular to Heisenberg models with classical spin can be found in [MV] and [DLT2].

Consider the elliptical region

$$E = \{x : (x, C^2 x) \leq 1\}$$

in R^2, where C is positive diagonal. If a ball strikes ∂E at a point x_0 from a direction y_0, $\| y_0 \| = 1$, then, after reflection, the ball strikes ∂E at a second point x_1 from a direction y_1.

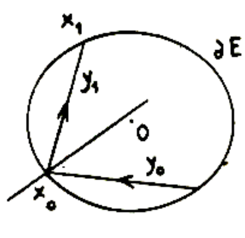

Figure 1.

The billiard map is the map $\psi_b : Y_+ \to Y_+$, where $\{(x, y) | x \in \partial E, \| y \| = 1, (x, Cy) > 0\}$, taking (x_0, y_0) to (x_1, y_1). It is a remarkable discovery by Moser and Veselov ([MV],[V]) that ψ_b can be described (up to signs, see below) as follows. Define the matrix polynomial

$$L_0(\lambda) = y_0 \otimes y_0 + \lambda x_0 \wedge y_0 - \lambda^2 C^2, (x_0, y_0) \in Y$$

where $x \otimes y = xy^T$ and $x \wedge y = xy^T - yx^T$, factor,

$$L_0(\lambda) = (\lambda C + y_0 \otimes C^{-1}x_0)(-\lambda C + C^{-1}x_0 \otimes y_0),$$

(note that $\| C^{-1}x_0 \| = 1$) exchange factors,

$$L_0'(\lambda) = (-\lambda C + C^{-1}x_0 \otimes y_0)(\lambda C + y_0 \otimes C^{-1}x_0)$$

and factor again,

$$L_1(\lambda) = (\lambda C + y_0' \otimes C^{-1}x_0')(-\lambda C + C^{-1}x_0' \otimes y_0')$$

with $y_0' = C^{-1}x_0$ and $\| C^{-1}x_0' \| = 1$. This construction defines a mapping

$$\phi : Y_+ \to Y_+$$

$$(x_0, y_0) \to (x_0', y_0')$$

with the property that $\phi^2 = -\psi_b$.

Consider now the function

$$A_0 : i\mathbf{R} \to Mat(\mathbf{C}, n)$$

$$\lambda \to \frac{L_0(\lambda)}{1 - \lambda^2}$$

so that $A_0(\lambda) > 0$ for $\lambda \neq 0$, and look for a factorization

$$e^{t\ \log A_0(\lambda)} = g_+(t, \lambda) g_-(t, \lambda)$$

where the matrix functions g_+ and g_- are determined by the properties below.

(i) $g_\pm(t, \lambda)$ have invertible analytic extensions to $Re\lambda > 0$, $Re\lambda < 0$ respectively,

(ii) $g_\pm(t, \lambda)$ are continuous and bounded in $\{Re\lambda \geq 0\} \setminus 0$, $\{Re\lambda \leq 0\} \setminus 0$, respectively.

(iii) as $\lambda \to 0$ in $Re\lambda > 0$, $Re\lambda < 0$ respectively,

$$0 < \eta \leq \left| \frac{det g_\pm(t, \lambda)}{\lambda^{(n-1)t}} \right| \leq \eta^{-1} < \infty,$$

for some constant independent of t,

(iv) $g_\pm(t,\lambda) \to C^t$ as $\lambda \to \infty$.

Theorem: *The factorization above exists and is unique. Moreover,*

(v) $g_-(t,\lambda) = g_+^*(t,\lambda), \lambda \in iR \setminus 0$.

The expression

$$A(t,\lambda) = g_+(t,\lambda)^{-1} A_0(\lambda) g_+(t,\lambda)$$

is always of the form

$$A(t,\lambda) = \frac{y(t) \otimes y(t) + \lambda x(t) \wedge y(t) - \lambda^2 C^2}{1-\lambda^2}$$

with $(x(t), y(t)) \in Y_+$ *and solves the evolution equation*

$$\frac{d}{dt} A(t,\lambda) = [(\pi_- log A(t,.))(\lambda), A(t,\lambda)]$$

$$A(0,\lambda) = A_0(\lambda),$$

where

$$\pi_- X(\lambda) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \lim_{r\to\infty} \int_{-ir}^{ir} \frac{X(\lambda')d\lambda'}{\lambda' - (\lambda - \epsilon)}.$$

Finally, the induced flow

$$(x_0, y_0) \to (x(t), y(t)) \in Y^+$$

interpolates the mapping ϕ *at integer times, i.e.,*

$$(x(1), y(1)) = \phi(x_0, y_0).$$

The proof of the theorem begins by showing by standard techniques that for the perturbed loop $A_0^\delta(\lambda) = A_0(\lambda) + \delta \cdot I, \delta > 0, \lambda \in iR$, consisting of strictly positive matrices, there is a standard (unique) factorization $exp(t\ log A_0^\delta(\lambda)) = g_+^\delta(t,\lambda) g_-^\delta(t,\lambda)$ with the usual requirements of analicity on both half-spaces (the analogous to properties (i) and (ii) with additional continuity at 0) and prescribed asymptotic behavior (the analogous to property (iv)). A rather delicate limiting procedure obtains the factorization in the case $\delta = 0$. The fact that the evolution defined through the factorization is the solution of the differential equation above again follows by standard techniques in (loop) group factorizations and a limiting procedure. Finally, the interpolation claim is a consequence of the explicit solution of the differential equation for $t = 1$, combined with the Moser-Veselov description of ϕ by (polynomial) factorization of $A(\lambda)$.

There is more than one way to introduce a sympletic structure for which the maps ϕ and ψ_b are canonical (and the evolution equation is completely integrable).

Theorem: *Let ω_+ be the (non-degenerate) restriction of the standard two-form in $\mathbf{R}^{2n}, \omega = \sum_{i=1}^n dx_i \wedge dy_i$, to the submanifold $Y_+ \subset \mathbf{R}^{2n}$. Consider the Hamiltonian*

$$H(x,y) = \frac{1}{4\pi i}\int_{C_R}[tr(\hat{A}(\lambda)log\hat{A}(\lambda) - \hat{A}(\lambda)) + 1](\frac{1-\lambda^2}{\lambda^2})d\lambda,$$

where

$$\hat{A}(\lambda) = \frac{\hat{y}\otimes\hat{y} + \lambda\hat{x}\wedge\hat{y} - \lambda^2 C^2}{1-\lambda^2}, \hat{x} = \frac{x}{\| C^{-1}x \|}, \hat{y} = \frac{y}{\| y \|}$$

and C_R is the contour in Figure 2, for any sufficiently large R.

The flow generated by H on (Y_+, ω_+) coincides with the induced interpolating flow for ϕ. Moreover, the Hamiltonians

$$I_j = \frac{1}{2\pi i}\int_{C_R}(trA(\lambda)^j - trA(0)^j)\frac{(1-\lambda^2)d\lambda}{\lambda^2}, 2 \le j \le n$$

are $n-1$ commuting integrals of H in Y_+, so the flow generated by H on Y_+ is completely integrable.

The proof of the theorem is a calculation which obtains the evolutions of $x(t)$ and $y(t)$ under the flows induced by H and the $I_j's$ (the lengthy details can be found in [DLT2]). Note that the phase space Y_+ is finite dimensional, but the rather explicit solution of the H-flow requires imbedding Y_+ into an infinite-dimensional space of functions $\{B(\lambda)\}$. The integrals I_j are equivalent to the eigenvalues of the matrix polynomial $L(\lambda) = y \otimes y + \lambda x \wedge y - \lambda^2 C^2$. Those integrals were used in [MV] to show the integrability of the map ϕ for an (apparently) different sympletic structure. By the way, when $n = 2$, the form ω_+ coincides with the 2-form ω_B used by Birkhoff ([B]) in his study of the billiard map (which happens to be canonical with respect to ω_B).

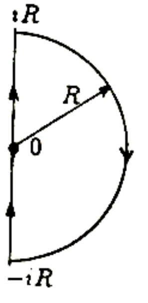

Figure 2

The interpolating flow has another sympletic interpretation. For adequate choices of (loop) group G and R-matrix structure, together with a representation of the dual of the R-Lie algebra, the (finite dimensional) coadjoint orbit through $A_0(\lambda)$ admits a Lie-Poisson symplectic structure for which the interpolating flow is canonical and belongs to a large class of commuting Hamiltonian flows that can be solved by group factorization techniques. The differential equation and its explicit solution then become an example of the formalism relating group factorizations and R-matrix structures ([STS],[DLT2]).

References

[B] G.D. Birkhoff, *Dynamical Systems*, Colloquium Publications, vol.9, AMS, Providence, RI, 1927 (revised edition, 1966).

[BS] J. Bona, R. Scott, *Solutions of the Korteweg-de Vries equation in fractional order Sobolev spaces*, Duke Math.J., 43 (1976), 87-99.

[DLT1] P.Deift, L.C.Li, C.Tomei, *Toda flows with infinitely many variables*, J. of Fun. Anal., 64 (1985), 358-402.

[DLT2] P.Deift, L.C.Li, C.Tomei, *Loop groups, discrete versions of some classical integrable systems and rank 2 extensions*, preprint, NYU, 1990, to appear Memoirs AMS.

[DNT] P.Deift, T.Nanda, C.Tomei, *Differential equations for the symmetric eigenvalue problem*, SIAM J. of Num. Anal., 20 (1982), 1-22.

[I] R.J.Iorio, Jr., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. PDE's, 11 (1986) 1031-1081.

[K] T.Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equations*, Studies in Applied Mathematics, vol. 8, Academic Press, 1983, 93-128.

[L] P.D.Lax, *Integrals of non linear equations of evolution and solitary waves*, CPAM, 21 (1968), 467-490.

[MV] J.Moser, A.P.Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, preprint, ETH, Zurich, 1989.

[P] B.Parlett, *The symmetric eigenvalue problem*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[S] W.W.Symes, *The QR algorithm and scattering for finite nonperiodic Toda lattice*, Physica 4D (1982), 275-280.

[STS] M.A. Semenov-Tian-Shanskii, *What is a classical R-matrix?* Func. Anal. Priloz., 17 (1983), 17-33 (Russian), English translation, Fun. Anal. Appl., 17 (1984), 259-272.

[V] A.P. Veselov, *Integrable systems with discrete time and difference operations*, Fun. Anal. Appl., 22 (1988), 1-13(Russian).

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