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A MINIMAL GENERATOR OF $\pi_6(G_2)$

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There is a simple theorem of Elie Cartan [C-E] stating that for any symmetric pair of compact Lie groups G,H, there is a canonical embedding of G/H as a totally geodesic submanifold of G. This inclusion does not constitute a section of the homogeneous bundle in general.

In this talk I will describe an example of a *non*-symmetric pair $(G_2, SU(3))$ with quotient S^6, that is included as a *minimal* submanifold in G_2, the automorphism group of the Cayley algebra. The bundle

$$SU(3) \quad \cdots \quad G_2 \to S^6$$

is the reduction of the bundle of orthonormal frames of the six-sphere and the killing form of G_2 induces a symmetric metric on S^6, by riemannian submersion. Details of what follows will appear in [C-R].

Consider the seven-sphere of unit Cayley numbers and the map it induces on $\mathbb{R}^8$ by Cayley conjugation: Each α in S^7 goes to σ_α in $SO(7)$ with $\sigma_\alpha(x) = \alpha x \overline{\alpha}$. This map $\alpha \longmapsto \sigma_\alpha$ generates $\pi_7 SO(7) \cong \mathbb{Z}$ and it gives rise through the property of *triality* to the generator of π_7 Spin (7), that sends α in S^7 to τ_α in Spin (7) with $\tau_\alpha(x) = \alpha x \alpha^2$ [T-Y-S].

An element of S^7, different from 1 or -1, can be written uniquely as $\alpha = \cos\theta + \sin\theta\, J$, where $J^2 = -1$ and $0 \leq \theta \leq \pi$, i.e., J lives in the equator S^6 consisting of all complex structures in S^7.

It can be easily seen that the intersection $\sigma_\alpha(S^7) \cap G_2$ consists of the parallel circle $\theta = \frac{2\pi}{3}$ plus the unit element. For such α in S^7 we have $\alpha^3 = 1$ and it turns out that the map from S^6 to G_2 that sends the complex structure J to $\cos(\frac{2\pi}{3}) + \sin(\frac{2\pi}{3})\, J$ is a minimal embedding [H

- L] that generates $\pi_6(G_2)$, which is therefore $\mathbb{Z}_3$. Moreover, the image of this parallel circle is a singular orbit of the conjugate action of G_2 on itself

$$G_2 \times G_2 \longrightarrow G_2$$
$$(A \ , \ B) \longmapsto ABA^{-1}$$

The other kind of non trivial singular orbit of this action is the symmetric space $G_2/_{SO(4)}$ of dimension 8.

Both these cojugate orbits are exponential images of non-trivial, singular, Adjoint orbits of G_2 on its lie algebra $\widehat{G}_2$. The two kinds of Adjoint orbits now are ten dimensional and moreover they have the same cell structure as follows from a theorem of R. Bott [B]. They are, however, *not* homotopy equivalent: Each is the quotient of G_2 by a *different* subgroup isomorphic to $U(2)$.

These two Adjoint orbits differ in their *third* homotopy group, which is zero for one and $\mathbb{Z}_3$ (again) for the other. Starting from the fact that $\pi_6 G_2 \cong \mathbb{Z}_3$ one can show that $\pi_6 SU(3) \cong \mathbb{Z}_6$ and $\pi_6 SU(2) \cong \mathbb{Z}_{12}$, using the system of principal bundles over the seven sphere. Traditionally one uses the opposite route [M].

One of the reasons for looking for algebraically explicit expressions representing generators of homotopy groups of the spaces involved in the classical bundles over the seven-sphere is related to the following problem $[R_1]$; $[R_2]$

Classify the differentiable structures on the seven - sphere by the (exotic) free actions of $SU(2)$ on $S^7 \times SU(2)$ or of $SO(7)$ on $SO(8)$.

In our case one is looking for an expression for the generator of $\pi_7 Sp(2) \cong \mathbb{Z}$. In this line, a generator of $\pi_7 SU(4)$ can be written by using the Bott periodicity theorem and successive projections $[R_3]$,[D-F-N],[H]. This procedure however does not put in evidence the relations between the various geometries involved in the problem.

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