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SOME VARIANTS OF FLOER COHOMOLOGY

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I would like to talk about a joint work with K. Fukaya and K. Ohta on some variants of the Floer cohomology and its application to almost definite 4-manifolds with boundary.

Let X be an oriented closed (smooth) 4-manifold. We write

$$Q : H_2(X,Z) \times H_2(X,Z) \to Z$$

for the intersection form of X. Let b^+ be the rank of a maximal positive subspace of $H_2(X,Z)$. S. K. Donaldson proved the following theorem by using gauge theory.

Theorem 1 (Donaldson's Theorem A) *Suppose X is closed and $b^+ = 0$. We assume X is simply connected for simplicity. Then for any classes α_1 and α_2 of $H_2(X,Z)$,*

$$Q(\alpha_1,\alpha_2) + \frac{1}{2}\sum_{Q(e,e)=-1} Q(\alpha_1,e)Q(\alpha_2,e) = 0.$$

This equality and an elementary algebric consideration imply

$$Q \cong (-1) \oplus (-1) \oplus \cdots \oplus (-1)$$

If X is not closed but its boundary is a homology 3-sphere, then a generalization of theorem A is know as "folk theorem". Let Y be an oriented homology 3-sphere and $I^*(Y)$ be the Floer cohomology groups of Y. The period of $I^*(Y)$ is $8(I^{*+8}(Y) \cong I^*(Y))$. There is a homomorphism $D : I^{-1}(Y) \longrightarrow Z$ which satisfies

Theorem 2 (Folk Theorem) *Suppose Y is the boundary of a simply conected (for simplicity) oriented 4-manifold X. Let Q be the intersection form of X. If $b^+ = 0$, then for any $\alpha_1, \alpha_2 \in H_2(X, Z)$, there is a class $q(X, \alpha_1, \alpha_2)$ of $I^{-1}(Y)$ such that*

$$Q(\alpha_1, \alpha_2) + \frac{1}{2} \sum_{Q(e,e)=-1} Q(\alpha_1, e)Q(\alpha_2, e) = Dq(X, \alpha_1, \alpha_2).$$

Corollary 3 *Suppose X is as above. If Q is not isomorphic to the standard negative definite form, then $I^{-1}(Y) \neq 0$.*

Remark. If $D : I^{-1}(Y) \longrightarrow Z$ is an isomorphism, then the above formula gives $q(X, \alpha_1, \alpha_2)$ itself. This is a "Donaldson's polynomial invariant with values in the floer cohomology" or " relative polynomial invariant". An example of such a case is given by the Poincaré homology 3-sphere wich is a boundary of a simply connecterd manifold with $Q \cong (-E_8)$. We write $(-E_8)$ for this manifold. Since $I^{\cdot}(S^3) = 0$, one gets Donaldson's Theorem A from the Folk Theorem by deleting a small ball from the given closed negative 4-manifold.

For manifolds with $b^1 = 1$ or 2, Donaldson proved:

Theorem 4 (Donaldson's Theorem B) *Suppose X is a closed oriented simply connected (for simplicity) spin 4-manifold with $b^+ = 1$. Then for any $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4 \in H_2(X, Z)$*

$$Q(\alpha_1, \alpha_2)Q(\alpha_3, \alpha_4) + Q(\alpha_1, \alpha_3)Q(\alpha_2, \alpha_4) + Q(\alpha_1, \alpha_4)Q(\alpha_2, \alpha_3) \equiv 0 \bmod 2$$

The above equality and an elementary algebraic consideration imply that the rank of Q is 2 and

$$Q \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 5 (Donaldson's Theorem C) *Suppose X is a closed oriented simply connected (for simplicity) spin 4-manifold with $b^+ = 2$. Then for any*

$\alpha_1 \cdots, \alpha_6 \in H_2(X, Z)$

$$Q(\alpha_1,\alpha_2)Q(\alpha_3,\alpha_4)Q(\alpha_5,\alpha_6) + (\text{similar terms}) \equiv 0 \bmod 2$$

In this case Q has rank 4 and

$$Q \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To extend these theorems to manifolds with boundary, we need some variants of the Floer cohomology:there exists Z_2-vextor spaces $I^*_{(i)}(Y)$ and $I^*_{(k,1)}(Y)(* \in Z, i \in Z_2, k, 1 \in Z_4, k-1 \equiv 0 \bmod 2)$ and homomorphisms $D: I^{-2}_{(0)}(Y) \longrightarrow Z_2$ and $D: I^{-3}_{(k,0)}(Y) \longrightarrow Z_2 (k = 1, 3 \in Z_4)$ which satisfy:

Theorem 6 *Suppose an oriented homology 3-sphere Y is the boundary of a simply connected (for simplicity) oriented spin 4-manifold X. If $b^+ = 1$, then for $\alpha_1, \cdots, \alpha_4 \in H_2(X, Z)$, there is $q(X, \alpha_1, \cdots, \alpha_4) \in I^{-2}_{(0)}(Y)$ such that*

$$Q(\alpha_1,\alpha_2)Q(\alpha_3,\alpha_4) + (\text{similar terms}) \equiv Dq(X,\alpha_1,\cdots,\alpha_4) \bmod 2.$$

If $b^+ = 2$, then for $\alpha_1, \cdots, \alpha_6 \in H_2(X, Z)$, there is $q(X, \alpha_1, \cdots, \alpha_6) \in I^{-3}_{(k,0)}(Y)(k = -2\ (sign(X)/8) - 3)$ such that

$$Q(\alpha_1,\alpha_2)Q(\alpha_3,\alpha_4)Q(\alpha_5,\alpha_6) + (\text{similar terms}) \equiv Dq(X,\alpha_1,\cdots,\alpha_6) \bmod 2.$$

There is an exact sequence (a spectral sequence) which gives relations between $I^*_{(i)}(Y)((I^*_{(k,1)}))$ and the Z_2-coefficient Floer cohomology $I^*(Y, Z_2)$ (see below). In particular, in the situation of the above theorem,

Corollary 7: If $b^+ = 1$ *and rank* $(Q) \neq 2$, *then* $I^*(Y, Z_2) \neq 0$ *for* $i = -2$ *or* -1 *and if* $b^* = 2$ *and rank*$(Q) \neq 4$, *then* $I^*(Y, Z_2) \neq 0$ *for* $i = -3, -2$ *or* -1.

Remark. If D is an isomorphism, then the above formula gives the relative invariant itself. Such an example is given by $(-E_8)\sharp m(S^2 \times S^2)(m = 1, 2)$ whose boundary is the Poincaré homology 3-sphere. See example below.

In what follows we give some properties of the variants of Floer cohomology . Let Y be an oriented homology 3-sphere and $I^*(Y, Z_2)$ the Z_2-coefficient Floer cohomology of Y. There is an isomorphism $\beta : I^*(Y, Z_2) \longrightarrow I^{*+8}(Y, Z_2)$.

Properties of $I^*_{(i)}$:

(1) As invariants of Y, we have Z_2-vector spaces $I^*_{(i)}(* \in Z)$ and also a homomorphism

$$u_{(i)} : I^*(Y, Z_2) \longrightarrow I^{*+2}(Y, Z_2)$$

for $i \in Z_2$.

(2) There is a natural homomorphism

$$I^*_{(i)}(Y) \longrightarrow I^{*+8}_{(i+1)}(Y)$$

for $i \in Z_2$). Hence $I^*_{(i)}(Y)$ has period 16. We also have the relation

$$\beta u_{(i)} \beta^{-1} = u_{(i+1)} : I^*(Y, Z_2) \longrightarrow I^{*+2}(Y, Z_2).$$

(3) There is a long exact sequence:

$$\longrightarrow I^{*-2}(Y, Z_2) \xrightarrow{u_{(i)}} I^*(Y, Z_2) \longrightarrow I^{*-1}_{(i)}(Y) \longrightarrow$$

for $i \in Z_2$.

(4) There is a natural homomorphism $D : I^{-2}_{(0)}(Y) \longrightarrow Z_2$

(5) $$u_{(0)} u_{(1)} = u_{(1)} u_{(0)} = 0 : I^*(Y, Z_2) \longrightarrow I^{*+4}(Y, Z_2).$$

Properties of $I^*_{(k,l)}(Y)$:

(1) As invariants of Y, we have Z_2-vector spaces $I^*_{(k,1)}(* \in Z)$ for $k, 1 \in Z_4$ with $k - l \equiv 1 \bmod 2$.

(2) there is a natural homomorphism

$$I^*_{(k,l)}(Y) \longrightarrow I^{*+8}_{(k+1,l+1)}(Y).$$

Hence $I^*_{(k,l)}(Y)$ has period 32.

(3) There is a spectral sequence convergent to $I^*_{(k,l)}(Y)$ with

$$(E_2, d_2) = \left(I^*(Y, Z_2) \oplus I^*(Y, Z_2) \oplus I^*(Y, Z_2), \begin{pmatrix} 0 & 0 & 0 \\ u_{(k)} & 0 & 0 \\ 0 & u_{(l)} & 0 \end{pmatrix}\right).$$

(4) There is a natural homomorphism $D : I^{-3}_{(0,l)}(Y) \longrightarrow Z_2 (l = 1,3)$.

Examples:

(i) Poincarè homology 3-sphere $\sum(2,3,5)$.

The Floer cohomology of $Y = \sum(2,3,5)$ is calculated by Fintushel and Stern.

$$I^*(Y) = \begin{cases} Z & \text{if} * \equiv -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

The universal coefficient theorem implies that

$$I^*(Y, Z_2) = \begin{cases} Z_2 & \text{if} * \equiv -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Then all the maps in the exact sequence and the spectral sequence above are trivial and we have

$$I^*_{(i)}(Y) = \begin{cases} Z_2 & \text{if} * \equiv -2, -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$I^*_{(k,l)}(Y) = \begin{cases} Z_2 & \text{if} * \equiv -3, 2, -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Brieskorn homology 3-sphere $\sum(2,3,5,7)$.

A calculation shows that $u_{(0)}$ and $u_{(1)}$ are non-trivial for $\sum(2,3,5,7)$.

As an application, Fintushel and Stern showed

Theorem 8 (Fintushel-Stern) *If a homotopy K3-surface has an embedded $\sum(2,3,7)$, then its Donaldson invariant is non-trivial.*

The homology 3-sphere $Y = \sum(2,3,7)$ satisfies

$$I^{-3}(Y, Z_2) = I^{-2}(Y, Z_2) = I^0(Y, Z_2) = 0, I^{-1}(Y, Z_2) = Z_2.$$

By using the variants of the Floer homologies and the spectral sequence, one can show that this condition is enough to get their result.

Theorem 9 *If a homotopy K3-surface has an embedded homology 3-sphere whose Z_2-coefficient Floer cohomology satisfies the above condition, then its*

Donaldson invariant is non-trivial.

I would like to observe that the long exact sequence is an analogue of a Thom-Gysin exact sequence in a reasonable sense.

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