HARMONIC MAPS AND SOLITON THEORY

Francis E. Burstall

1 Introduction

The study of harmonic maps of a Riemann sphere into a Lie group or, more generally, a symmetric space has been the focus for intense research by a number of Differential Geometers and Theoretical Physicists. As a result, these maps are now quite well understood and are seen to correspond to holomorphic maps into some (perhaps infinite-dimensional) complex manifold. For more information on this circle of ideas, the Reader is referred to the articles of Guest and Valli in these proceedings.

In this article, I wish to report on recent progress that has been made in understanding harmonic maps of 2-tori in symmetric spaces. This work has its origins in the results of Hitchin [9] on harmonic tori in S^3 and those of Pinkall-Sterling [13] on constant mean curvature tori in $\mathbf{R}^3$ (equivalently, harmonic tori in S^2). In those papers, the problem again reduces to one in Complex Analysis or, more properly, Algebraic Geometry but of a rather different kind: here the fundamental object is an algebraic curve, a *spectral curve*, and the equations of motion reduce to a linear flow on the Jacobian of this curve. This is a familiar situation in the theory of completely integrable Hamiltonian systems (see, for example [1,14]) and the link with Hamiltonian systems was made explicit in the study of Ferus-Pedit-Pinkall-Sterling [7] who showed that the non-superminimal minimal 2-tori in S^4 arise from solving a pair of commuting Hamiltonian flows in a suitable loop algebra. It turns out that a similar programme can be carried out for a large class of harmonic tori in Lie groups and symmetric spaces and this is what I shall describe below.

These results were obtained in collaboration with D. Ferus, F. Pedit and U. Pinkall. The detailed proofs will appear elsewhere.

2 Harmonic maps

Let $\phi : (M, g) \to (N, h)$ be a (smooth) map of Riemannian manifolds. The *energy* $E(\phi)$ of ϕ is given by

$$E(\phi) = \tfrac{1}{2} \int_M |d\phi|^2 \, dvol_M.$$

We say that ϕ is *harmonic* if it extremizes the energy on every precompact subdomain of M. This is the case precisely when ϕ satisfies the corresponding Euler-Lagrange equations:

$$\mathrm{trace}_g \nabla d\phi = 0,$$

or, in local co-ordinates,

$$g^{ij} \left\{ \frac{\partial^2 \phi^\alpha}{\partial x_i \partial x_j} - {}^M\Gamma_{ij}^k \frac{\partial \phi^\alpha}{\partial x_k} + {}^N\Gamma_{\beta\gamma}^\alpha(\phi) \frac{\partial \phi^\beta}{\partial x_i} \frac{\partial \phi^\gamma}{\partial x_j} \right\} = 0,$$

where the Γ are the Christoffel symbols of M and N. Thus harmonic maps are solutions of a system of semi-linear elliptic partial differential equations.

Harmonic maps have been studied from many points of view and there is now a vast literature on the subject, surveys of which may be found in [5,6]. In particular, there is a well-developed analytic theory of existence and regularity of weak solutions which, in case M is compact and N is non-positively curved, guarantees the existence of an essentially unique harmonic map in every homotopy class. This, in turn, has led to many applications to the geometry and topology of negatively curved manifolds. In these notes, however, we shall discuss harmonic maps in a situation where these techniques fail to tell the whole story: this is the case of harmonic maps of a Riemann surface into a positively curved manifold.

When $\dim M = 2$, the energy functional enjoys a number of special properties reminiscent of the Yang-Mills functional in 4 dimensions: it is conformally invariant and, when N is Kähler, there is a topological charge and instanton

solutions (i.e., $\pm$-holomorphic maps). These phenomena account for the interest of Theoretical Physicists in harmonic maps—they study them to gain insight into the (more complicated) Yang-Mills functional.

The analogy with Yang-Mills becomes even more striking when we take as our target N a compact Riemannian symmetric space. These are homogeneous spaces G/K where G is a compact Lie group equipped with an involution τ such that

$$(G^\tau)_0 \subset K \subset G^\tau.$$

In this setting, the harmonic map equations have a gauge-theoretic formulation as a kind of Yang-Mills-Higgs equation with gauge group K[1].

Among the examples of such symmetric spaces are spheres, projective spaces and Grassmannians (real, complex and quaternionic) as well as less familiar exceptional spaces such as $G_2/SO(4)$. Moreover, any Lie group can be viewed as a symmetric space $G \times G/\Delta G$.

In fact, one may reduce everything to this latter case by observing that the map $G/K \to G$ given by

$$gK \mapsto g^\tau g^{-1}$$

is a well-defined totally geodesic immersion [3]. Since harmonicity of a map is preserved under post-composition by a totally geodesic map, this reduces the study of harmonic maps into symmetric spaces to that of harmonic maps into Lie groups.

Thus we shall examine harmonic maps of a Riemann surface into a Lie group and, in particular, harmonic maps of a 2-torus.

3 Commuting flows

Since the energy is conformally invariant for 2-dimensional domains, harmonic maps of a 2-torus into a Lie group may be viewed as harmonic maps of $\mathbf{R}^2$

[1] I learnt this from (unpublished) 1982 notes of J.H. Rawnsley, see also [9].

which are doubly periodic with respect to some lattice. So let us consider harmonic maps of $\mathbf{R}^2$ into a compact Lie group G.

Let θ be the (left) Maurer-Cartan form on G and let α be a g-valued 1-form on $\mathbf{R}^2$. Since $\mathbf{R}^2$ is simply connected, we know that $\alpha = \phi^*\theta$ for some map $\phi : \mathbf{R}^2 \to G$ if and only if α satisfies the Maurer-Cartan equations:

$$d\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0 \qquad (1)$$

and then ϕ is unique up to left multiplication by a constant element of G. Equip G with a bi-invariant metric and then $\phi : \mathbf{R}^2 \to G$ is harmonic if and only if $\alpha = \phi^*\theta$ is co-closed:

$$d^*\alpha = 0. \qquad (2)$$

Thus it suffices to consider g-valued 1-forms α on $\mathbf{R}^2$ satisfying (1) and (2).

The fundamental observation of Uhlenbeck [17] (see also [18,19]) is that these equations may be combined into a single Maurer-Cartan equation on introduction of a *spectral parameter*. For this, identify $\mathbf{R}^2$ with the complex line $\mathbf{C}$ and let

$$\alpha = \alpha' + \alpha''$$

be the type decomposition of α: thus α' is a $\mathbf{g}^{\mathbf{C}}$-valued $(1,0)$-form and $\alpha'' = \overline{\alpha'}$. For $\lambda \in \mathbf{C}^{\cdot}$, set

$$A_\lambda = \tfrac{1}{2}(1 - \lambda)\alpha' + \tfrac{1}{2}(1 - \lambda^{-1})\alpha'' \qquad (3)$$

so that each A_λ is a $\mathbf{g}^{\mathbf{C}}$-valued 1-form which is g-valued for $\lambda \in S^1$. We now have

Theorem 3.1 [17] *A g-valued 1-form α on $\mathbf{R}^2$ satisfies (1) and (2) if and only if, for each $\lambda \in S^1$,*

$$dA_\lambda + \tfrac{1}{2}[A_\lambda \wedge A_\lambda] = 0.$$

This result is the starting point of the analysis of harmonic maps to G from both S^2 and T^2. For harmonic maps of S^2, the next step is to integrate this

family of Maurer-Cartan equations to obtain a holomorphic map into the loop group (c.f., the contribution of Valli to these proceedings). For us, however, we begin by showing how the forms A_λ may be obtained by solving a pair of commuting ordinary differential equations.

We have seen that to find a harmonic map, it is necessary and sufficient to find a family of 1-forms A_λ of the form (3) with each A_λ solving the Maurer-Cartan equations. To put this result in context, we introduce some notation:

Let $\Omega\mathbf{g}$ be the space of based loops in $\mathbf{g}$, i.e.,

$$\Omega\mathbf{g} = \{\xi : S^1 \to \mathbf{g}: \quad \xi(1) = 0\}.$$

Then $\Omega\mathbf{g}$ is a Lie algebra under point-wise bracket. Any loop $\xi \in \Omega\mathbf{g}$ has a unique representation

$$\xi(\lambda) = \sum_{n \neq 0} \xi_n(1 - \lambda^n)$$

with each $\xi_n \in \mathbf{g}^{\mathbf{C}}$ and $\xi_{-n} = \overline{\xi_n}$. We use this to define a filtration of $\Omega\mathbf{g}$ by finite-dimensional subspaces

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega\mathbf{g}$$

with

$$\Omega_d = \{\xi: \xi_n = 0 \text{ for } |n| > d\}.$$

We also introduce subalgebras $\Omega^\pm$ of $\Omega\mathbf{g}$ by

$$\Omega^+ = \{\xi: \xi_n = 0 \text{ for } n < 0\}$$
$$\Omega^- = \{\xi: \xi_n = 0 \text{ for } n > 0\}.$$

Observe that $\Omega^\pm$ are mutually conjugate (where the conjugation is with respect to the real form $\Omega\mathbf{g}$) and that

$$\Omega\mathbf{g}^{\mathbf{C}} = \Omega^+ \oplus \Omega^-.$$

A family of 1-forms A_λ of the form (3) is then the same as an $\Omega\mathbf{g}$-valued 1-form A with $A^{(1,0)}$ taking values in $\Omega^+ \cap \Omega_1^{\mathbf{C}}$ and theorem 3.1 tells us that any

such A satisfying the Maurer-Cartan equations in $\Omega\mathfrak{g}$ gives rise to a harmonic map $\phi : \mathbf{R}^2 \to G$ with $\phi^*\theta = A_{-1}$.

To produce such A, fix $d \in \mathbf{N}$ and introduce vector fields X_1, X_2 on Ω_d by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1 - \lambda)\xi_d].$$

These are easily seen to be well-defined and we have

Theorem 3.2 *The X_i are complete and commute. Thus, if X_i^t denotes the flow on Ω_d generated by X_i, then*

$$(t^1, t^2) \cdot \xi = X_1^{t^1} X_2^{t^2}(\xi)$$

defines an action of $\mathbf{R}^2$ on Ω_d.

Fix an initial condition $\xi_0 \in \Omega_d$ and let $\xi : \mathbf{R}^2 \to \Omega_d$ be given by

$$\xi(t^1, t^2) = (t^1, t^2) \cdot \xi_0.$$

Then $A_\lambda = 2i(1 - \lambda)\xi_d dz - 2i(1 - \lambda^{-1})\xi_{-d} d\bar{z}$ satisfies the Maurer-Cartan equations and so gives rise to a harmonic map $\mathbf{R}^2 \to G$.

Otherwise said: if $\xi : \mathbf{R}^2 \to \Omega_d$ solves

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1 - \lambda)\xi_d] \tag{4}$$

then $4i\xi_d dz = \phi^*\theta^{(1,0)}$ for a harmonic map $\phi : \mathbf{R}^2 \to G$ and solutions of (4) exist on all $\mathbf{R}^2$ for any choice of initial condition.

The flows on Ω_d have a Hamiltonian origin: indeed $\Omega\mathfrak{g}$ is a Poisson manifold and the flows X_j on each Ω_d are (the restrictions of) the Hamiltonian flows of two Poisson commuting functions. We briefly sketch this development.

Recall that a Poisson manifold is a manifold M together with a skew bilinear map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f, g) \mapsto \{f, g\}$, satisfying, for all $f, g, h \in C^\infty(M)$,

   (i) the derivation property:

$$\{f, gh\} = \{f, g\}h + g\{f, h\};$$

(ii) the Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$

In particular, $C^\infty(M)$ is a Lie algebra with bracket $\{\,,\,\}$. Moreover, the derivation property gives us a map $C^\infty(M) \to C^\infty(TM)$, $f \mapsto X_f$ by

$$X_f g = \{f,g\}$$

and one sees from the Jacobi identity that this map is a Lie algebra homomorphism. We call X_f the *Hamiltonian vector field* of f. For more information on Poisson manifolds, the Reader is referred to the book of Libermann-Marle [12].

A canonical example of a Poisson manifold (originally due to Lie himself!) arises as follows: let $\mathbf{g}$ be a Lie algebra and $\mathbf{g}^*$ its dual. Then $\mathbf{g}^*$ has a Poisson structure given by

$$\{f,g\}(x) = \langle x, [df_x, dg_x]\rangle,$$

where we have identified $T_x^*\mathbf{g}^*$ with $\mathbf{g}$. (It is an illuminating exercise to prove that $\{\,,\,\}$ so defined does indeed satisfy the Jacobi identity).

We use this to define a slightly unusual Poisson structure on $\Omega\mathbf{g}$. First introduce a second Lie bracket $[\,,\,]_R$ on $\Omega\mathbf{g}$ by

$$[\xi,\eta]_R = 2i[\xi^+,\eta^+] - 2i[\xi^-,\eta^-],$$

where $\xi^\pm$ are the components of ξ in $\Omega^\pm$. It is straight-forward to check that $[\,,\,]_R$ satisfies the Jacobi identity. Now identify $\Omega\mathbf{g}$ with $\Omega\mathbf{g}^*$ via the L^2-inner product and so obtain a Poisson bracket on $\Omega\mathbf{g}$ by

$$\{f,g\}_R(\xi) = (\xi, [\nabla f_\xi, \nabla g_\xi]_R)_{L^2},$$

where ∇f is the L^2-gradient of f.

The remarkable fact about this Poisson structure is that invariant functions on $\Omega\mathbf{g}$ Poisson commute with respect to it: here a function on $\Omega\mathbf{g}$ is said to be invariant if

$$[\xi, \nabla f_\xi] = 0$$

for all $\xi \in \Omega g$—this is just the infinitesimal version of the condition that f be invariant under the adjoint action of the loop group.

We can now see where our commuting vector fields come from: for $d \in \mathbf{N}$, define f_1, f_2 by

$$(f_1 - if_2)(\xi) = \int_{S^1} \lambda^{1-d}(\xi, \xi).$$

The f_j are easily seen to be invariant and a calculation shows that the corresponding Hamiltonian vector fields restrict to the X_j of theorem 3.2 on Ω_d.

The construction of this Poisson structure and, indeed, the construction of solutions to the Maurer-Cartan equations from the Hamiltonian flows of invariant functions is part of a general theory, that of r-matrices, that is well known in the integrable systems literature [15,16]. For brief expositions of the theory in this and a related context, see [2,7].

4 Conserved quantities

We have seen that harmonic maps $\mathbf{R}^2 \rightarrow G$ may be obtained by solving ordinary differential equations to find $\xi : \mathbf{R}^2 \rightarrow \Omega_d$ satisfying

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1 - \lambda)\xi_d].$$

Since ξ is real, we also have the conjugate equation

$$\frac{\partial \xi}{\partial \bar{z}} = -[\xi, 2i(1 - \lambda^{-1})\xi_{-d}].$$

These equations are in Lax form, that is, of the form

$$\dot{\xi} = [\xi, A(\xi)],$$

where A is an Ωg-valued 1-form dependent on ξ. Such equations have many conserved quantities: indeed, if $P : \Omega g \rightarrow \mathbf{C}$ is an invariant function, then $P \circ \xi$ is constant. For example, let $\langle \, , \, \rangle$ denote the (invariant) L^2 inner product on Ωg. Then

$$d\langle \xi, \xi \rangle = 2\langle \xi, d\xi \rangle = 2\langle \xi, [\xi, A] \rangle = -2\langle [\xi, \xi], A \rangle = 0,$$

so that the ξ evolves on spheres in Ω_d (this is why the X_j are complete).

Again, if $p : \mathbf{g}^{\mathbf{C}} \to \mathbf{C}$ is an invariant polynomial then, for each $\lambda \in \mathbf{C}^*$, $p(\xi(\lambda))$ is independent of z. Comparing coefficients then allows us to conclude, for instance, that $p(\xi_d)$ is constant. Moreover, in case that $\mathbf{g}$ is a matrix algebra, this also implies that the characteristic polynomial of each $\xi(\lambda)$ is independent of z. It is essentially the divisor of this polynomial that constitutes the spectral curve to be discussed below.

Meanwhile, let us see what these ideas tell us about the harmonic maps that we have constructed: recall that the harmonic map ϕ constructed from ξ in theorem 3.2 satisfies

$$\phi^* \theta(\frac{\partial}{\partial z}) = 4i\xi_d$$

so that $\phi^* \theta(\frac{\partial}{\partial z})$ lies in a level set of any invariant polynomial on $\mathbf{g}^{\mathbf{C}}$. In fact, this argument may be refined slightly to give:

Proposition 4.1 $\xi_d : \mathbf{R}^2 \to \mathbf{g}^{\mathbf{C}}$ *(and hence $\phi^* \theta(\frac{\partial}{\partial z})$) takes values in a single* $\mathrm{Ad}\, G^{\mathbf{C}}$*-orbit in* $\mathbf{g}^{\mathbf{C}}$.

In particular, if such a ϕ is non-constant then $d\phi$ never vanishes.

What kind of harmonic maps of $\mathbf{R}^2$ satisfy such restrictions? To answer this, we recall an old argument of Chern-Goldberg [4]: if M is a Riemann surface and N is a Riemannian manifold, the harmonic map equations may be written

$$\phi^{-1} \nabla^N_{\frac{\partial}{\partial \bar{z}}} \phi_* \frac{\partial}{\partial z} = 0.$$

From this, it is easy to conclude that if P is a parallel section of $S^k T^* N$ and ϕ is harmonic then $(\phi^* P)^{(k,0)}$ is a holomorphic differential. In the case at hand, let p be an homogeneous invariant polynomial on $\mathbf{g}^{\mathbf{C}}$ of degree k. Then $p \circ \theta$ is parallel for any bi-invariant metric and we conclude that $p(\phi^* \theta(\frac{\partial}{\partial z}))dz^k$ is a holomorphic differential.

There are two cases where this gives a lot of information. Firstly, if M is the Riemann sphere, there are no non-vanishing holomorphic differentials and we conclude that $\phi^* \theta(\frac{\partial}{\partial z})$ lies in the zero-set of any invariant polynomial.

Otherwise said, in this case, each $\phi^*\theta(\frac{\partial}{\partial z})$ is nilpotent. On the other hand, when M is a 2-torus, each power of the canonical bundle is canonically trivial (Liouville's Theorem) and we conclude that $\phi^*\theta(\frac{\partial}{\partial z})$ takes values in a single common level set of the invariant polynomials on $\mathbf{g}^{\mathbf{C}}$.

In general, this is insufficient to allow us to deduce that $\phi^*\theta(\frac{\partial}{\partial z})$ takes values in a single $\mathrm{Ad}\, G^{\mathbf{C}}$-orbit. However, results of Kostant [10] and Kostant-Rallis [11] allow us to conclude that certain level sets comprise a single orbit. As a special case of such arguments, we have

Proposition 4.2 *Let $\phi : T^2 \to G$ be harmonic and factor through a rank one symmetric space. If ϕ is non-conformal at one (and hence every) point, then $\phi^*\theta(\frac{\partial}{\partial z})$ takes values in a single $\mathrm{Ad}\, G^{\mathbf{C}}$-orbit of semi-simple elements of $\mathbf{g}^{\mathbf{C}}$.*

In any case, a fairly large class of harmonic 2-tori satisfy the necessary conditions of proposition 4.1 to arise from our construction. In the next section, we shall discuss sufficient conditions.

5 Harmonic maps of finite type

We baptise the harmonic maps we have been discussing in the following

Definition A harmonic map $\phi : \mathbf{R}^2 \to G$ is of *finite type* if, for some $d \in \mathbf{N}$, there is a map $\xi : \mathbf{R}^2 \to \Omega_d$ such that

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1 - \lambda)\xi_d], \tag{5}$$

$$4i\xi_d = \phi^*\theta(\frac{\partial}{\partial z}). \tag{6}$$

Substituting (6) into (5) and taking the conjugate equation as well, (5) becomes

$$d\xi = [\xi, A_\lambda] \tag{7}$$

where, as usual, $A_\lambda = \frac{1}{2}(1 - \lambda)\alpha' + \frac{1}{2}(1 - \lambda^{-1})\alpha''$.

We have seen that if ϕ is of finite type then $\phi^*\theta(\frac{\partial}{\partial z})$ takes values in a single $\mathrm{Ad}\, G^{\mathbf{C}}$-orbit in $\mathbf{g}^{\mathbf{C}}$. A partial converse is contained in the following theorem.

Theorem 5.1 *Let $\phi : \mathbf{R}^2/\Lambda \to G$ be a harmonic map of a 2-torus such that $\phi^* \theta(\frac{\partial}{\partial z})$ takes values in a single $\mathrm{Ad}\, G^{\mathbf{C}}$-orbit of semisimple elements in $\mathbf{g}^{\mathbf{C}}$. Then ϕ is of finite type.*

In particular, from proposition 4.2, we get the rather satisfying

Corollary 5.2 *Let $\phi : \mathbf{R}^2/\Lambda \to G/K$ be a non-conformal harmonic map of a 2-torus into a rank one symmetric space. Then, viewed as a map into G, ϕ is of finite type.*

We remark that the theorem excludes all harmonic maps of tori that holomorphically cover a 2-sphere since these would have nilpotent $\phi^* \theta(\frac{\partial}{\partial z})$. Indeed, I conjecture that very few (if any) harmonic 2-spheres will come from our constructions in view of the above mentioned restrictions on the number of zeros of $d\phi$.

The key step in the proof of theorem 5.1 is to find a *formal* solution to (6) and (7), that is, to find $Y = \sum_{k \geq 0} \lambda^{-k} Y_k$ with $Y_k : \mathbf{R}^2/\Lambda \to \mathbf{g}^{\mathbf{C}}$ satisfying (in the sense of formal Laurent series)

$$dY = [Y, A_\lambda];$$

$$4i Y_0 = \phi^* \theta(\frac{\partial}{\partial z}).$$

The hypothesis that $\phi^* \theta(\frac{\partial}{\partial z})$ lies in a single orbit of semisimple elements allows us to recursively define an appropriate formal gauge transformation under which these equations take a very simple form for which a solution is readily found. In this way, we obtain a formal solution Y.

One now observes that each Y_k is a *Jacobi field*, that is, a solution to the linearization of the harmonic map equations at ϕ. This is a linear elliptic partial differential equation and so, since the torus is compact, the Y_k span a finite dimensional space. Using this, one eventually finds a ξ satisfying (6) and (7) as a linear combination of the polynomial parts (and their conjugates) of various $\lambda^N Y$.

6 The spectral curve

We have seen that a substantial class of harmonic tori may be obtained by integrating a pair of commuting Hamiltonian flows in a loop algebra. Moreover, we have seen that these equations possess a large number of conserved quantities and we might expect therefore that the Hamiltonian system under discussion is completely integrable in the sense of Liouville-Arnold, reducing to a linear flow on a torus. Indeed, most of the known integrable systems have a similar representation as Lax equations on a loop algebra. In this section, we will show how our Hamiltonian flows give rise to a linear flow on a certain complex torus. This development is essentially well known in the integrable systems literature but the methods of [1] or [14] do not apply to our case since our spectral curve is necessarily singular. This being the case, we shall give a complete proof of the linearity of our flow by applying the method of Griffiths [8].

We begin by defining the spectral curve. For simplicity, we take $\mathbf{g} = \mathbf{u}(N)$. View λ as an affine co-ordinate on $\mathbb{P}^1$ and let $\mathcal{O}(2d)$ be the line bundle corresponding to the divisor $d\{0\}+d\{\infty\}$. Thus $\mathcal{O}(2d)$ trivialises on $U_+ = \mathbb{P}^1\backslash \{\infty\}$, $U_- = \mathbb{P}^1 \backslash \{0\}$ with transition function $g_{+-}(\lambda) = \lambda^{2d}$ on $U_+ \cap U_- = \mathbb{C}^*$. We denote by σ_d the holomorphic section of $\mathcal{O}(2d)$ with divisor $d\{0\} + d\{\infty\}$ given by $\lambda^{\pm d}$ on $U_\pm$.

Let $\xi \in \Omega_d$. Then $X = \sigma_d \xi$ is a holomorphic section of $\mathcal{O}(2d) \otimes \mathbf{gl}(N, \mathbb{C})$. Our spectral curve will be the normalisation of the divisor of the characteristic polynomial of X. To give this invariant meaning, let Y be the total space of $\mathcal{O}(2d) \xrightarrow{\pi} \mathbb{P}^1$ and pull X back to a section of $\pi^{-1}\mathcal{O}(2d)$. Now $\pi^{-1}\mathcal{O}(2d)$ has a tautological section η whence $\eta\mathrm{Id}_N - X$ is a well-defined section of $\pi^{-1}\mathcal{O}(2d) \otimes \mathbf{gl}(N, \mathbb{C})$ and $Q_X = \det(\eta\mathrm{Id}_N - X) \in H^0(Y, \otimes^N \pi^{-1}\mathcal{O}(2d))$. Let C_0 be the divisor of Q_X: this is a necessarily singular algebraic curve in Y since X vanishes at $\lambda = 1$. We call the normalisation of C_0 the *spectral curve* of X and denote it by C.

Now let $\xi : \mathbf{R}^2 \to \Omega_d$ solve

$$d\xi = [\xi, 2i(1 - \lambda)\xi_d dz - 2i(1 - \lambda^{-1})\xi_{-d}d\bar{z}] \qquad (8)$$

and assume, in addition that, at one (and hence, by proposition 4.1, every) point, ξ_d has N distinct eigenvalues. The above discussion of conserved quantities implies that Q_X is independent of $z \in \mathbf{R}^2$ so that the spectral curve is the same for each $\xi(z)$. However, there is a quantity on C which does evolve with (8): the eigenvalues of X are independent of z but the eigenspaces need not be. By our assumption on ξ_d, X has N distinct eigenvalues at $\lambda = \infty$ and so the eigenspaces are generically one dimensional. Thus, off a finite number of points of C_0 we may define, for each $z \in \mathbf{R}^2$, a holomorphic map into $\mathbb{P}^{N-1}$ whose value at $(\lambda, \eta) \in C_0$ is the η-eigenspace of $X_\lambda(z)$. This map extends to a holomorphic map $C \to \mathbb{P}^{N-1}$ or, equivalently, a holomorphic line sub-bundle $\tilde{L}_z$ of the trivial $\mathbf{C}^N$ bundle over C. If we now set $L_z = \tilde{L}_z \otimes \tilde{L}_0^{-1}$, then each L_z has degree zero.

The space of degree zero line bundles on C is the *Jacobian* $J(C)$ of C and is a complex torus. Indeed, in the long exact sequence in cohomology induced by the exponential short exact sequence of sheaves

$$0 \to \mathbf{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0,$$

the degree zero line bundles correspond, via the cocycles given by their transition functions, to the kernel of $H^0(\mathcal{O}_C^*) \to H^2(C, \mathbf{Z})$ whence

$$J(C) \cong H^1(\mathcal{O}_C)/H^1(C, \mathbf{Z}).$$

Thus we have obtained a map $L : \mathbf{R}^2 \to J(C)$. I claim that this map integrates a pair of *linear* flows on $J(C)$. For this, we must show that $dL_z : \mathbf{R}^2 \to H^1(\mathcal{O}_C)$ is independent of z.

Let Z be a (real) constant vector field on $\mathbf{R}^2$. Then

$$d_Z\xi = [\xi, (1 - \lambda)b + (1 - \lambda^{-1})\bar{b}] \qquad (9)$$

where $b = c\xi_d$ for some constant c. To calculate $d_Z L$, we begin by fixing a Leray cover $\{U_\alpha\}$ on C. Then $\tilde{L}_x$ is trivialised by local sections $\nu_\alpha \in \mathcal{O}^*(U_\alpha) \otimes \mathbf{C}^N$. If we define $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ by

$$\nu_\alpha = g_{\alpha\beta} \nu_\beta$$

then $\tilde{L}_x$ corresponds to the cocycle $\{g_{\alpha\beta}\} \in H^1(\mathcal{O}_C^*)$. Moreover, there are $\rho_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ such that

$$e^{\rho_{\alpha\beta}} = g_{\alpha\beta}$$

and, letting $\dot{}$ stand for differentiation along Z, it is easy to see that $d_Z L = \{\dot{\rho}_{\alpha\beta}\} : \mathbf{R}^2 \to H^1(\mathcal{O}_C)$.

The idea of Griffiths [8] is to identify the cocycle $\{\dot{\rho}_{\alpha\beta}\}$ with a certain residue that is easily computed. For this we use π to pull all the data on $\mathbb{P}^1$ back to C. In particular, we view λ as a meromorphic function on C and $B = (1 - \lambda)b + (1 - \lambda^{-1})\bar{b}$ as a meromorphic $\mathbf{gl}(N, \mathbf{C})$-valued function on C.

Lemma 6.1 *With $\{U_\alpha\}$ as above, there are meromorphic functions μ_α on U_α such that*

$$B\nu_\alpha = \mu_\alpha \nu_\alpha - \dot{\nu}_\alpha, \qquad (10)$$

and, moreover,

$$\mu_\alpha - \mu_\beta = \dot{\rho}_{\alpha\beta} \qquad (11)$$

on $U_\alpha \cap U_\beta$.

Proof. Pulling ξ and σ_d back to C, we have

$$X\nu_\alpha = \sigma_d \xi \nu_\alpha = \eta \nu_\alpha$$

and differentiating along Z gives

$$\sigma_d \dot{\xi} \nu_\alpha + \sigma_d \xi \dot{\nu}_\alpha = \eta \dot{\nu}_\alpha$$

or, using (9) and rearranging,

$$\sigma_d \xi (B\nu_\alpha + \dot{\nu}_\alpha) = \eta(B\nu_\alpha + \dot{\nu}_\alpha).$$

Since the eigenspaces of X are one dimensional, this gives

$$B\nu_\alpha + \dot{\nu}_\alpha = \mu_\alpha \nu_\alpha$$

for meromorphic functions μ_α.

Finally, since $\nu_\alpha = g_{\alpha\beta}\nu_\beta$, we have

$$\begin{aligned}
\mu_\alpha \nu_\alpha = B\nu_\alpha + \dot{\nu}_\alpha &= B(g_{\alpha\beta}\nu_\beta) + \dot{g}_{\alpha\beta}\nu_\beta + g_{\alpha\beta}\dot{\nu}_\beta \\
&= g_{\alpha\beta}\mu_\beta\nu_\beta + \dot{g}_{\alpha\beta}\nu_\beta = \mu_\beta\nu_\alpha + \dot{\rho}_{\alpha\beta}\nu_\alpha
\end{aligned}$$

so that (11) holds. □

We now introduce the sheaf of principal parts associated with an effective divisor: let $D = \sum n_i p_i$ be an effective divisor and z_i a local co-ordinate centred at p_i. A *principal part* at p_i is an expression of the form

$$z_i^{-n_i} a_{n_i} + \cdots + z_i^{-1} a_1.$$

The sheaf of collections of such principal parts, one for each p_i, denoted $\mathcal{O}_D(D)$ can be identified with the quotient sheaf $\mathcal{O}_C/\mathcal{O}_C(D)$ where $\mathcal{O}_C(D)$ is the sheaf of meromorphic functions on C with pole divisors $\leq D$. In the case at hand, let $D = \lambda^{-1}\{0\} + \lambda^{-1}\{\infty\}$. Then each component of B is an element of $H^0(\mathcal{O}_C(D))$ and we deduce that the $\{\mu_\alpha\}$ of lemma 6.1 give rise to a well-defined section $\mu \in H^0(\mathcal{O}_D(D))$.

Part of the long exact sequence induced by

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0$$

is the connecting homomorphism $\delta : H^0(\mathcal{O}_D(D)) \to H^1(\mathcal{O}_C)$ and it is immediate from (11) and the definition of δ that

Theorem 6.2 [8] $\delta(\mu) = \{\dot{\rho}_{\alpha\beta}\} = d_z L.$

Thus it suffices to examine the principal parts of the μ_α. For this, we note that since ξ_d (and hence ξ_{-d}) have N distinct eigenvalues, C_0 is smooth near $\lambda = 0, \infty$ by the Implicit Function Theorem. Thus $D = \sum p_i + \sum q_i$ where the

p_i, q_i are $2N$ distinct points with $\pi(p_i) = 0$, $\pi(q_i) = \infty$. Moreover, λ is a local co-ordinate near each p_i and $w = 1/\lambda$ is a co-ordinate near each q_i.

Let us examine μ near some q_i: we have

$$B\nu_\alpha = \mu_a\nu_\alpha - \dot{\nu}_\alpha$$

while trivialising $\pi^{-1}\mathcal{O}(2d)$ on $\pi^{-1}(U_-)$ gives

$$(\xi_d + w\xi_{d-1} + \cdots + w^{2d}\xi_{-d})\nu_\alpha = \eta\nu_\alpha$$

so that

$$\xi_d\nu_\alpha = \eta\nu_\alpha + wR\nu_\alpha$$

with R holomorphic near q_i. Recall that $B = (1-w^{-1})c\xi_d + (1-w)\overline{c\xi}_d$ whence, up to holomorphic terms,

$$\mu_a\nu_\alpha = -w^{-1}c\xi_d\nu_\alpha = -c(\eta/w)\nu_\alpha - cR\nu_\alpha = -c(\eta/w)\nu_\alpha.$$

Thus the principal part of μ_a at q_i is $-c\eta/w$ which is independent of z. Similar reasoning at the p_i completes the argument to show that $d_Z L$ is constant and so establishes the linearity of the flow on the Jacobian.

It remains to invert the construction and produce harmonic maps from suitable data on an algebraic curve (c.f., [9,7]). I shall return to this elsewhere.

References

[1] M. Adler and P. van Moerbeke, *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*, Adv. Math. **38** (1980), 318–379.

[2] F.E. Burstall, *Harmonic tori in Lie groups*, Geometry and topology of submanifolds, III (L. Verstraelen and A. West, eds.), World Sci., Singapore, 1991, pp. 73–80.

[3] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.

[4] S.S. Chern and S.I. Goldberg, *On the volume-decreasing property of a class of real harmonic mappings*, Amer. J. Math. **97** (1975), 133–147.

[5] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. Lond. Math. Soc. **10** (1978), 1–68.

[6] ———, *Another report on harmonic maps*, Bull. Lond. Math. Soc. **20** (1988), 385–524.

[7] D. Ferus, F. Pedit, U. Pinkall, and I. Sterling, *Minimal tori in S^4*, T.U. Berlin preprint, 1990.

[8] P.A. Griffiths, *Linearizing flows and a cohomological interpretation of Lax equations*, Amer. J. Math. **107** (1985), 1445–1483.

[9] N.J. Hitchin, *Harmonic maps from a 2-torus to the 3-sphere*, J. Diff. Geom. **31** (1990), 627–710.

[10] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **86** (1963), 327–402.

[11] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[12] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics*, Reidel, Dordrecht, 1987.

[13] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. **130** (1989), 407–451.

[14] A.G. Reyman and M.A. Semenov-Tian-Shansky, *Reduction of Hamiltonian systems, affine Lie algebras and Lax equations, II*, Invent. Math. **63** (1981), 423–432.

[15] ———, *Compatible Poisson structures for Lax equations: an r-matrix approach*, Phys. Let. A **130** (1988), 456–460.

[16] M.A. Semenov-Tian-Shansky, *What is a classical r-matrix*, Funkz. Anal. i Prilozh. **17** (1983), 17–33.

[17] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.

[18] V.E. Zakharov and A.V. Mikhaĭlov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Sov. Phys. JETP **47** (1978), 1017–1027.

[19] V.E. Zakharov and A.B. Shabat, *Integration of non-linear equations of mathematical physics by the method of inverse scattering, II*, Funkts. Anal. i Prilozh. **13** (1978), 13–22.

Francis E. Burstall
School of Mathematical Sciences, University of Bath
Bath, BA2 7AY, United Kingdom