Matemática Contemporânea, Vol. 18, 175–194
http://doi.org/10.21711/231766362000/rmc1811

TRANSVERSELY HOLOMORPHIC FOLIATIONS AND CR STRUCTURES

Howard Jacobowitz *

Abstract

This paper relates elliptic and CR structures by generalizing results of Haefliger and Sundararaman.

1. Statement of results

We start with a bundle V of complex tangent vectors on a manifold M,

$$V \subset \mathbb{C} \otimes TM.$$

Definition 1 *V is involutive if $[V, V] \subset V$.*

We mean this on the sheaf level: Let U be an open subset of M and denote by $\Gamma(U, V)$ the set of smooth sections of $V|_U$. Then V is involutive if

$$X, Y \in \Gamma(U, V) \Rightarrow [X, Y] \in \Gamma(U, V).$$

Complex structures and CR structures are well-known examples. Involutive structures are also known as formally integrable structures. The basic reference for these structures is [14].

*The author was a Visitor at the Institute for Advanced Study for some of the time of this research and wishes to express his gratitude for this opportunity. This work was also partially supported by NSF Grant # DMS94-04494. The author would like to thank Professor Haefliger for his thoughtful comments on a previous version of this paper.
AMS Subject Classifiation: Primary 35F99, 32V05, 57T36 Secondary 58J10.
Key words and phrases: involutive, CR structure, elliptic structure, transversally holomorphic foliations.

Definition 2 *(M, V) is an elliptic structure if V is involutive, $V + \overline{V} = \mathbb{C} \otimes TM$ and $d = rank_{\mathbb{C}} V \cap \overline{V}$ is greater than zero.*

Note that d is a constant over the manifold. When $d = 0$, (M, V) is a complex structure (see Theorem A). So in our convention, a complex structure is not elliptic.

Here is a simple and useful class of elliptic structures:

Let M be a complex manifold and $\pi : B \to M$ a fiber bundle over M. Let $T^{1,0}$ be the global sub-bundle of $\mathbb{C} \otimes T^*M$ generated by the local coordinates $\{dz_1, \dots, dz_n\}$ and let $W = \pi^*(T^{1,0})$. Then $V = W^{\perp} \subset \mathbb{C} \otimes TB$ is involutive. In particular, the Hopf foliation provides an elliptic structure on S^3. Note that not all elliptic structures are locally trivial fiber bundles. This may be seen by considering Seifert fibrations. See also [9].

We shall use the convention of writing an elliptic structure as (M^{2n+d}, V) to mean that

$$rank_{\mathbb{C}} V = n + d, \; rank_{\mathbb{C}} V \cap \overline{V} = d.$$

Let $U \subset TM$ be defined by $\mathbb{C} \otimes U = V \cap \overline{V}$. Note that $[U, U] \subset U$ and so U defines a $2n$-codimension foliation F, which we call the foliation associated to the elliptic structure. We sometimes also refer to this foliation as the elliptic foliation.

If we focus on the foliation F rather than on the sub-bundle V we are led to the equivalent concept of a transversely holomorphic foliation. This is a well-studied particular case of the structures introduced by Haefliger [4].

Definition 3 *A co-dimension $2n$ foliation F on M^{2n+d} is called transversely holomorphic if there exists a covering $M = \cup_j U^j$, and local coordinate charts $\Phi_j : U^j \to \mathbb{C}^n \times \mathbb{R}^d$ such that the leaves of F are locally given by the sets $\{z = c\}$ and with transition functions $F_{jk} = \Phi_j \circ \Phi_k^{-1}$ of the form*

$$\begin{aligned} z^j &= f_{jk}(z^k) \\ t^j &= g_{jk}(z^k, \overline{z}^k, t^k) \end{aligned}$$

where f_{jk} *is holomorphic in an open neighborhood of* $\Phi_k(U^j \cap U^k)$ *and* g_{jk} *is* C^∞ *on* $\Phi_k(U^j \cap U^k)$. *We will often identify* U^k *with the open subset* $\Phi_k(U^k)$ *and speak of coordinates* (z^k, t) *on* U^k.

The equivalence of elliptic structures and transversely holomorphic foliations is a consequence of the Newlander-Nirenberg Theorem (see §2). In particular, the components of z are annihilated by the local sections of V.

Besides elliptic structures, we also will be considering a second type of involutive structure. Again let

$$V \subset \mathbb{C} \otimes TM.$$

Definition 4 *(M, V) is a CR structure if* V *is involutive,* $V \cap \overline{V} = \{0\}$ *and* $V \oplus \overline{V} \neq \mathbb{C} \otimes TM$.

This last condition excludes complex structures.

Definition 5 *Let* (M, V) *be a CR structure. A function* F *defined on some open set* $U \subset M$ *is a CR function on* U *if* $Lf = 0$ *for all* $L \in V_q$, $q \in U$.

The classic example is a real hypersurface in a complex manifold. The construction is local, so let $M^{2n-1} \subset \mathbb{C}^n$. Set

$$V = (\mathbb{C} \otimes TM) \cap T_{0,1}(\mathbb{C}^n). \tag{1}$$

Here

$$T_{0,1}(\mathbb{C}^n) = linear\ span\{\frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}\}.$$

Then, $[V, V] \subset V$, $V \cap \overline{V} = \{0\}$ and $rank_{\mathbb{C}}\ V + \overline{V} = 2n - 2 < 2n - 1$. Thus (M, V) is a CR structure. Note that the restriction of any holomorphic function to M is clearly a CR function.

The definition of V given in Equation 1 makes sense for any submanifold $M^k \subset \mathbb{C}^n$. It is clear that $[V, V] \subset V$ and that $V \cap \overline{V} = \{0\}$. So V is a

CR structure whenever it has constant rank. This is a limitation on how M is placed in $\mathbb{C}^n$, except for the case above where M is of codimension one. We return to this in §2. Note that the restriction of any holomorphic function to M^k is again a CR function.

We shall use the convention of writing a CR structure as (M^{2n+d}, V) to mean that

$$rank_{\mathbb{C}} V = n.$$

So for a real hypersurface in $\mathbb{C}^{n+1}$ we write (M^{2n+1}, V).

We return to the Hopf foliation of $S^3 \subset \mathbb{C}^2$. The CR structure is given by

$$V_0 = \{L \in \mathbb{C} \otimes T\mathbb{C}^2|_{S^3} : Lz_1 = 0,\ Lz_2 = 0,\ L|z|^2 = 0\}$$

and the elliptic structure is given by

$$V = \{L \in \mathbb{C} \otimes T\mathbb{C}^2|_{S^3} : z_2 L z_1 - z_1 L z_2 = 0,\ L|z|^2 = 0\}.$$

So $V_0 \subset V$.

Definition 6 *An elliptic structure (M^{2n+d}, V) dominates a CR structure (M^{2n+d}, V_0) if $V_0 \subset V$.*

Recall that our convention is

$$rank_{\mathbb{C}}\, V = n + d \text{ and } rank_{\mathbb{C}} V_0 = n.$$

Lemma 1 *If the elliptic structure (M^{2n+d}, V) dominates the CR structure (M^{2n+d}, V_0), then*

$$V = V_0 \oplus (\mathbb{C} \otimes TF)$$

where F is the foliation associated to V.

Corollary 1 *If the elliptic structures (M^{2n+d}, V_1) and (M^{2n+d}, V_2) both dominate the CR structure (M^{2n+d}, V_0) and if these elliptic structures have the same foliation then $V_1 = V_2$.*

Mendoza and Treves (personal communication) noticed that many interesting elliptic structures dominate CR structures. We first show that a concept introduced by Haefliger and Sundararaman essentially gives a necessary and sufficient condition for a given elliptic structure to dominate some C^ω CR structure. We then define CR foliations and explain their connection to this question.

First we note that an elliptic structure is always equivalent to a C^ω elliptic structure. (Or, what is the same thing, a transversely holomorphic foliation is always equivalent to a C^ω transversely holomorphic foliation.) See the next section for this and for other well-known results that we will be using.

In particular, the leaves of the associated foliation are C^ω immersed manifolds in M. Thus they may be complexified to yield a manifold M' of dimension $2n+2d$ containing M, and locally unique in a neighborhood of M. This manifold is foliated by leaves which have the structure of complex manifolds of dimension d. It is natural to ask if M' has a complex structure in which this foliation is holomorphic. This motivates the following definition.

Definition 7 *[5] A transversely holomorphic foliation F on M^{2n+d} is complexifiable if there exists a complex manifold $\hat{M}$, $dim_{\mathbb{C}}\hat{M} = n+d$, with a holomorphic foliation $\hat{F}$ by leaves of complex dimension d and a C^ω embedding*

$$\Phi : M \to \hat{M}$$

such that

1. For each leaf $\hat{L}$ in $\hat{M}$ there is a leaf L in M such that

$$\hat{L} \cap \Phi(M) = \Phi(L).$$

Further $\Phi(L)$ is totally real in $\hat{L}$.

2. If f is holomorphic near some point $\Phi(p) \in \hat{M}$ and constant on the leaves, then $f \circ \Phi$ is a holomorphic function of z^k whenever $p \in U^k$.

The definition of totally real is recalled in the next section. We have added to the definition of [5] the requirement that Φ is C^ω. This is natural, in light of the preceding discussion and avoids the distraction of an extra argument to reduce to this case.

When M is a C^∞ manifold, we say that a CR structure is C^ω if there exists an C^ω atlas for M, compatible with the original C^∞ atlas, in which V_0 is a C^ω bundle.

Theorem 1 *An elliptic structure (M^{2n+d}, V) dominates some C^ω CR structure (M^{2n+d}, V_0) if and only if (M^{2n+d}, V) is complexifiable.*

This result is essentially due to [5]. In §3 we sketch a new proof based on CR structures. Further, it is asserted in [5] that there exist transversely holomorphic foliations that are not complexifiable. However there is a gap in their argument. So the existence of noncomplexifiable transversely holomorphic foliations, while highly likely, remains open.

Conversely, we ask when is a given CR structure dominated by an elliptic structure. The Newlander-Nirenberg Theorem implies that if there do not exist "enough" CR functions then the CR structure cannot be dominated by an elliptic structure. To avoid such local obstructions, we restrict our attention to C^ω CR structures. Then we can always find, locally, an elliptic V with $V_0 \subset V$. For now there exist, in the neighborhood of any given point, CR functions $f_1, \ldots, f_{n+d}$ such that

$$df_1 \wedge \ldots \wedge df_{n+d} \wedge d\overline{f_1} \wedge \ldots \wedge d\overline{f_n} \neq 0$$

(see §2) with the CR structure given by $V_0 = \{df_1, \ldots, df_{n+d}\}^\perp$. So set $V = \{df_1, \ldots, df_n\}^\perp$. Then, near the given point, V is an elliptic structure and $V_0 \subset V$.

Thus the existence of V is a global question. Here is one formulation of the topology involved.

Let (M^{2n+d}, V_0) be a CR structure and let F be a foliation by leaves of dimension d. Define the sub-bundle $H \subset TM$ by

$$\mathbb{C} \otimes H = V_0 \oplus \overline{V_0}.$$

Note that $rank_{\mathbb{R}} H = 2n$.

Definition 8 *(M^{2n+d}, V_0, F) is a CR foliation if for each $p \in M$*

1. *$T_pF \oplus H_p = T_pM$ and*
2. *there exists a neighborhood U of p and CR functions $f_1, \ldots, f_n$ such that*

$$df_1 \wedge \ldots \wedge df_n \wedge d\overline{f}_1 \ldots \wedge d\overline{f}_n \neq 0$$

and

$$\{q \in U : f(q) = f(p)\}$$

is the connected component containing p of the intersection of U with the leaf through p.

We say that a CR structure (M, V_0) admits a CR foliation if there is some foliation F such that (M, V_0, F) is a CR foliation. We also call F a CR foliation, when the CR structure is understood.

The existence of the foliation F places a restriction on the CR structure (M, V_0). This is a global restriction; clearly such a foliation always exists locally.

It is known that every orientable 3-manifold admits a CR structure but most such manifolds do not admit CR foliations. (The first fact follows from [8] and [7] and the second from [2] and [3]). So the global restriction is quite strong.

Theorem 2 *A C^ω CR structure (M^{2n+d}, V_0) is dominated by some elliptic structure (M^{2n+d}, V) if and only if (M^{2n+d}, V_0) admits a CR foliation.*

Remark. The elliptic structure is complexifiable (by Theorem 1).

Proof of Theorem 2. Assume (M, V_0) is dominated by some elliptic structure (M, V). We claim that the foliation F of the elliptic structure is a CR foliation. From Lemma 1 we have

$$V = V_0 \oplus (\mathbb{C} \otimes TF).$$

Since $V+\overline{V}=\mathbb{C}\otimes TM$ while $V_0\oplus\overline{V_0}=\mathbb{C}\otimes H$, we see that

$$\mathbb{C}\otimes TM=(\mathbb{C}\otimes H)\oplus(\mathbb{C}\otimes TF).$$

Thus

$$TM=H\oplus TF.$$

Further, for any $p\in M$, there exist coordinates $(z_1,\dots,z_n,t_1,\dots,t_d)$ such that

$$F=\{z=c\}\quad\text{near}\quad p.$$

Here we use the equivalence of elliptic structures and transversely holomorphic foliations. Since $V_0\subset V$, the functions $z_1,\dots,z_n$ are CR functions. Clearly, also

$$dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z_1}\wedge\ldots\wedge d\overline{z_n}\neq 0.$$

Thus (M,V_0,F) is a CR foliation.

Conversely, assume that (M,V_0,F) is a CR foliation. Define

$$V=V_0\oplus(\mathbb{C}\otimes TF).$$

Then $rank_{\mathbb{C}}V=n+d$, $rank_{\mathbb{C}}V\cap\overline{V}=d$ and $V+\overline{V}=\mathbb{C}\otimes TM$. Thus V is an elliptic structure provided it is involutive.

Finally, near any point $p\in M$ we have CR functions $f_1,\dots,f_n$ such that $df\wedge d\overline{f}\neq 0$ and $\{f=c\}$ gives F . Thus

$$V_0\subset\{df\}^{\perp}\quad\text{and}\quad TF\subset\{df\}^{\perp}.$$

Hence $V\subset\{df\}^{\perp}$ and by a dimension count $V=\{df\}^{\perp}$. It follows that V is involutive and hence is an elliptic structure that dominates V_0.

2. Preliminaries

First we recall the definition and some basic properties of almost complex structures.

Definition 9 *Let X be an even dimensional manifold and $J : TX \to TX$, a bundle map. (X, J) is an almost complex structure if $J^2 = -I$.*

The eigenvalues of J are $\pm i$. Extend J by complex linearity to $J : \mathbb{C} \otimes TX \to \mathbb{C} \otimes TX$ and let $T_{(1,0)}$ be the i eigenvector space and $T_{(0,1)}$ the $-i$ eigenvector space. These bundles are each of rank equal to $\frac{1}{2} dim X$. Note that

$$T_{(0,1)}|_p = \{Z \in \mathbb{C} \otimes TM_p : Z = w + iJw, \quad w \in TM_p\}.$$

One could alternatively define an almost complex structure by focusing on $V = T_{(0,1)}$:

Definition 10 *(X, V) is an almost complex structure if V is a sub-bundle of $\mathbb{C} \otimes TX$ with $V \oplus \overline{V} = \mathbb{C} \otimes TX$.*

We recover J by showing the map of $TX \to TX$ given by $\Re v \to \Im v$ for $v \in V$ is well defined. We use $\Re$ and $\Im$ for the real and imaginary parts of functions, vectors, etc.

Definition 11 *An almost complex structure (X, V) is called integrable if V is involutive.*

The Newlander-Nirenberg Theorem establishes that a manifold with an integrable almost complex structure is complex.

Theorem A. [10] *Let $V \subset \mathbb{C} \otimes TX$ be an involutive bundle with $V \oplus \overline{V} = \mathbb{C} \otimes TX$. Then X admits a complex structure with $V = T_{0,1}(X)$.*

In other words, there exist an open covering $M = \bigcup_j U^j$ and local coordinates $(z_1^j, \ldots, z_n^j)$ on U^j such that

$$V|_{U^j} = linear\ span \left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}$$

and on the overlap $U^j \cap U^k$ the map $z^j \to z^k$ is biholomorphic.

A version of the Newlander-Nirenberg Theorem ([11] or [14], page 291) applies to elliptic structures.

Theorem B. *An elliptic structure* (M^{2n+d}, V) *admits a co-dimension 2n transversely holomorphic foliation* F *with* $\mathbb{C} \otimes TF = V \cap \overline{V}$ *and, in local coordinates for the transversely holomorphic foliation,* $V = \{dz_1, \dots, dz_n\}^{\perp}$.

The converse is trivial: If F is a transversely holomorphic foliation then the locally defined bundles *linear span*$\{\partial_{\overline{z_1}}, \dots \partial_{\overline{z_n}}, \partial_{t_1}, \dots, \partial_{t_n}\}$ fit together to give a global elliptic V.

In the definition of a transversely holomorphic foliation, we required that $g_{ij} \in C^{\infty}$. In fact, every transversely holomorphic foliation is of class C^{ω}.

Theorem C. [5] *Let* M *be a* C^{∞} *manifold and let* F *be a transversely holomorphic foliation. There exists a* C^{ω} *structure on* M *compatible with the given* C^{∞} *structure, a covering* $M = \bigcup U^j$ *and local coordinate charts* $\Phi_j : U^j \to \mathbb{C}^n \times \mathbb{R}^d$ *with transition functions* $F_{jk} = \Phi_j \circ \Phi_k^{-1}$ *of the form*

$$\begin{aligned} z^j &= f_{jk}(z^k) \\ t^j &= g_{jk}(z^k, \overline{z}^k, t^k) \end{aligned}$$

where f_{jk} *is holomorphic in an open neighborhood of* $\Phi_k(U^j \cap U^k)$ *and* g_{jk} *is* C^{ω} *on* $\Phi_k(U^j \cap U^k)$.

Now let M be a C^{∞} manifold and V a C^{∞} sub-bundle of $\mathbb{C} \otimes TM$. If V is elliptic then there exists a C^{ω} structure on M, compatible with the original C^{∞} structure, in which V becomes a C^{ω} bundle. Such a result does not hold, for example, for CR structures. There exist C^{∞} CR structures which cannot be given by a C^{ω} bundle V. In fact, C^{ω} CR structures have the following property which is definitely not common to most C^{∞} CR structures. See [12] for the first counter-examples.

Theorem D. [1] *Any* C^{ω} *CR structure is locally realizable.*

More explicitly, this theorem asserts the following: Let V_0 be a C^ω CR structure on M^{2n+d}. Then for each point $p \in M$ there exists some neighborhood U of p and CR functions $f_1, \dots, f_n, g_1, \dots g_d$ on U such that

$$df_1 \wedge \dots \wedge df_n \wedge \overline{df_1} \wedge \dots \wedge \overline{df_n} \wedge dg_1 \wedge \dots \wedge dg_d \neq 0 \text{ at } p.$$

Thus a C^∞ CR structure which admits a C^ω structure must have "many" CR functions. However, there do exist C^∞ CR structures with "few" CR functions [12] or even no CR functions, except the constants [13], [6]. These C^∞ CR structures cannot be made C^ω and the ones with too "few" CR functions cannot be dominated by elliptic structures.

We have seen that whenever there is an induced CR structure on $M^n \subset \mathbb{C}^N$, then the restriction to M of a holomorphic function gives a CR function. In the C^ω case the converse is also true.

Theorem E. *Let $M^n \subset \mathbb{C}^N$ be a CR submanifold, p a point of M, and g a CR function in some M-neighborhood of p. If M and g are real analytic, then there exists a function h holomorphic in a $\mathbb{C}^N$-neighborhood of p such that h restricted to M coincides with g.*

We have also seen that if M is a real hypersurface in $\mathbb{C}^N$, then

$$V_0 = (\mathbb{C} \otimes TM) \cap T_{0,1}(\mathbb{C}^N)$$

always defines a CR structure and, more generally, that if $M^n \subset \mathbb{C}^N$ then (M, V_0) is a CR structure provided $rank_{\mathbb{C}} V_0$ is a constant. Here is a useful condition guaranteeing this. First, let $J : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be some complex structure on the vector space $\mathbb{R}^{2N}$. This just means that $J^2 = -I$. Each real plane $P^n \subset \mathbb{R}^{2N}$ contains a maximal complex sub-plane $P_{\mathbb{C}}$ given by

$$P_{\mathbb{C}} = P \cap JP.$$

The dimension of $P_{\mathbb{C}}$ is upper semi-continuous – it cannot increase under small perturbations. Further, for $n > N$ the minimal and generic complex dimension is $n - N$, while for $n \leq N$ this minimal dimension is zero.

For any $M^n \subset \mathbb{C}^N$, let

$$H = TM \cap JTM.$$

(It is easy to see that this is equivalent to our previous definition of H.)

Definition 12 *If $n > N$ and $\mathrm{rank}_{\mathbb{R}} H = 2(n-N)$, then M is said to be generic. If $n \leq N$ and $H = \{0\}$ then M is said to be totally real.*

Finally, the maximal complex plane H is related to the induced CR structure V_0 in a very natural way: If $\{L_1, \dots, L_n\}$ is a basis for V_0 at some point $p \in M$, then $\{\Re L_1, \Im L_1, \dots, \Re L_n, \Im L_n\}$ is a basis for H at that point. Hence when M is generic, V_0 is of constant rank and so defines a CR structure.

To further explain generic and totally real and to present some results used in §3, we have the following result.

Lemma 2 *For $M^{2n+d} \subset \mathbb{C}^{n+d}$, the following are equivalent*

1. *M is generic at p.*

2. *After a complex rotation at p, M is given locally by*

$$w_j = F_j(z, \overline{z}, t) \qquad j = 1, \dots, d$$

with $(z, w) \in \mathbb{C}^{n+d}$, $t \in \mathbb{R}^d$, $F : \mathbb{C}^n \times \mathbb{R}^d \to \mathbb{C}^d$ where for each fixed z the submanifold $\{(z, w) : w = F(z, \overline{z}, t)\}$ is totally real.

3. Proof of Theorem 1

We fix an open covering of M and local coordinates as in the definition of transversely holomorphic foliations.

Theorem 3 *(M, F) is complexifiable if and only if there exists a real analytic global vector-valued one-form $\omega : TM \to TF$ such that $\omega|_{TF}$ is the identity and in U^j*

$$\omega = \sum \omega_a \otimes \frac{\partial}{\partial t_a}$$

with

$$d\omega_a \cong 0 \quad mod\{\omega_1, \ldots, \omega_d, dz_1, \ldots, dz_n\}.$$

Remark. This is essentially the result of Haefliger and Sundararaman. We have only added the observation that the relation between J and ω can be made more explicit than is done in [5] and so we end up with a global real-valued ω.

We now prove Theorem 1 from Theorem 3.

Proof of Theorem 1. Let (M, F) be complexifiable. Then we have $\omega : TM \to TF$. Define an involutive structure on U^k by

$$V_0^k = \{dz_1^k, \ldots, dz_n^k, \omega_1^k, \ldots, \omega_d^k\}^\perp.$$

The elliptic structure on U^k is given by $V = \{dz_1^k, \ldots, dz_n^k\}^\perp$. So $V_0^k \subset V$.

Lemma 3 *V_0^k is a C^ω CR structure on U^k and $V_0^k = V_0^j$ on $U^k \cap U^j$.*

Proof: From the fact that $\omega|_{TF}$ is the identity, we see that the $2n + d$-form $\omega \wedge dz \wedge d\bar{z}$ is never zero. Hence V_0^k has constant rank (equal to n) and $V_0^k \cap \overline{V_0^k} = \{0\}$. Further, from

$$d\omega \cong 0 \quad mod\{\omega, dz\}$$

we see that $[V_0, V_0] \subset V_0$. Thus V_0^k is also a CR structure. Further, recall that the original elliptic structure could always be taken to be C^ω. Thus V_0^k is also a C^ω structure. That $V_0^k = V_0^j$ on $U^k \cap U^j$ follows from the facts that ω is globally well-defined and the transition functions $z^j = H_{jk}(z^k)$ are holomorphic.

For the converse, let the elliptic structure (M, V) dominate a CR structure (M, V_0). To show that (M, V) is complexifiable, we show how to construct the one-form ω.

We start with a general observation. Let F be a foliation of co-dimension $2n$ on a manifold M of dimension $2n + d$ and let H be a distribution on M of

$2n-$planes transverse to F. Then there exists a unique 1-form $\omega : TM \to TF$ such that

$$\begin{aligned} \omega^\perp &= H \\ \omega|_{TF} &= identity. \end{aligned}$$

Now if (M, V) dominates (M, V_0), then $H \subset TM$ defined by $\mathbb{C} \otimes H = V_0 \oplus \overline{V}_0$ is transverse to F. So we have such a form $\omega : TM \to TF$. Using the usual local coordinates (z, t) for the elliptic structure in some U, we write $\omega = \sum \omega_a \otimes \frac{\partial}{\partial t_a}$.

Since ω annihilates H, it also annihilates V_0. Further, since $V_0 \subset V$, dz also annihilates V_0. Thus from

$$\omega \wedge dz \wedge d\overline{z} \neq 0$$

we conclude

$$V_0 = \{dz_1, \ldots, dz_n, \omega_1, \ldots, \omega_d\}^\perp$$

in U^k. From $[V_0, V_0] \subset V_0$ it follows that

$$d\omega_a \cong 0 \quad mod\{dz, \omega\}.$$

We then apply Theorem 3 to derive that (M, V) is complexifiable.

It is possible to prove Theorem 1 without recourse to Theorem 3. First, if the elliptic structure (M^{2n+d}, V) is complexifiable, then the fact that the leaves of F are totally real implies that M is a generic submanifold of $\hat{M}$ and hence has an induced CR structure. Conversely, if (M^{2n+d}, V_0) is a C^ω CR structure with $V_0 \subset V$, then the local realizability result, Theorem D, may be used to construct a compatible global complex structure on $\hat{M}$. We want to provide a few details about this approach.

First assume (M, F) is complexifiable and let $\Phi : M^{2n+d} \to \hat{M}$ be the complexification. Thus $\hat{M}$ is a complex manifold of complex dimension $n + d$ and is holomorphically foliated by leaves of complex dimension d. For each leaf L of the elliptic foliation F on M, there is a leaf $\hat{L}$ of the holomorphic foliation $\hat{F}$ on $\hat{M}$ such that

$$\Phi(L) \subset \hat{L}.$$

In particular, for each $p \in M$, there exists some open neighborhood U of p in M and some open neighborhood $\hat{U}$ of $\Phi(p)$ in $\hat{M}$ and some holomorphic function $f : \hat{U} \to \mathbb{C}^n$ such that each component of $\hat{L} \cap \hat{U}$ is of the form $f^{-1}(c)$ and each component of $L \cap U$ is of the form $(f \circ \Phi)^{-1}(c)$. Recall that Φ is a C^ω map.

Lemma 4 *$\Phi(M)$ is a generic submanifold of $\hat{M}$ and so*

$$V_0' = (\mathbb{C} \otimes T(\Phi(M))) \cap T_{0,1}\hat{M}$$

is a C^ω CR structure on $\Phi(M)$.

Remark. It follows that $V_0 = \{X \in \mathbb{C} \otimes T(M),\ \Phi_*(X) \in V_0'\}$ is a C^ω CR structure on M dominated by the elliptic structure V. Thus (M, F) complexifiable implies V dominates a C^ω CR structure.

Proof: Choosing local coordinates $(z, \zeta) \in \mathbb{C}^{n+d}$ for $\hat{M}$ with the leaves of $\hat{F}$ given by $\{z = c\}$, we get local coordinates $(z, t) \in \mathbb{C}^n \times \mathbb{R}^d$ for M and $\Phi : M \to \hat{M}$ becomes

$$\Phi(z, t) = \big(z, f(z, \overline{z}, t)\big).$$

For each fixed z, the set

$$\{\zeta \in \mathbb{C}^d : \zeta = f(z, \overline{z}, t)\}$$

is a totally real sub-manifold. Thus, as a consequence of Lemma 2, $\Phi(M)$ is generic.

Now we need to show, conversely, that if the elliptic structure (M^{2n+d}, V) dominates some C^ω CR structure (M^{2n+d}, V_0), then (M, F) is complexifiable.

Since F is C^ω, each leaf may be complexified to yield a manifold M', $dim\ M' = 2n + 2d$, with a foliation F'. Each leaf of F' is of dimension $2d$ and has a complex structure. We claim that M' admits a complex structure in which F' is a holomorphic foliation and such that the induced complex structure on each leaf of F' agrees with this original complex structure.

We fix a covering of M, $M = \bigcup_j U^j$, by sufficiently small open sets. The foliation F is given by $\{f^j = c\}$, where $f_1^j, \ldots, f_n^j$ are functions on U^j with $Lf_k^j = 0$, $L \in V$, and $df_1^j \wedge \ldots \wedge d\bar{f}_n^j \neq 0$. Since $V_0 \subset V$, each f_k^j is also a CR function.

Lemma 5 *Assume that for each j there exist C^ω CR functions $g_1^j, \ldots, g_d^j$ on U^j such that*

$$df^j \wedge \overline{df^j} \wedge dg^j \neq 0 \text{ on } U^j$$

and for each j and k there exist functions F_{kj} and G_{kj} holomorphic on

$$f^j(U^j \cap U^k) \oplus g^j(U^j \cap U^k) \subset \mathbb{C}^{n+d}$$

such that

$$\begin{aligned} f^k &= F_{kj}(f^j) \\ g^k &= G_{kj}(f^j, g^j). \end{aligned}$$

Let h^j be the holomorphic extension to M' of g^j. Then the local charts $\{f^j, h^j\}$ define a complex structure on M' which is the desired complexification.

Remark. We use f^j to also denote the extension of f^j to an open subset of M', defined by taking f^j constant on the leaves of F'.

Proof: We only need to show

$$df^j \wedge \overline{df}^j \wedge dh^j \wedge \overline{dh}^j \neq 0;$$

everything else then follows. It suffices to do this for a point $p \in M$. We suppress the superscripts.

So assume that at some point $p \in M$ we have

$$df \wedge d\overline{f} \wedge dh \wedge d\overline{h} = 0. \tag{2}$$

Choose a basis $X_1, \ldots, X_{2n}, T_1, \ldots, T_d$ for T_pM such that

$$dg_k(X_j) = 0 \text{ for } k = 1, \ldots, d \text{ and } j = 1, \ldots, 2n$$

and such that each T_j is tangent to the leaf through p. Here we are using

$$df \wedge d\overline{f} \wedge dg \neq 0.$$

Note that also $dh_k(X_j) = 0$. Let $J : T_pL \to T_pL'$ correspond to the complexification of this leaf. Then

$$\{X_1, \ldots, X_{2n}, T_1, \ldots, T_d, JT_1, \ldots, JT_d\}$$

is a basis for T_pM'. From (2) we have

$$((df \wedge d\overline{f})(X_1 \wedge \ldots \wedge X_{2n}))((dh \wedge d\overline{h})(T_1 \wedge \ldots \wedge JT_d)) = 0.$$

Next, from $df \wedge d\overline{f} \wedge dg \neq 0$, we see that the first factor is not zero. Thus

$$(dh \wedge d\overline{h})(T_1 \wedge \ldots \wedge JT_d) = 0$$

from which it follows (since $(T_k + iJT_k)h = 0$) that

$$dh((T_1 - iJT_1) \wedge \ldots \wedge (T_d - iJT_d)) = 0$$

and hence (again using $(T_k + iJT_k)h = 0$)

$$(dh_1 \wedge \ldots \wedge dh_d)(T_1 \wedge \ldots \wedge T_d) = 0$$

and so also

$$(dg_1 \wedge \ldots \wedge dg_d)(T_1 \wedge \ldots \wedge T_d) = 0.$$

But this contradicts $df \wedge d\overline{f} \wedge dg \neq 0$.

So to prove Theorem 1, we need to show that given f^j as described immediately before Lemma 5, there exist g^j satisfying the assumptions of this lemma. We work in a fixed U^j and again drop the superscripts on f and g.

Lemma 6 *Let* (U^{2n+d}, V_0) *be a* C^ω *CR structure and* $f : U \to \mathbb{C}^n$ *a CR map with* $df \wedge d\overline{f} \neq 0$. *Fix some point* $p \in U$. *If* $\{q : f(q) = f(p)\}$ *is transverse to* H_p *then there exists a neighborhood* U_1 *of* p *and a CR mapping* $g : U_1 \to \mathbb{C}^d$ *for which* $df \wedge d\overline{f} \wedge dg \neq 0$. *Further,* g *is real analytic provided* f *is.*

Comparing this lemma with Theorem D we see that here the first n components are specified and we wish to find the remaining components.

Proof: By Theorem D, there is some C^ω CR map $U^{2n+d} \to \mathbb{C}^{n+d}$, so we may assume $U^{2n+d} \subset \mathbb{C}^{n+d}$ and $p_0 = 0 \in \mathbb{C}^{n+d}$. Let T_0 denote the tangent space to the leaf at $p = 0$. Our assumption is that T_0 is transverse to H_0. But $rank_{\mathbb{R}} H_0 + rank_{\mathbb{R}} T_0 = 2n + d$. Thus

$$H_0 \oplus T_0 = T_0 M$$

so $JT_0 \cap T_0 = \{0\}$ and T_0 is a totally real d-plane in $\mathbb{C}^{n+d}$. Thus we may find coordinates $(z_1 \dots z_n, w_1 \dots w_d)$ with $w = u + iv$ such that M is a graph over (z, u). That is

$$M = \{(z,\, u + i\psi(z, \overline{z}, u)\} \quad \text{and} \quad d\psi(0) = 0.$$

Note that $df|_{T_0} = 0$, so, at the origin, $df_1 \wedge \cdots \wedge df_n = \lambda dz_1 \wedge \ldots \wedge dz_n$ with $\lambda \neq 0$. Thus for $g_\ell = u_\ell + i\psi_\ell(z, \overline{z}, u)$ we have at the origin

$$df \wedge d\overline{f} \wedge dg = |\lambda|^2 dz \wedge d\overline{z} \wedge du \neq 0.$$

This proves the Lemma.

We need the transverse condition; this is demonstrated by simple examples.

Returning to $f^j : U^j \to \mathbb{C}^n$, we apply this lemma to obtain $g^j : U^j \to \mathbb{C}^d$ such that

$$df^j \wedge d\overline{f}^j \wedge dg^j \neq 0.$$

We claim that on $U^j \cap U^k$ we have

$$f^k = F_{kj}(f^j)$$
$$g^k = G_{kj}(f^i, g^j)$$

with F_{kj} holomorphic near $f^j(U^j \cap U^k)$ and G_{kj} holomorphic near $f^j(U^j \cap U^k) \oplus g^j(U^j \cap U^k)$. To see this, we may start with $U^j \subset \mathbb{C}^{n+d}$ and $f^j = z$, $g^j = w$. Then f^k is CR on $U^j \cap U^k$ and hence is the restriction of some holomorphic F_{kj} (Theorem E). Provided we shrink each U a little before starting, we can obtain that F_{kj} is holomorphic in a $\mathbb{C}^{n+d}$ neighborhood of the submanifold $f^j(U^j \cap U^k)$.

Thus

$$f^k = F_{kj}(z, w)|_M = F_{kj}(f^j, g^j).$$

In the same way

$$g^k = G_{kj}(f^j, g^j)$$

with G_{kj} holomorphic. However, $\{f = c\}$ gives the foliation F and it follows that F^{kj} is independent of g. So we are done.

Finally, we use Lemma 5 to complete the proof of Theorem 1.

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Department of Mathematical Sciences
Rutgers University
Camden, NJ 08102
USA