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PROJECTIVE DIMENSIONS FOR ONE POINT EXTENSION ALGEBRAS

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Let A be a finite dimensional k-algebra and let M be a finitely generated A-module. The algebra $B = A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$, with the usual matrix operations, is called the one-point extension of A by M. It is well-known that the finitely generated B-modules can be identified with triples (k^t, X, f), where X is an A-module and $f\colon M \otimes_k k^t \longrightarrow X$ is an A-morphism.

When dealing, for instance, with necessary conditions for an one-point extension $B = A[M]$ to be quasitilted, one needs good criterions for deciding the injective and the projective dimensions of a B-module (k^t, X, f) in terms of (homological) properties of X and of f. Denote by $\mathrm{pd}_A X$ and $\mathrm{id}_A X$ the projective and the injective dimensions of an A-module X, respectively. In [3](III.2.2), it has been proved that $\mathrm{id}_B(k^t, X, f) \leq 1$ if and only if $\mathrm{id}_A X \leq 1$ and $\mathrm{Ext}^1_A(M, X) = 0$. On the other hand, only necessary conditions for $\mathrm{pd}_B(k^t, X, f) \leq 1$ were established in general. The purpose of this short note is to establish a criterion for $\mathrm{pd}_B(k^t, X, f)$ to be at most one, generalizing known results from [3]. Let $\tilde{X} = (k^t, X, f)$ be an indecomposable B-module. We shall see in section 1 below that the morphism f induces naturally a morphism $\Theta_f\colon\ \mathrm{Hom}_A(\mathrm{Ker}f, -) \longrightarrow \mathrm{Ext}^2_A(\mathrm{Coker}f, -)$. Our criterion can be stated as follows.

Theorem. *Let A be an algebra with $gl\,dim A \leq 2$, $B = A[M]$ and $\tilde{X} = (k^t, X, f)$ an indecomposable B-module with $f \neq 0$. Then $pd_B\tilde{X} \leq 1$ if and only if*

(a) $Kerf$ is a projective A-module; and

(b) The sequence $Hom_A(Kerf, -) \xrightarrow{\Theta_f} Ext^2_A(Cokerf, -) \longrightarrow 0$ is exact.

We shall prove this result in section 1 below, while in section 2 we shall see some applications to the class of quasitilted algebras. The results of this paper were completed during exchange visits México - São Paulo. The authors wish to thank their Institutions, FAPESP in Brazil and CONACyT in México for support. The first author also acknowledges support by CNPq.

1. Main result

For a given non zero morphism $f\colon Y \longrightarrow X$ in mod$A$, we shall denote by ξ_f the exact sequence $0 \longrightarrow \mathrm{Ker} f \xrightarrow{\iota} Y \xrightarrow{f} X \xrightarrow{\pi} \mathrm{Coker} f \longrightarrow 0$, where ι and π denote the natural inclusion and projection, respectively. We shall first see that ξ_f induces naturally a morphism $\mathrm{Hom}_A(\mathrm{Ker} f, -) \xrightarrow{\Theta_f} \mathrm{Ext}^2_A(\mathrm{Coker} f, -)$. In fact, let $p\colon P_0 \longrightarrow X$ be the projective cover of X in mod$A$ and consider in mod$A$ the exact sequence

$$0 \longrightarrow K \longrightarrow Y \oplus P_0 \xrightarrow{(f,p)} X \longrightarrow 0$$

where $K = \mathrm{Ker}(f,p)$. From the snake lemma, we have the following exact sequences

$$(*) \qquad 0 \longrightarrow \mathrm{Ker} f \longrightarrow K \longrightarrow \Omega \longrightarrow 0 \quad \text{and}$$

$$(**) \qquad 0 \longrightarrow \Omega \longrightarrow P_C \longrightarrow \mathrm{Coker} f \longrightarrow 0$$

where $\Omega = \Omega^1(\mathrm{Coker} f)$ is the first syzygy of $\mathrm{Coker} f$ and $P_C \longrightarrow \mathrm{Coker} f$ is the projective cover of $\mathrm{Coker} f$ in mod$A$. Using $(**)$ and the fact that P_C is projective, we infer that, for each $i \geq 1$, the connecting morphism $\delta_i\colon \mathrm{Ext}^i_A(\Omega, -) \longrightarrow \mathrm{Ext}^{i+1}_A(\mathrm{Coker} f, -)$ is an isomorphism. Using $(*)$, we get the connecting morphism $(\mathrm{Ker} f, -) \xrightarrow{\gamma} \mathrm{Ext}^1_A(\Omega, -)$. Define now $\Theta_f = \delta_1 \gamma$.

We shall now prove our main result.

Theorem 1.1. *Let A be an algebra with gl dim$A \leq 2$, $B = A[M]$ and*

$\tilde{X} = (k^t, X, f)$ an indecomposable B-module with $f \neq 0$. Then $pd_B \tilde{X} \leq 1$ if and only if

(a) $Kerf$ is a projective A-module; and

(b) The sequence $Hom_A(Kerf, -) \xrightarrow{\Theta_f} Ext^2_A(Cokerf, -) \longrightarrow 0$ is exact.

Proof: The fact that $\mathrm{pd}_B \tilde{X} \leq 1$ implies that $\mathrm{Ker} f$ is projective in $\mathrm{mod} A$ is proved in [3](III.2.1). For the convenience of the reader, however, we shall show it here. We shall use the notation above, considering $Y = M^t$. In particular, denote by K the kernel of the morphism (f, p), where $p \colon P_0 \longrightarrow X$ is the projective cover of X in $\mathrm{mod} A$. Since $(0, P_0, 0)$ and (k, M, id) are projective B-modules, then the commutative diagram

shows that $\mathrm{pd}_B \tilde{X} \leq 1$ if and only if K is a projective A-module. Suppose first that $\mathrm{pd}_B \tilde{X} \leq 1$. Then K is a projective A-module and since $\mathrm{gl\,dim} A \leq 2$ we infer that $\mathrm{pd}_A \Omega \leq 1$. From $(*)$ we get that $\mathrm{Ker} f$ is a projective A-module and $\mathrm{Hom}_A(\mathrm{Ker} f, -) \xrightarrow{\gamma} \mathrm{Ext}^1_A(\Omega, -) \longrightarrow 0$ is an exact sequence. So it implies that $\mathrm{Hom}_A(\mathrm{Ker} f, -) \xrightarrow{\Theta_f} \mathrm{Ext}^2_A(\mathrm{Coker} f, -) \longrightarrow 0$ is also exact as required.
Conversely, if $\mathrm{Ker} f$ is a projective A-module and the sequence
$\mathrm{Hom}_A(\mathrm{Ker} f, -) \xrightarrow{\Theta_f} \mathrm{Ext}^2_A(\mathrm{Coker} f, -) \longrightarrow 0$ is exact, then $\mathrm{Ext}^1_A(K, -) = 0$ and K is therefore a projective A-module. This finishes the proof.

□

The following corollary has been proved in [3](III.2.1).

Corollary 1.2. *Let A be an algebra with $gl\,dim A \leq 2$, $B = A[M]$ and $\tilde{X} = (k^t, X, f)$ an indecomposable B-module. Assume that $pd_A(Cokerf) \leq 1$. Then $pd_B \tilde{X} \leq 1$ if and only if $Kerf$ is a projective A-module.*

2. Quasitilted algebras

An algebra A is called *quasitilted* provided: (i) $\mathrm{gl\,dim} A \leq 2$; and (ii) for each indecomposable A-module X, either $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$ (see [3]). In [1] and [2] we have discussed the situation where an one-point extension $B = A[M]$ is quasitilted. Here, we will show that our main theorem can be used to get some further criterion for such an algebra B to be quasitilted.

Proposition 2.1. *Let A be an algebra with gl dim$A \leq 2$, M a hereditary projective A-module and $B = A[M]$. Assume furthermore that if $\tilde{X} = (k^t, X, f)$ is an indecomposable B-module, then either $pd_A X \leq 1$ or $id_A X \leq 1$. Then B is a quasitilted algebra.*

Proof: Since $\mathrm{gl\,dim} A \leq 2$ and M is a hereditary projective A-module, we infer that $\mathrm{gl\,dim} B \leq 2$. Let $\tilde{X} = (k^t, X, f)$ be an indecomposable B-module. Suppose first that $t = 0$. Then X is an indecomposable A-module and, since A is quasitilted, we have that either $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$. Correspondingly, $\mathrm{pd}_B \tilde{X} \leq 1$ or $\mathrm{id}_B \tilde{X} \leq 1$ and the result is proved in this case. Assume now $t \neq 0$. If $\mathrm{id}_A X \leq 1$, then $\mathrm{id}_B \tilde{X} \leq 1$ because $\mathrm{Ext}^1_A(M, X) = 0$. Suppose now that $\mathrm{pd}_A X \leq 1$ and consider the exact sequences

$$(*) \qquad 0 \longrightarrow Ker f \longrightarrow M^t \longrightarrow C \longrightarrow 0 \quad \text{and}$$

$$(**) \qquad 0 \longrightarrow C \longrightarrow X \longrightarrow Coker f \longrightarrow 0$$

Since M is a hereditary projective A-module, $\mathrm{Ker} f$ is a projective A-module. Observe that $\mathrm{Ext}^1_A(M^t, -) = 0 = \mathrm{Ext}^2_A(X, -)$, and then, by the above construction, we infer that the sequence

$$\mathrm{Hom}_A(\mathrm{Ker} f, -) \xrightarrow{\Theta_f} \mathrm{Ext}^2_A(\mathrm{Coker} f, -) \longrightarrow 0$$

is exact. The result now follows from our main theorem.

□

For the next result we need some notations. For a quasitilted algebra A we denote by $\mathcal{L}_A$ (respectively, by $\mathcal{R}_A$) the subcategory of mod$A$ consisting of all indecomposable modules Y such that all its predecessors (respectively, all its successors) X have $\text{pd}_A X \leq 1$ (respectively, $\text{id}_A X \leq 1$) (see [3]).

Corollary 2.2. *Let A be a quasitilted algebra and M be a hereditary projective A-module such that $Hom_A(M, \mathcal{R}_A \setminus \mathcal{L}_A) = 0$. Then $B = A[M]$ is a quasitilted algebra.*

Proof: Just observe that if $\tilde{X} = (k^t, X, f)$ is an indecomposable B-module with $t > 0$, then $\text{pd}_A X \leq 1$.

□

Proposition 2.3. *Let A be an algebra with $gl\,dim A \leq 2$, $M \in modA$, $M = M_1 \oplus M_2$, $pd_A M \leq 1$, $B = A[M]$ and $B_i = A[M_i]$. Assume that each indecomposable B-module $\tilde{X} = (k^t, X, f)$ with $id_B \tilde{X} = 2$ is isomorphic to $(k^t, X_1 \oplus X_2, f_1 \oplus f_2)$ with $pd_{B_i}(k^t, X_i, f_i) \leq 1$, $i = 1, 2$. Then B is quasitilted.*

Proof: Since $\text{gl}\,\text{dim} A \leq 2$ and $\text{pd}_A M \leq 1$, we infer that $\text{gl}\,\text{dim} B \leq 2$. Therefore, to show that B is quasitilted, it suffices to show that each indecomposable $\tilde{X}$ with $\text{id}_B \tilde{X} = 2$ has $\text{pd}_B \tilde{X} \leq 1$. Let $\tilde{X} = (k^t, X, f)$ be an indecomposable B-module with $\text{id}_B \tilde{X} = 2$. By the hypothesis, $\tilde{X}$ is isomorphic to $(k^t, X_1 \oplus X_2, f_1 \oplus f_2)$ with $\text{pd}_{B_i}(k^t, X_i, f_i) \leq 1$, $i = 1, 2$. By 1.1, $\text{Ker} f_i$ is a projective A-module and the corresponding ξ_i induces an exact sequence $\text{Hom}_A(\text{Ker} f_i, -) \xrightarrow{\Theta_i} \text{Ext}^2_A(\text{Coker} f_i, -) \longrightarrow 0$, for $i = 1, 2$. So $\text{Ker}(f_1 \oplus f_2)$ is a projective A-module and $\xi_1 \oplus \xi_2$ induces an exact sequence $\text{Hom}_A(\text{Ker}(f_1 \oplus f_2), -) \xrightarrow{\Theta} \text{Ext}^2_A(\text{Coker}(f_1 \oplus f_2), -) \longrightarrow 0$. Hence $\text{pd}_B \tilde{X} \leq 1$, as required.

□

We end this note with an example which shows that an one-point extension of a quasitilted algebra by a hereditary projective module can be non-quasitilted.

Example 2.4. Let A be the radical square zero algebra given by the quiver

It is not difficult to see that A is a tilted algebra. Consider now the A-module $M = P_1 \oplus P_4$, where P_i denotes the indecomposable projective A-module corresponding to the vertex i. Clearly, M is a hereditary projective A-module. Let $\alpha\colon P_4 \longrightarrow I_4$ be a nonzero A-morphism, where I_4 is the indecomposable injective A-module corresponding to the vertex 4, and consider the B-module $Y = (k, P_1 \oplus I_4, \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right))$. It is not difficult to see that Y is an indecomposable B-module and it has both projective and injective dimensions equal to 2, and therefore $B = A[M]$ is not quasitilted.

□

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