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VALUE DISTRIBUTION OF THE GAUSS MAP OF COMPLETE MINIMAL SURFACES IN $\mathbf{R}^m$

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Let M be an (oriented) nonflat complete minimal surface immersed in $\mathbf{R}^3$. By definition, the Gauss map g of M is the map which maps each point $p \in M$ to the point $g(p) \in S^2$ such that $\overrightarrow{Og(p)} = n_p$, for the positive unit normal vector n_p of M at p. The unit sphere S^2 may be canonically identified with the 1-dimensional projective space $P^1(C)$ and M may be considered as an open Riemann surface with conformal metric. By the assumption of minimality of $M, g : M \to P^1(C)$ is anti-holomorphic. In the following, we call the conjugate of the map g the classical Gauss map of M.

In 1961, R. Osserman showed that g cannot omit a set of positive logarithmic capacity([8]). Afterwards, F. Xavier proved that g cannot omit 7 points([13]). In 1988, I have obtained the following final result of this subject;

Theorem A. *The Gauss map of a nonflat complete minimal surface immersed in* $\mathbf{R}^3$ *can omit at most four values*([3]).

Actually, there are many examples of nonflat complete minimal surfaces immersed in $\mathbf{R}^3$ whose Gauss maps omit exactly four values. Among these, Scherk's surface is most famous.

Moreover, I revealed some relations between this result and the defect relation in Nevanlinna theory on value distribution of meromorphic functions, and gave some modified defect relations for the Gauss maps of complete minimal surfaces([5]).

We next consider a nonflat complete minimal surface $x = (x_1, \ldots, x_m) : M \to \mathbf{R}^m$ immersed in $\mathbf{R}^m$. It is well-known that the set of all oriented

2-planes in $\mathbf{R}^m$ which contain the origin can be identified with the quadric

$$Q_{m-2}(C) := \{(w_1 : \cdots : w_m); w_1^2 + \cdots + w_m^2 = 0\} \subset P^{m-1}(C).$$

By definition, the (generalized) Gauss map G of M is the map which maps each $p \in M$ to the point in $Q_{m-2}(C)$ corresponding to the oriented tangent plane of M at p. By the assumption of minimality of M, each x_i is harmonic. So, if we set

$$f_i := \frac{\partial x_i}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - \sqrt{-1}\frac{\partial}{\partial v}\right) x_i$$

for each holomorphic local coordinate $z = u + \sqrt{-1}v$, then f_i is holomorphic. As is easily seen, G is locally given by $G = (f_1 : \cdots : f_m)$ and so a holomorphic map into $P^{m-1}(C)$.

In [6], I have given the following generalization of Theorem A.

Theorem B. *In the above situation, assume that G is nondegenerate, namely, $G(M) \not\subseteq H$ for any hyperplane H in $P^{m-1}(C)$. If G omits $q(> m)$ hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(C)$ which are located in general position, namely any m of which are linearly independent, then $q \leq m(m+1)/2$.*

Here, the number $m(m+1)/2$ is best possible for all odd numbers and some small even numbers m([4]).

Theorems A and B are given as easy consequences of the more general modified defect relations for holomorphic maps into $P^n(C)$, which will be explained in the following.

Let M be an open Riemann surface. A function u defined on M excluding a discrete set is said to have mild singularities if, around each point $a \in M$, we can write

$$(\#) \qquad |u| = |z-a|^{\sigma} \left|\log \frac{1}{|z-a|}\right|^{\tau} u^*$$

with some $\sigma, \tau \in \mathbf{R}$, a positive C^∞-function u^* and a holomorphic local coordinate z. For such a function u, we define the divisor $\nu_u : M \to \mathbf{R}$ of u by

$$\nu_u(a) := \text{the number } \sigma \text{ of the expression } (\#) \text{ for some } \tau \text{ and } u^*$$

for each $a \in M$. Here, a divisor on M means a map $v : M \to \mathbf{R}$ whose support $| v | := \{z; v(z) \neq 0\}$ is discrete.

For a nonzero meromorphic function ψ on $M, v_\psi(a)$ is just the order of ψ at a.

To a divisor v we correspond the $(1,1)$-current $[v]$ defined by

$$[v](\varphi) = \sum_{z \in M} v(z)\varphi(z) \qquad (\varphi \in \mathcal{D}),$$

where $\mathcal{D}$ denotes the set of all C^∞-functions on M with compact supports. In some cases, a $(1,1)$-form Ω on M is regarded as a $(1,1)$-current defined by

$$\Omega(\varphi) = \int_M \varphi^\Omega (\varphi \in \mathcal{D}).$$

For two $(1,1)$-currents Ω_1 and Ω_2, by $\Omega_1 \prec \Omega_2$ we mean that there is a bounded continuous nonnegative function k with mild singularities and a nonnegative integer-valued divisor v such that

$$\Omega_1 + [v] = \Omega_2 + dd^c \log k^2,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$.

Let $f : M \to P^n(C)$ be a holomorphic map and

$$H : a_0 w_0 + \cdots + a_n w_n = 0$$

be a hyperplane in $P^n(C)$ with $f(M) \not\subset H$. Take a representation $f = (f_0 : \cdots : f_n)$ on M which is reduced, namely whose components are holomorphic functions with common zeros. Set $F(H) := a_0 f_0 + \cdots + a_n f_n$ and $v(f, H) := v_{F(H)}$. The n-truncated pull-back $f^*(H)^{[n]}$ of H as a divisor is defined by

$$f^*(H)^{[n]} := [\min(v(f,H), n)].$$

We have always $f^*(H)^{[n]} \prec \Omega_f$, where Ω_f denotes the pull-back of the Fubini-Study metric on $P^n(C)$.

Definition 1. *We define the modified H-defect of H of f by*

$$D_f(H) := 1 - \inf\{\eta \geq 0; f^*(H)^{[n]} \prec \eta \Omega_f \text{ on } M - K \text{ for a compact set } K\}.$$

The modified H-defect has the following properties.

Proposition 2.
(i) $0 \leq D_f(H) \leq 1$.
(ii) If there exists a bounded nonzero holomorphic function g on M excluding a compact set K such that $v_g \geq \min(v(f,H),n)$ on $M-K$, or particularly if $\#f^{-1}(H) < \infty$, then $D_f(H) = 1$.
(iii) If $v(f,H) \geq m$ at every $a \in f^{-1}(H) - K$ for some compact set K, then $D_f(H) \geq 1 - n/m$.

Proof. The assertion (i) is trivial, and (ii) is also obvious because

$$f^*(H)^{[n]} \prec \eta\Omega_f + dd^c \log |\, g\,|^2$$

on $M - K$ for $\eta = 0$. We have also

$$f^*(H)^{[n]} \prec (n/m) dd^c \log \|\, f\,\|^2 + dd^c \log k^2$$

on $M-K$ for the bounded function $k := \left(\frac{|F(H)|}{\|f\|}\right)^{n/m}$, where

$$\|\, f\,\| = (|\, f_0\,|^2 + \cdots + |\, f_n\,|^2)^{1/2}$$

for a reduced representation $f = (f_0 : \cdots : f_n)$. This gives (iii).

We now recall the classical value distribution theory of holomorphic maps of $\Delta_{R,\infty} := \{z; R \leq |\, z\,| < \infty\}$ into $P^n(C)$. The order function of f and the counting function (truncated by n) of a hyperplane H for f are defined by

$$T_f(r) = \int_R^r \frac{dt}{t} \int_{R \leq |z| \leq t} \Omega_f \qquad (R < r < +\infty),$$

$$N_f(r)^{[n]} = \int_R^r \frac{dt}{t} \int_{R \leq |z| \leq t} f^*(H)^{[n]} \qquad (R < r < +\infty),$$

respectively. The classical defect (truncated by n) is defined by

$$\delta_f(H)^{[n]} = 1 - \limsup_{r \to \infty} \frac{N_f(r)^{[n]}}{T_f(r)}.$$

We can prove the following relation between the classical defect and the modified H-defect.

Proposition 3. *Let f be a nondegenerate holomorphic map of an open Riemann surface M into $P^n(C)$. Assume that there is a biholomorphic map Φ of an open set in M onto a neighborhood of $\Delta_{R,\infty}$ such that the restriction $\tilde{f} := f \circ \Phi^{-1} \mid \Delta_{R,\infty}$ has an essential singularity at ∞. Then, for every hyperplane H,*

$$0 \geq D_f(H) \leq \delta_f(H)^{[n]} \leq 1$$

This is shown by the same argument as in [5,§1].

We state here the classical defect relation which are given by Cartan, Ahlfors and Weyl and improved by Nochka.

Theorem 4. *Let $f : \Delta_{R,\infty} \to P^n(C)$ be a holomorphic map with an essential singularity at ∞. Assume that $f(\Delta_{R,\infty})$ is not included in any $(k-1)$-dimensional projective linear subspace of $P^n(C)$. Then, for arbitrary hyperplanes $H_j (1 \leq j \leq q)$ in general position,*

$$\sum_{j=1}^{q} \delta_f(H_j)^{[n]} \leq 2n - k + 1.$$

Now, we give some definitions in order to state the modified defect relation. We call ds^2 a pseudo-metric on M if it is locally written $ds^2 = \lambda_z^2 \mid dz \mid^2$ with a nonnegative function λ_z with mild singularities. A continuous pseudo-metric ds^2 means a pseudo-metric such that the above λ_z is continuous. For a pseudo-metric ds^2 we define the Ricci form by $Ric_{ds^2} := -dd^c \log \lambda_z^2$ considered as a current, the total curvature of M by $C(M) := \int_M Ric_{ds^2}$ and the Gaussian curvature by $K_{ds^2} := -\frac{1}{\lambda_z^2} \frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda_z^2$. The curvature is called to be strictly negative if there exists a positive constant C such that $K_{ds^2} \leq -C$. We call an open Riemann surface M to be of finite type if M is biholomorphic with a compact Riemann surface $\bar{M}$ with finitely many points removed, a holomorphic map f of such a Riemann surface M into $P^n(C)$ to be transcendental if f has no holomorphic extension to $\bar{M}$. We give another definition.

Definition 5. *We define the H-order of a holomorphic map $f : M \to P^n(C)$*

by

$$p_f := \inf\{p \geq; -Ric_{ds^2} \prec p\Omega_f \text{on } M - K \text{ for some compact set} K\}.$$

The modified defect relation is stated as follows:

Theorem 6. *Let (M, ds^2) be an open Riemann surface with a complete continuous pseudo-metric ds^2 and $f : M \to P^n(C)$ a holomorphic map. Assume that $f(M)$ is included in some k-dimensional projective linear subspace of $P^n(C)$ and not included in any $(k-1)$-dimensional projective linear subspace. If M is not of finite type, or else f is transcendental, then for any hyperplanes $H_1, \ldots, H_q$ in general position*

$$\sum_{j=1}^{q} D_f(H_j) \leq 2n - k + 1 + \frac{p_f k(2n-k+1)}{2}.$$

For the proof of Theorem 6, we need the following Theorem on pseudometrics with negative curvature.

Theorem 7. *Let (M, ds^2) be as in Theorem 6 and $d\tau^2$ a continuous pseudometric on M whose curvature is strictly negative on $M - K$ for a compact set K. If there exist a constant p with $0 < p < 1$, a divisor v and a bounded function k with mild singularities such that $v(a) \geq 1 - p$ for each $a \in |v|$ and*

$$-Ric_{ds^2} + [v] = p(-Ric_{d\tau^2}) + dd^c \log k^2$$

on $M - K$, then M is of finite type.

The proof of Theorem 7 is mainly due to the generalized Schwarz lemma and the classical Huber's Theorem. For the proof of Theorem 6 we construct a suitable pseudo-metric $d\tau^2$ satisfying the conditions of Theorem 7 for some p and estimate the constant p. We omit the details.

As an application of Theorem 6, we have the following modified defect.

Theorem 8. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and G the Gauss map of M. Then, for any* *hyperplanes $H_1, \ldots, H_q$ located in general position*

$$\sum_{j=1}^{q} D_G(H_j) \le \frac{m(m+1)}{2}.$$

Proof. By Chern-Osserman's Theorem ([1]), M is not of finite type, or else G is transcendental. On the other hand, we can write locally

$$ds^2 = 2 \| f \|^2 | dz |^2,$$

which implies $p_G \le 1$. The Gauss map $G : M \to P^{m-1}(C)$ satisfies the conditions of Theorem 6 for some k with $1 \le k < m$. So, we have

$$\begin{aligned} \sum_{j=1}^{q} D_G(H_j) &\le 2(m-1) - k + 1 + \frac{k(2(m-1) - k + 1)}{2} \\ &= \frac{m(m+1) - (m-k-1)(m-k-2)}{2} \le \frac{m(m+1)}{2}. \end{aligned}$$

In [1], Chern-Osserman proved that the Gauss map G of a nonflat complete minimal surface M with finite total curvature immersed in $\mathbf{R}^m$ cannot omit $(m-1)(m+2)/2 (< m(m+1)/2)$ hyperplanes in general position if G is nondegenerate. Recently, Ru noted that the 'nondegeneracy' assumption of this result can he dropped. Theorem B is now an immediate consequence of Theorem 8 and Proposition 2, (ii).

For a minimal surface in $\mathbf{R}^3$, we have the following modified defect relation.

Theorem 9. *Let M be a nonflat complete minimal surface with infinite total curvature and $g : M \to P^1(C)$ the classical Gauss map. Then, for arbitrary distinct values $\alpha_1, \ldots, \alpha_q \in P^1(C)$, we have*

$$\sum_{j=1}^{q} D_g(\alpha_j) \le 4.$$

Proof. In this case, we have $p_g \le 2$, because ds^2 is written as

$$ds^2 = | h |^2 (| g_0 |^2 + | g_1 |^2)^2 | dz |^2$$

with a nowhere zero holomorphic function h if we take a reduced representation $g = (g_0 : g_1)$. Consider the case $n = k = 1$ and $p_g \leq 2$ in Theorem 6. We have $\sum_{j=1}^{q} D_g(\alpha_j) \leq 2 + 2 = 4$. This gives Theorem 9.

For the case where M has finite total curvature, the Gauss map g can omit at most 3 values by Osserman's result. Therefore, we have Theorem A.

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