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HYPERSURFACES IN THE COMPLEX HYPERBOLIC SPACE

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A.D.Alexandrov ([1]) proved that the geodesic spheres are the only compact embedded hypersurfaces with constant mean curvature in a simply connected space of constant curvature (with the additional hypothesis of the hypersurface being contained in a hemisphere in the spherical case). Since in a two point homogeneous space the geodesic spheres are homogeneous hypersurfaces and therefore with constant mean curvature, it is natural to ask if Alexandrov's Theorem can be extended to these spaces. We answer this question affirmatively for the complex hyperbolic space CH^n. For simplicity, we work just in the 2-dimensional (complex) case. We prove:

Theorem 1. *Let M be a compact embedded hypersurface with constant mean curvature of the complex hyperbolic space CH^2. Then M is a geodesic sphere.*

In a recent paper, J. L. Barbosa, M. P. do Carmo and J. Eschenburg proved that the geodesic spheres in CH^2 are stable ([2]). If we take the geodesics spheres passing through a given point and centered at a given geodesic of the space we obtain a one parameter family of geodesic spheres which, as the center tends to infinity, converges to a so-called horosphere of CH^2. It can be proven that horospheres have constant mean curvature and this brings naturally the question of determining the stability of such hypersurfaces. As it happens in the real hyperbolic space, we prove that the horospheres in CH^2 are all stable. Considering in CH^2 a Riemannian metric whose sectional curvature vary between -4 and -1, in ([3]) we prove:

*Comunication research presented by the first author.

Theorem 2. *The horospheres of CH^2 are hypersurfaces with constant mean curvature equal to 4/3 and are all stable. Furthermore, they are the orbits of the Heinsenberg's group (a 3-dimensional non commutative nilpotent Lie group) which acts by isometries in CH^2 whithout fixed points (therefore, the horospheres inherit a natural Lie group structure). In particular, the horospheres are (extrinsically) homogeneous submanifolds of CH^2. Any two horospheres of CH^2 are congruent.*

Besides the geodesic spheres and the horospheres, other nice examples of hypersurfaces with constant mean curvature in CH^2 are the equidistant hypersurfaces, defined in the following way: the hyperbolic plane H^2 can be isometrically embedded in a unique way (up to congruences)as a totally geodesic submanifold of CH^2 (this follows from the characterization of the totally geodesics submanifolds of a symmetric space). Therefore, given $c > 0$, an equidistant hypersurface P_c is defined as the boundary of the tubular neighborhood with radius $\sinh^{-1}(c)$ of H^2. In ([3]) we give a detailed description of such hypersurfaces. In particular, we prove:

Theorem 3. *An equidistant hypersurface P_c in CH^2 is a homogeneous hypersurface with constant mean curvature $(1+4c^2)/(3c\sqrt{1+c^2})$. It is stable if $c \geq \sqrt{2}/2$ and unstable otherwise. Two equidistant hypersurfaces are congruent iff they have the same mean curvature.*

We remark that the function $(1+4c^2)/(3c\sqrt{1+c^2})$ attains the minimun absolute value at $c = \sqrt{2}/2$ that is $P_{\sqrt{2}/2}$ is the equidistant hypersurface whose mean curvature is the smallest one.

References

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