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TOPOLOGICAL OBSTRUCTIONS TO THE EMBEDDABILITY OF CR-MANIFOLDS

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Introduction

Any Riemannian manifold can be isometrically embedded into an Euclidean space. The analogous statement for complex manifolds is false. Aside from the obvious fact, by the maximum principle, that compact complex manifolds cannot be holomorphically embedded into $\mathbf{C}^N$, a manifold admits a holomorphic embedding into $\mathbf{C}^N$ which is proper, if and only if it is a Stein manifold. We study the question of embeddability for structures modeled after boundaries of complex manifolds.

Boundaries of complex manifolds were studied by Poincaré in 1907 [P], where he showed that the polydisc is not biholomorphic to the ball in two complex dimensions, by reducing the problem to the boundaries of those domains. The abstract definition of the boundary as a CR-structure on a manifold is essentially in E. Cartan [C], where a classification of the 3-dimensional homogeneous structures is given. A CR-structure over a manifold of real dimension 3 is given locally by a complex vector field L, defined up to a nonvanishing function, which is not purely real nor imaginary at any point. In other words, a CR-structure is a subbundle of the complexified tangent bundle all of whose fibers are not purely imaginary nor real. A hypersurface N embedded in a complex manifold M has a natural CR-structure given by the intersection $T^{1,0}M \cap CTN$. A simple count of dimensions shows that this intersection is of complex dimension one. In particular, considering the standard sphere in $\mathbf{C}^2, S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \ / \ \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$ the standard CR-structure is given by the global vector field

$$L = z_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial \bar{z}_1}$$

We consider only strictly-pseudoconvex CR-manifolds. In the 3 dimensional case this simply means that $L, \bar{L}, [L, \bar{L}]$ are linearly independent at every point. This is an open condition, therefore small deformations of the complex vector field are still strictly pseudoconvex.

Definition 1. We say that a CR-structure L on a manifold N is embeddable if there exists an embedding $f : N \to \mathbf{C}^n$ for some n, where $f_*(L) \in T^{1,0}\mathbf{C}^n$. A CR-structure on N is locally embeddable if, given $p \in N$, there exists a neighborhood of p which is embeddable.

Definition 2. A complex function $f : N \to \mathbf{C}$ on a manifold with CR-structure L is said to be a CR-function if $L(f) = 0$.

In [BM] Boutet de Monvel showed that any compact, orientable d-dimensional, $d \geq 5$, strictly pseudoconvex CR-manifold is embeddable in some $\mathbf{C}^N$. By a result of Harvey and Lawson [HL], this manifold is the boundary of a subvariety in $\mathbf{C}^N$. On the other hand, 3-dimensional compact orientable CR-manifolds are not necessarily embeddable. The first example is attributed to Andreotti in [R]. This example is actually on the list of homogeneous structures of Cartan, but at that time it was unclear whether it was embeddable.

Strictly pseudoconvex CR-manifolds arise naturally as boundaries of Stein spaces. By a theorem of Andreotti-Frankel-Milnor [M1], the fundamental groups of a Stein manifold and its boundary are isomorphic in dimensions > 2. This is not the case in complex dimension 2. We give constructions of non-embeddable three dimensional CR-manifolds arising from covers of boundaries of Stein manifolds which have the property that the fundamental groups of the manifold and its boundary are distinct.

A more radical phenomenom occurs in 3-dimensions, namely, that even locally it might not be embeddable, as was shown for the first time by Nirenberg

[N]. This situation was further analysed by Jacobowitz and Treves [JT 1,2] to show that, in fact, non-embeddable CR-structures are, in some sense, dense in the space of CR-structures over a 3-dimensional manifold in the higher dimensional case. It is known that local embeddability occurs always in dimensions ≥ 7 [K] [A], but the 5-dimensional case is not settled. We also show that local non-embeddability can be obtained by pasting together non-embeddable structures on S^3 arising from the series of quotient singularities A_n, providing a more geometrical explanation for the local non-embeddability than the techniques in [N] and [JT 1,2]. Most of the results announced here appeared in [F,F1], to which we refer for the proofs.

Covers of boundaries of Stein manifolds

Let X be a topological space. Suppose X is a relatively compact subset of a larger space Y. We say that an open set $V \subset X$ is a neighborhood of ∂X if it is the intersection of a neighborhood of ∂X in Y and X. A distinguished neighborhood of ∂X is defined to be such a neighborhood which doesn't have relatively compact components in X. If X is not a subspace of a larger space, we still can define a distinguished neighborhood of the boundary of X as an open set V such that $V = X - K$, where K is a compact in X and there are no relatively compact components of $X - K$. We say then that V is a neighborhood of ∂X, as we will consider only distinguished neighborhoods.

Observe that this definition doesn't imply that the neighborhood is connected. As an example, consider the cylinder $S^1 \times (0,1)$ which clearly has neighborhoods of the boundary which have two connected components.

We are interested in boundaries of Stein spaces. As any open Riemann surface is a Stein space, there are no restrictions on the possible boundaries of 1-dimensional Stein spaces. In particular, as in the case of the cylinder, the boundary might not be connected. The situation changes completly in higher dimensions.

Theorem 1. [M1]*Let S be a connected Stein manifold of dimension strictly* *greater than 1. Then a neighborhood of ∂S is connected. If dimension of S is greater than 2, then $\pi_1(S) = \pi_1(\partial S)$*

In dimension 2, we have the following

Theorem 2. *Let S be a simply conected Stein manifold of dimension 2. Let V be a neighborhood of ∂S. Suppose that $\pi : \mathbf{M} \to \mathbf{V}$ is a finite nonramified cover of complex manifolds. Then holomorphic functions of $\mathbf{M}$ are pull-backs of holomorphic functions on $\mathbf{V}$.*

The set $\mathbf{V}$ above can be thought of as a tubular neighborhood of a strictly pseudoconvex CR-manifold, and it can actually be confounded with this manifold. To formulate the version for CR-manifolds, we recall that a function ϕ, on a complex manifold is said to be a strictly plurisubharmonic exaustion function if its level sets are strictiy pseudoconvex CR-manifolds.

Corollary 1. *Let ϕ be a strictly plurisubharmonic exaustion function of a simply connected Stein manifold S of dimension 2. Let V be a neighborhood of ∂S and $\pi : M \to V$ a finite nonramified cover. Then the compact connected level sets of $\pi^*(\phi)$ are non-embeddable CR-manifolds. In fact CR functions are pull-backs of CR-functions of the level set of ϕ in V.*

Remarks:

1. The theorem is still true if we delete small parts of the manifold $\mathbf{V}$. This fact is very important, for later use in the pasting of CR-manifolds, Analogously, it is not fundamental that we cover the whole CR-manifold of the boundary, but a large enough piece of it. See [F] for details.

2. The theorem is still true if S is not simply connected, but has different fundamental group than its boundary, as in example 1.

Example 1. Consider the Reinhardt domain

$$S = \{(z_1, z_2) \in \mathbf{C}^2 \ / \ (|\, z_1 \,| -1)^2 + (|\, z_2 \,| -1)^2 < 1/2\}$$

Observe that it is homeomorphic to $S^1 \times S^1 \times D^1$, whereas ∂S is homeomorphic to $S^1 \times S^1 \times S^1$. It is easy to see that it is a Stein manifold with a strictly pseudoconvex boundary. S is not simply connected but has different fundamental group than its boundary. We describe two completely different possible 2-coverings of ∂S.

i) The difference of the topology is concentrated in the last disc. If we consider a finite cover of the last S^1 we obtain a CR-manifold whose CR-functions are pull-backs of CR-functions on ∂S.

ii) Consider now a double cover of the first S^1, we can give it explicitly as

$$M = \{(z_1, z_2, w) \in \mathbf{C}^3 \ / \ w^2 = z_1 \ \text{ and } \ (|\, z_1 \,| -1)^2 + (|\, z_2 \,| -1)^2 = 1/2\}$$

In this case we see clearly that M is embedded into $\mathbf{C}^3$.

Example 2. Singularities. Consider a polynomial $f(x_1, x_2, x_3)$ which vanishes at the origin and $\mathbf{A}$ the algebraic set with an isolated singularity at the origin

$$\mathbf{A} = \{(x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ f(x_1, x_2, x_3) = 0\}$$

Define the Stein neighborhood $\mathbf{S}^\epsilon = \mathbf{A} \cap B_\epsilon$ where B_ϵ, is a small ball centered at the origin. In the following we will fix an arbitrarily small ϵ, and denote the Stein neighborhood simply by $\mathbf{S}$, dropping the superscript. A smoothing of $\mathbf{S}$ is the family

$$\mathbf{S}_c = \{(x_1, x_2, x_3) \in B_\epsilon / f(x_1, x_2, x_3) = c\}$$

In particular we will consider the quotient singularities. Those are precisely the ones for which $\pi_1(\partial \mathbf{S})$ is finite. Let $G \subset SU(2)$ be a finite subgroup. We consider $SU(2)$ acting on $\mathbf{C}^2$ as a matrix group. It is a classical result that $G \backslash \mathbf{C}^2$ can be given a structure of an analytic space with one isolated singularity. In fact, the ring of polynomials in $\mathbf{C}[z_1, z_2]$ which are invariant by G is generated by 3 polynomials and those can be used to construct a map $\pi : \mathbf{C}^2 \to \mathbf{C}^3$ that induces an isomorphism of the quotient to a hypersurface in $\mathbf{C}^3$. Observe then that $G \backslash S^3$ is diffeomorphic to ∂S, the boundary of a

Stein neighborhood of the isolated singularity. As a specific example let G be the cyclic subgroup of order k given by

$$G_k = \{\begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix} \in SU(2) \mid g^k = 1\}$$

The invariant polynomials are generated by z_1^k, z_2^k and $z_1 . z_2$, so by the observations above $G_k \backslash \mathbf{C}^2$ is isomorphic to the hypersurface defined by the equation $x_1 \cdot x_2 - x_3^k = 0$.

$$\mathbf{A}_{k-1} = \{(x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ x_1 \cdot x_2 - x_3^k = 0\}$$

The map $\pi : \mathbf{C}^2 \to \mathbf{C}^3$ defines a non-ramified cover outside the origin, that is $\pi : \mathbf{C}^2 - \{0\} \to \mathbf{A}_{k-1} - \{0\}$ is a k-sheeted holomorphic cover. Analogous descriptions can be given for the groups D_k, E_6, E_7, E_8. See [Lo]

If we construct $\mathbf{S}_c$ as indicated above, then a neighborhood $\mathbf{V}_c$ of $\partial \mathbf{S}_c$ will have a cover diffeomorphic to the space between two concentric spheres in $\mathbf{C}^2$. Therefore on a tubular neighborhood of $S^3 \subset \mathbf{C}^2$ there exists a family of complex structures which we denote by $\mathbf{M}_c$ such that $\pi : \mathbf{M}_c \to \mathbf{V}_c$ is a holomorphic cover. Observe that $\mathbf{V}_c$ and $\mathbf{S}_c$ satisfy the hypotheses of theorem 2 by [M2], therefore we get the following

Theorem 3. *For each finite subgroup G of $SU(2)$ we get families of complex structures on a tubular neighborhood of $S^3 \subset \mathbf{C}^2$ such that holomorphic functions are invariant under G.*

Corollary 2. *For each finite subgroup G of $SU(2)$ there is a family L_c^G of G-invariant CR-structures on S^3, deformation of the standard structure, such that CR-functions are invariant under G for $c \neq 0$.*

The following example is attributed to Andreotti in [R], see also [AS] and [B]. It corresponds to the proposition above in the case of the quadratic quotient singularity

$$\mathbf{A}_1 = \{(x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ x_1 \cdot x_2 - x_3^2 = 0\}$$

Let $\mathbf{M} = \{(z_1, z_2) \in \mathbf{C}^2 \ / \ \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$ be the sphere $S^3 \subset \mathbf{C}^2$. For a small complex number c define the map $\pi_c : \mathbf{M} \to \mathbf{C}^3$ given by

$$\begin{aligned} x_1 &= z_1^2 + t z_2^2 \\ x_2 &= z_2^2 + t z_1^2 \\ x_3 &= z_1 z_2 - t z_1 z_2 \end{aligned}$$

It is then clear that the map has its image inside the manifold

$$S_c = \{(x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ x_1 \cdot x_2 - x_3^2 = c\}$$

Moreover the image of S^3 is the real projective space P^3 given by the intersection

$$S_c \cap \{(x_1, x_2, x_3) \in \mathbf{C}^3 \ / \ \mid x_1 \mid^2 + \mid x_2 \mid^2 + 2 \mid x_3 \mid^2 = 1\}$$

It is easy to see that the CR-structure on S^3 induced by π_c is given by the vector field $Z = L + cL$. From corollary 2, CR-functions are invariant under the map $(z_1, z_2) \to (-z_1, -z_2)$.

Pasting CR-structures

One of the most fundamental facts in 1 dimensional complex manifolds is that we can make connected sums, this fact is essentially a consequence of the fact that a disc in $\mathbf{C}$ is biholomorphic to the complement of any disc centered at the compactification point of $\mathbf{C}$, and is reflected by the observation that there exist inversions of the annulus mapping one boundary component to the other.

This situation is no longer true in higher dimensions. Essentially because there are no one point compactifications of $\mathbf{C}^n, n > 1$. Fortunately, things are different for CR-structures. Analogously to the 1-dimensional complex case, the quadric, which is the standard open model, admits a 1-point compactification which is $S^3 \subset \mathbf{C}^2$ and an open ball in the quadric is CR-biholomorphic to a complement of a neighborhood of the compactification point.

The quadric in C^2 is the real 3-dimensional manifold defined as

$$Q = \{(z_1, z_2) \in C^2 \,/\, Im z_2 = | z_1 |^2\}$$

and its CR-structure is given by the complex vector field

$$L = i\frac{\partial}{\partial z_1} - 2\bar{z}_1\frac{\partial}{\partial z_2}$$

The quadric Q is CR-biholomorphic to $S^3 - \{p\} \subset C^2$, where $p = (1,0) \in C^2$, by the following transformations

$$\eta = z_2 - i/z_2 + 1 \qquad \varsigma = 2z_1/z_2 + i$$

Consider the point $(0,0) \in Q$ and the neighborhood

$$B_\delta = \{(z_1, z_2) \in Q \,/\, | z_1 |^2 + x_2^2 < \delta\}$$

By the transformation $(z_1, z_2) \to (tz_1, t^2 z_2), t \in R^+$ the neighborhood B_δ is dilated. Given a neighborhood of p it is clear then that, for a suficiently large t, B_δ is CR-biholomorphic to the complement of a closed set contained in this neighborhood.

Proposition 1. *Let M and M' be CR-manifolds which have balls B and B' with the standard structure. Then they admit a pasting which doesn't change the structure in the complement of these balls.*

Now, pasting together the examples of non-embeddable CR-structures on S^3 of corollary 2, we will obtain an example of a CR-structure which is not locally embeddable. Observe that if we paste a sphere we don't change the topology of a manifold.

Theorem 4. *There exists a CR-structure on B_δ arbitrarily near the standard one, obtained by pasting non-embeddable structures on S^3 which acumulate at the origin of B_δ, such that, any germ of CR-function at the origin has vanishing first derivatives.*

Remarks: 1. We can improve the theorem above using the same technique to show that given any point P in an embeddable CR-structure of dimension 3, there exists a deformation obtained by pasting spheres, so that CR-functions have vanishing first derivatives at P. See [F].

2. In higher dimensions we can find nonembeddable strictly pseudoconvex open CR-manifolds. In particular, we were able to find arbitrarily small deformations of the standard strictly pseudoconvex structure on an arbitrarily small neighborhood of S^5 which are not embeddable. Also, if we allow different Levi forms, the topological restrictions expressed in theorem 1 do not hold.

3. In example 2, we see that surface singularities can give rise to simply connected Stein manifolds with non-simply connected boundaries. It is an open question to classify all Stein manifolds with this property.

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